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CONVERGENCE AND CONVERGENCE RATE TO FRACTIONAL BROWNIAN MOTION FOR WEIGHTED RANDOM SUMS

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Abstract. We consider infinite sums of weighted i.i.d. random variables, with finite variance and arbitrary distribution, and derive a necessary and sufficient conditions for the weak convergence (in function space with uniform topology) of normalized sums to fractional Brownian motion (FBM). We consider also convergence rates questions. Using the embedding suggested by the Komlós–Major–Tusnády strong approximations method, we derive (under certain conditions on the weights) estimates for the quality of the functional approximation to FBM.

1. Introduction

In this paper we study approximations to a fractional Brownian motion, with Hurst parameter $H > 1/2$. The approximating processes are based on a stationary sequence of random variables $\{X_j, j \in \mathbb{Z}\}$ obtained by weighted sums of i.i.d. random variables $\{\xi_k, k \in \mathbb{Z}\}$:

$$X_j = \sum_{k=-\infty}^{\infty} a_{j-k}\xi_k. \tag{1}$$

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The coefficients $\{a_k\}$ are deterministic, and, depending on their behavior, the sequence $\{X_j\}$ may be long-range dependent, in the sense that its correlation decays polynomially.

Recall that a standard fractional Brownian motion (FBM) is defined as a continuous Gaussian process $B_H = \{B_H(t), t \geq 0\}$ with stationary increments such that

$$\mathbf{E}B_H(t) = 0, \quad \mathbf{E}B_H^2(t) = Lt^{2H}.$$

We call L the variance parameter, and H the Hurst parameter. We shall also use the symbol FBM(H, L) to denote a fractional Brownian motion with Hurst parameter H and variance parameter L. In fact, H is the self-similarity index; that is, the scaled process $\{B_H(\lambda t), t \geq 0\}$ is identical in distribution to $\{\lambda^H B_H(t), t \geq 0\}$, for any $\lambda > 0$. It turns that $0 < H < 1$. The case $H = 1/2$ corresponds to ordinary Brownian motion. Otherwise, H can be smaller or larger than $1/2$ yielding, respectively, negative or positive correlated increments. We are interested in the latter case only, hence we consider $1/2 < H < 1$.

We study the role that the weights $\{a_j\}$ play in establishing a functional approximation to FBM(H, L) by scaled sums of variables of the form (1). We introduce the "random walk"

$$S_0 := 0, \quad S_n := X_1 + \cdots + X_n, \quad n = 1, 2, \ldots,$$

where $\{X_j\}$ are defined by (1), and the scaled process

$$Z_{n,H}(t) = \frac{S_{[nt]}}{n^H} = \frac{1}{n^H} \sum_{j=1}^{[nt]} X_j, \quad t \in [0, \infty), \tag{2}$$

where $[x]$ denotes the largest integer not exceeding the real number x. The main condition on the coefficients $\{a_k\}$ is that

$$V_n^2 := \sum_{k \in \mathbb{Z}} \big(a_{k+1} + \cdots + a_{k+n}\big)^2 \sim Ln^{2H}, \quad \text{as} \quad n \to \infty. \tag{3}$$

The notation $\alpha_n \sim \beta_n$, as $n \to \infty$, is used in the sense that $\lim_{n\to\infty} \alpha_n/\beta_n = 1$.

Condition (3) makes the process $\{Z_{n,H}\}$ long-range dependent and asymptotically $H-$self similar. In fact, (1) and (2) is one of the most natural ways of introducing long-range dependence. We can use this as a prototype for simulations of processes with long-range dependence and for approximating fractional Brownian motion. In time series theory, see Box et al. [4], $\{X_n\}$ is called a linear stationary process (a possibly non-standard terminology).

Our first result, Theorem 1, concerns the limit of the sequence of processes $Z_{n,H}$, as $n \to \infty$: we show that the limit is FBM(H, L) if and only if (3) holds. We remark that by "limit" of $Z_{n,H}$ we mean weak limit of their induced probability measures on the space $D[0, \infty)$. As usual, this is the space of functions from $[0, \infty)$ into $\mathbb{R}$ that are everywhere right-continuous and with left limits. A number of topologies can be introduced on this space. Since a FBM has continuous sample paths, it turns out that we can (and we shall) endow $D[0,\infty)$ with a strong topology, that of uniform convergence on compacta. The latter means that we treat $D[0,\infty)$ as a space with the metric (see Pollard [11] p.108, for example)

$$d(x,y) := \sum_{k=1}^{\infty} \min\{1, \sup_{0\le t\le k} |x(t)-y(t)|\}, \qquad x,y\in D[0,\infty). \tag{4}$$

When we write $Z_{n,H} \Rightarrow B_H$ we understand weak convergence of the corresponding probability measures in $D[0,\infty)$ under this topology.

A further generalization to Theorem 1 can be obtained by replacing the scaling factor n^H in (2) by an arbitrary nonnegative function $g(n)$. We want to find the most general assumptions on numbers $g(n) > 0$ under which we have the following convergence

$$Z_n(t) := (g(n))^{-1} \sum_{j=1}^{[nt]} X_j \Rightarrow \mathrm{FBM}(H,L) \quad \text{as} \quad n\to\infty. \tag{5}$$

The desired general necessary and sufficient conditions will be obtained in Theorem 2. This theorem generalizes the corresponding results of Davydov [6], in which where was a gap between necessary and sufficient conditions.

The quality of the approximation depends crucially on more detailed properties of the coefficients. To understand this, we study the rate of convergence of $Z_{n,H}$ toward a FBM. We do so by appealing to the Komlós-Major-Tusnády strong approximation results. The main result, Theorem 3, is that the processes $Z_{n,H}$ can be constructed jointly with a fractional Brownian motion (i.e. on the same probability space) such that the maximum deviation of $Z_{n,H}$ from the fractional Brownian motion, on a compact interval, is bounded asymptotically (in an almost sure sense) by some computable function of n, that depends on the weights $\{a_k\}$ and moments of ξ_0.

The work is motivated partly by the rapidly growing interest in modeling communication networks traffic by fractal processes, see e.g. Leland et al. [8], Beran et al. [1], and Willinger et al. [15]. An early reference on the importance of FBM in applications is the paper of Mandelbrot and van Ness [9]. See also the recent work of Norros [10] and Konstantopoulos and Lin [7] for applications in performance of queuing networks.

The paper is organized as follows. Section 2 states the main results on the functional convergence to FBM and some sufficient and applicable criteria. Section 3 is entirely devoted to the question of deriving general and applicable rates of convergence for the functional approximation theorem. Section 4 is devoted to the proofs of the rates of convergence. Everything is based on the KMT approximation, summarized in Theorem 4 of this section. Section 5 is devoted to the proof of the results on the functional convergence to FBM stated in Section 2.

2. Characterizing the convergence to a fractional Brownian motion

Throughout the paper we assume that

$$\mathbf{E}\xi_0 = 0, \quad \mathbf{E}\xi_0^2 = 1 \quad \text{and} \quad 0 < \sum_{k\in\mathbb{Z}} a_k^2 < \infty, \tag{6}$$

which ensures that $\{X_j, j \in \mathbb{Z}\}$ are square integrable. In fact, the infinite sums in (1) also converge almost surely by a theorem of Kolmogorov and Khinchine; see, e.g., Shiryayev [14, Theorem IV.2.1]. Observe that

$$S_n = \sum_{k\in\mathbb{Z}} \big(a_{-k+1} + \cdots + a_{-k+n}\big)\xi_k \quad \text{and} \quad V_n^2 = \mathbf{E}S_n^2 \quad \text{for all} \quad n \geq 0. \tag{7}$$

The first theorem below *characterizes* the convergence of (2) to a fractional Brownian motion.

Theorem 1. *Assume that conditions* (6) *hold. Let* $\{Z_{n,H} \geq 1\}$ *be the sequence of processes defined by* (2)*. Let* $FBM(H, L)$ *be a fractional Brownian motion with Hurst parameter* $H \in (1/2, 1)$ *and variance parameter* $L > 0$*. Then the following are equivalent:*

(i) $Z_{n,H} \Rightarrow FBM(H, L)$, *as* $n\to\infty$.
(ii) $V_n^2 \sim Ln^{2H}$, *as* $n\to\infty$.

The theorem gives an applicable criterion, in terms of the coefficients $\{a_k\}$, for approximating a fractional Brownian motion with a desired parameter $H > 1/2$. It will be proved in Section 5. Two sufficient criteria are worth mentioning. First, let

$$r(n) := \operatorname{cov}(X_0, X_n) \equiv \mathbf{E}(X_0X_n)$$

be the covariance function of the stationary process X.

Corollary 1. *If*

$$r(n) \sim LH(2H-1)n^{2H-2}, \quad \text{as} \quad n\to\infty, \tag{8}$$

then $Z_{n,H} \Rightarrow FBM(H, L)$.

To see this, just observe that (8) implies *(ii)* of Theorem 1, as in the proof of Theorem 7.2.11 of [13].

In practice, it is often desirable to specify the weights explicitly. It is also useful to obtain $\{X_j\}$ from $\{\xi_k\}$ causally. This means that $a_n = 0$ for $n < 0$. A typical case is the following:

$$a_n = \begin{cases} (n+1)^p - n^p, & n \geq 0, \\ 0, & n < 0, \end{cases} \qquad p \in (0, 1/2). \tag{9}$$

A similar example may be found in [6, Remark 5]:

$$a_n = \begin{cases} pn^{p-1}, & n \geq 0, \\ 0, & n < 0, \end{cases} \qquad p \in (0, 1/2). \tag{10}$$

These examples are partial cases of the following simple

Corollary 2. *If* $a_n = 0$ *for all* $n < 0$ *and*

$$(11) \qquad a_n \sim pn^{p-1} \qquad as \qquad n \to \infty \qquad for\ some \qquad p \in (0, 1/2),$$

then $Z_{n,H} \Rightarrow FBM(H_p, L_p)$ *with*

$$(12) \qquad H_p = p + 1/2 \quad and \quad L_p = \frac{1}{2p+1} + \int_0^\infty \big((x+1)^p - x^p\big)^2 dx.$$

Now we consider a more difficult question about the most general assumptions for convergence (5).

Theorem 2. *Assume that conditions* (6) *hold. Then the following statements are equivalent:*

(i) *There exist a number* $L > 0$ *and numbers* $g(n) > 0$ *such that convergence* (5) *holds.*
(ii) *Convergence* (5) *holds for* $L = 1$ *and numbers* $g(n) \equiv V_n > 0$.
(iii) *The function* $h(n) := V_n / n^H$ *is slowly varying.*

For a comprehensive account on the theory of regularly varying functions the reader is referred to the monograph of Resnick [12].
Remark: The fact that (iii) is necessary for (i) was proved by Davydov [6, Theorem 1], but under additional assumption that V_n is monotone. He proved in [6, Theorem 2], also that (iii) is sufficient for (i) but only if $\mathbf{E}\xi^4 < \infty$ when $H > 1/2$.

3. Rate of convergence via strong approximations

In this section, we work out a rate of convergence for the functional central limit theorem to fractional Brownian motion. We will use B_H to denote a FBM($p+1/2, L_p$), i.e. a FBM with the parameters introduced in Corollary 2. In particular, we have the notation $H = p + 1/2$, throughout this section.

It can be seen that the rate of convergence cannot depend only on the general condition (ii) of Theorem 1 that gave the normalizing factor in the FCLT, but on more delicate conditions for the coefficients $\{a_j\}$. We introduce the following notations:

$$A_m := a_0 + ... + a_m \quad \text{for} \quad m \geq 0 \tag{13}$$
$$A_m := -(a_{m+1} + ... + a_{-1}) \quad \text{for} \quad m < 0,$$

$$\Delta_n^{(1)} := \sum_{m=0}^{\infty} \big(A_{n+m} - A_m - (n+m+1)^p + (m+1)^p\big)^2,$$
$$\Delta_n^{(2)} := \sum_{m=-n}^{-1} \big(A_{n+m} - A_m - (n+m+1)^p\big)^2, \tag{14}$$
$$\Delta_n^{(3)} := \sum_{m<-n} (A_{n+m} - A_m)^2, \qquad \Delta_n = \Delta_n^{(1)} + \Delta_n^{(2)} + \Delta_n^{(3)},$$
$$\Delta_{\alpha,n} := \sum_{m\in\mathbb{Z}} \max\big\{|m|^{1/\alpha}, n^{1/\alpha}\big\}\big|a_m - a_{m+n}\big|.$$

We will suppose later on that

$$\mathbf{E}|\xi_0|^\alpha < \infty \quad \text{for some} \quad \alpha > 2. \tag{15}$$

The main result can now be stated as follows.

Theorem 3. *If* (15) *holds then for all* $H \in (0,1)$ *it is possible to construct a FBM* $B_H(\cdot)$ *such that*

$$S_{[t]} - B_H(t) = o\Big(\Delta_{\alpha,[t]}\Big) + O\Big(\sqrt{\big(1+\Delta_{[t]}\big)\log t}\Big), \quad \textit{as} \quad t \to \infty \quad \textit{a.s.}$$

Before proving the theorem, we state several interesting special cases for $p > 0$.

Corollary 3. *Assume that* (15) *holds and that the following conditions are fulfilled*

$$a_m \geq a_{m+1} > 0 \quad \textit{for} \quad m \geq 0 \quad \textit{and} \quad a_m = 0 \quad \textit{for} \quad m < 0, \tag{16}$$
$$\exists \beta \leq 1/2 + p \qquad \mu_n' := a_n - (n+1)^p + n^p = O(n^{\beta-3/2}), \tag{17}$$
$$\exists \gamma > 0 \quad \mu_n := A_n - (n+1)^p = O(n^{\gamma-1/2}), \qquad \textit{as} \quad n \to \infty. \tag{18}$$

Then, for $p > 0$*, the random processes constructed in Theorem* 3 *satisfy the following statement:*

$$S_{[t]} - B_H(t) = o\big(t^{p+1/\alpha}\big) + O\big(t^{\max\{\beta,\gamma\}}\sqrt{\log t}\big), \quad \textit{as} \quad t \to \infty \quad \textit{a.s.} \tag{19}$$

We can now state the result of the rate of convergence for the functional approximation of the process $Z_{n,H}(t)$, introduced in (2). Let $\tilde{Z}_{n,H}(t) = B_H(nt)/n^H$ be a FBM identical, in distribution, to B_H.

Corollary 4. *If the statement* (19) *holds and* $p > 0$*, then, with probability 1,*

$$\sup_{0\leq t<\infty} \frac{|Z_{n,H}(t) - \tilde{Z}_{n,H}(t)|}{r(t)} = o\big(n^{1/\alpha-1/2}\big) + O\big(n^{\max\{\beta,\gamma\}-p-1/2}\sqrt{\log n}\big), \tag{20}$$

as $n \to \infty$*, where*

$$r(t) = (1+t)^{p+1/\alpha} + (1+t)^{\max\{\beta,\gamma\}}\sqrt{1+\log(1+t)}. \tag{21}$$

In particular,

$$\sup_{0\le t\le 1}\big|Z_{n,H}(t) - \tilde{Z}_{n,H}(t)\big| = o\big(n^{1/\alpha-1/2}\big) + O\big(n^{\max\{\beta,\gamma\}-p-1/2}\sqrt{\log n}\big), \tag{22}$$

as $n \to \infty$*, if only* (19) *or* (20) *hold.*

Remark: If $\max\{\beta,\gamma\} < p+1/\alpha$, then we may omit the second summands in formulas (19), (20), (21) and (22); and if $\max\{\beta,\gamma\} \ge p+1/\alpha$, then we may omit the first summands in the listed above formulas.
Remark: If condition (17) holds with $\beta \ne 1/2$, then assumption (18) is automatically fulfilled with $\gamma = \max\{\beta, 1/2\}$ because in this case

$$\mu_n = \sum_{k=0}^{n}\mu'_k = O\Big(1+\sum_{k=1}^{n}k^{\beta-3/2}\Big) = O\Big(n^{\max\{\beta-1/2,0\}}\Big).$$

Hence, the estimates (19), (20) and (22) with $\max\{\beta,\gamma\} = 1/2$ are the best possible if no assumption (18) was made.
Examples: For some $p \in (0,1/2)$ consider

$$a_m := (m+1)^p - m^p \quad \text{for} \quad m \ge 0 \quad \text{and} \quad a_m := 0 \quad \text{for} \quad m < 0.$$

This is the example (9). We have convergence to $FBM(p+1/2, L_p)$, and, by Corollary 3 with $A_n \equiv (n+1)^p$ and $\beta = \gamma \le p$, we have the rate $o(t^{p+1/\alpha})$ in (19), the rate $o(n^{1/\alpha-1/2})$ in (22) and the same rate $o(n^{1/\alpha-1/2})$ in (20) with $r(t) = (1+t)^{p+1/\alpha}$. But if we choose

$$a_m := pm^{p-1} \quad \text{for} \quad m \ge 0 \quad \text{and} \quad a_m := 0 \quad \text{for} \quad m < 0,$$

as in (10), then, by the previous remark, we obtain the worse estimates with $\gamma = 1/2 > \beta = p - 1/2$.

4. Proofs of the rates of convergence

The following theorem is the standard KMT approximation. We stress that throughout this section, when not mentioned explicitly, all limits exist with probability 1 and are taken as $t \to \infty$ or as $n = [t] \to \infty$.

Theorem 4 (c.f. Corollary 1.1, p. 4 of [5])**.** *Suppose that condition* (15) *holds. Then it is possible to construct a Wiener process* $W = \{W(t), t \in \mathbb{R}\}$ **, such that*

(23)

$$\delta_n := \sum_{k=1}^{n}\xi_k - W(n) = o(n^{1/\alpha}) \quad \text{and} \quad \delta_{-n} := W(-n) - \sum_{k=0}^{n-1}\xi_{-k} = o(n^{1/\alpha}).$$

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*A Wiener process on the index set $\mathbb{R}$ is defined, as usual, by piecing together two independent Brownian motions; so, $W(0) = 0$, and $\{W(t), t \ge 0\}$ is independent of $\{W(-t), t \ge 0\}$.

For $k \in \mathbb{Z}$ and $n \in \mathbb{N}$ we will define

$$\widetilde{\xi}_k := W(k) - W(k-1) \quad \text{and} \quad \widetilde{S}_n = \sum_{k\in\mathbb{Z}} \big(A_{n-k} - A_{-k}\big)\widetilde{\xi}_k. \tag{24}$$

Then $\{\widetilde{\xi}_k\}$ and $\{\widetilde{S}_n\}$ may be treated as a "normal" version of $\{\xi_k\}$ and $\{\widetilde{S}_n\}$. It is convenient to introduce also

$$\begin{aligned} \widetilde{B}_n &= \sum_{k\in\mathbb{Z}} \Big(\big((n-k+1)^+\big)^p - \big((-k+1)^+\big)^p\Big)\widetilde{\xi}_k, \\ B_H(t) &= \int_{\mathbb{R}} \Big(\big((t-s)^+\big)^p - \big((-s)^+\big)^p\Big) dW(s), \end{aligned} \tag{25}$$

where we adopt the convention $0^0 = 0$. It is easy to verify (see, also, [9] or [9, p.321]) that for all $p = H - 1/2 \in (-1/2, 1/2)$ the process $B_H(\cdot)$ from (25) is a FBM$(p+1/2, L_p)$, i.e. a FBM with the parameters introduced in Corollary 2.

Throughout, we will be working on the probability space alluded in Theorem 4. Our first goal is to prove Theorem 3 with the process $B_H(\cdot)$, constructed by the representation (25).

We first extend the KMT approximation from partial sums of the i.i.d. r.v.'s $\{\xi_k\}$ to the partial sums of the weighted random variables $\{X_j\}$. This is done in the following:

Lemma 1. *Suppose that condition* (15) *holds. Then*

$$S_n - \widetilde{S}_n = o\big(\Delta_{\alpha,n}\big), \quad a.s.$$

Proof. We point out that

$$\varepsilon_n = \sup_{k\ge n} \Big(k^{-1/\alpha} \max_{m\le k} |\delta_m|\Big) \to 0, \quad \text{a.s.} \tag{26}$$

as it follows from (23). After performing summation by parts, we obtain from (7), (13) and (24) that

$$S_n - \widetilde{S}_n = \sum_{k\in\mathbb{Z}} \big(A_{n-k} - A_{-k}\big)\big(\xi_k - \widetilde{\xi}_k\big) = \sum_{k\in\mathbb{Z}} \delta_k \big(a_{n-k} - a_{-k}\big)$$

with $\delta_0 = 0$, because $\xi_k - \widetilde{\xi}_k \equiv \delta_k - \delta_{k-1}$ by (23). Hence, from (26),

$$\big|S_n - \widetilde{S}_n\big| \le \Delta_{\alpha,n}\varepsilon_n = o\big(\Delta_{\alpha,n}\big), \quad \text{a.s.}$$

as announced in the lemma. □

Next we compare $\widetilde{B}_n$ and $\widetilde{S}_n$.

Lemma 2. *If* $n \to \infty$, *then*

$$\widetilde{S}_n - \widetilde{B}_n = O\Big(\sqrt{\Delta_n \log n}\Big), \quad a.s.$$

Proof. From the expressions (24) and (25) we obtain

$$\Delta_n \equiv \mathbf{E}\big(\widetilde{S}_n - \widetilde{B}_n\big)^2$$
$$= \sum_{k\in\mathbb{Z}} \Big(A_{n-k} - A_{-k} - \big((n-k+1)^+\big)^p + \big((-k+1)^+\big)^p\Big)^2.$$

The random variables $\widetilde{S}_n - \widetilde{B}_n$ are normally distributed with zero means. Therefore, we have

$$\mathbf{P}\Big(\big|\widetilde{S}_n - \widetilde{B}_n\big| \geq 2\sqrt{\Delta_n \log n}\Big) \leq \exp\Big(-\Big(2\sqrt{\Delta_n \log n}\Big)^2 \Big/ \Big(2\Delta_n\Big)\Big) = n^{-2}.$$

Thus, the Borel-Cantelli lemma implies that

$$\varlimsup_{n\to\infty} \frac{\big|\widetilde{S}_n - \widetilde{B}_n\big|}{2\sqrt{\Delta_n \log n}} \leq 1, \quad \text{a.s.,}$$

and so $\big|\widetilde{S}_n - \widetilde{B}_n\big| = O\big(\sqrt{\Delta_n \log n}\big)$, a.s. □

The next lemma compares $\{\widetilde{B}_n, n \in \mathbb{Z}\}$ and $\{B_H(n), n \in \mathbb{Z}\}$, where the latter is the FBM sampled at integer times.

Lemma 3. *If $n \to \infty$ then*

$$\widetilde{B}_n - B_H(n) = O\big(\sqrt{\log n}\big), \quad a.s.$$

Proof. It follows from (25) that

$$\widetilde{B}_n - B_H(n) = \int_{\mathbb{R}} \big(g(n,s) - g(0,s)\big) dW(s),$$

where $0 \leq g(t,s) := \big((t-[s])^+\big)^p - \big((t-s)^+\big)^p$. But $(-[s])^+ \leq (-s+1)^+$ and $0^0 = 0$. Hence

$$g^2(t,s) \leq \Big(\big((-s+1)^+\big)^p - \big((-s)^+\big)^p\Big)^2 \qquad \forall\ s, p.$$

We then have

$$\mathbf{E}\big(B_H(n) - \widetilde{B}_n\big)^2 =$$
$$= \int_{\mathbb{R}} \big(g(n,s) - g(0,s)\big)^2 ds \leq 2\int_{\mathbb{R}} g^2(n,s)ds + 2\int_{\mathbb{R}} g^2(0,s)ds =$$
$$= 4\int_{\mathbb{R}} g^2(0,s)ds \leq 4\int_{\mathbb{R}} \Big(\big((-s+1)^+\big)^p - \big((-s)^+\big)^p\Big)^2 ds =$$
$$= 4\mathbf{E}B_H^2(1) \equiv 4L_p < \infty.$$

So, $B_H(n) - \widetilde{B}_n$ are normal with zero means and uniformly bounded variances and the Borel-Cantelli lemma implies that $\big|B_H(n) - \widetilde{B}_n\big| = O\big(\sqrt{\log n}\big)$, a.s. □

Finally, we examine the error introduced by sampling the FBM at integer times.

Lemma 4. *If $t \to \infty$ then*

$$B_H(t) - B_H([t]) = O\big(\sqrt{\log t}\big), \quad a.s.$$

Proof. We make use of the Berman [2] asymptotic formula:

$$K_0(x) := \mathbf{P}\Big(\sup_{0\le t\le 1} B_H(t) > x\Big) \sim \mathbf{P}\Big(B_H(1) > x\Big) \quad \text{as} \quad x \to \infty.$$

Hence,

$$K(x) := \mathbf{P}\Big(\sup_{0\le t\le 1} \big|B_H(t)\big| > x\Big) \le 2K_0(x) \sim \frac{2}{\sqrt{2\pi L_p x}} e^{-x^2/(2L_p)}$$

as $x \to \infty$. Thus, there exist constants C and C' such that

$$K(x) \le C' e^{-x^2/C} \quad \forall x > 0.$$

Next observe that $\sup_{0\le t\le 1} \big|B_H(t)\big|$ and $\sup_{n\le t\le n+1} \big|B_H(t) - B_H([t])\big|$, $n \in \mathbb{N}$, are identically distributed, and apply the last inequality to find

$$\mathbf{P}\Big(\sup_{n\le t\le n+1} \big|B_H(t) - B_H([t])\big| \ge \sqrt{2C\log n}\Big) \le K\big(\sqrt{2C\log n}\big) \le C' n^{-2}.$$

The Borel-Cantelli lemma applies once more, and establishes the lemma. □

Proof of Theorem 3. It follows immediately from Lemmas 1, 2, 3 and 4. □

Next, we deal with the special cases. We start with the proof of Corollary 3. This needs two auxiliary lemmas.

Lemma 5. *Assume that condition* (17) *holds and $p > 0$. Then*

$$\big|\mu'_m\big| \le C_0\big(m+1\big)^{\beta-3/2}, \quad \big|a_n\big| \le C_1\big(n+1\big)^{p-1} \quad \forall n > 0, \tag{27}$$

for some positive constants C_0 and C_1. And if (16) *and* (17) *are satisfied, then*

$$\Delta_{\alpha,n} = O\big(n^{p+1/\alpha}\big).$$

Proof. Inequalities (27) follow immediately from (17). Hence

$$\Delta^{(1)}_{\alpha,n} := n^{1/\alpha} \sum_{m=-n}^{n-1} \big|a_m - a_{m+n}\big| \le 2n^{1/\alpha} \sum_{m=0}^{2n-1} a_m \le 2C_1 n^{1/\alpha}(2n)^p. \tag{28}$$

Suppose now that $\{a_j, j = 0, 1, 2, \ldots\}$ are non-negative and non-increasing. After performing summation by parts, we have

(29)

$$\begin{aligned}\Delta^{(2)}_{\alpha,n} &:= \sum_{m=n}^{\infty} m^{1/\alpha}\big(a_m - a_{m+n}\big) \\ &= \sum_{m=n}^{2n-1} m^{1/\alpha} a_m + \sum_{m=2n}^{\infty} a_m\big(m^{1/\alpha} - (m-n)^{1/\alpha}\big) \\ &\le n \cdot n^{1/\alpha} \cdot C_1 (2n)^{p-1} + \sum_{m=2n}^{\infty} C_1 m^{p-1} \cdot \big(1/\alpha\big) m^{1/\alpha - 1} = O\big(n^{p+1/\alpha}\big).\end{aligned}$$

Compare definitions (14), (28) and (29) to obtain $\Delta_{\alpha,n} = \Delta^{(1)}_{\alpha,n} + \Delta^{(2)}_{\alpha,n} = O\big(n^{p+1/\alpha}\big)$. □

Lemma 6. *If the conditions* (17) *and* (18) *hold with* $a_m = 0$ *for* $m < 0$*, then*

$$\Delta_n = O\big(n^{\max\{\beta,\gamma\}}\big).$$

Proof. From (18) we immediately obtain

$$\Delta^{(2)}_n = \sum_{m=-n}^{-1} \big(\mu_{n+m}\big)^2 = \sum_{k=0}^{n-1} \mu_k^2 = nO\big(n^{2\gamma-1}\big), \tag{30}$$

$$\Delta^{(1,1)}_n := \sum_{m=0}^{n-1} \big(\mu_{n+m} - \mu_m\big)^2 \le 2 \sum_{m=0}^{2n-1} \mu_m^2 = nO\big(n^{2\gamma-1}\big). \tag{31}$$

Next, using (27) we have $\big|\mu_{m+n} - \mu_m\big| = \Big|\sum_{k=m+1}^{m+n} \mu'_k\Big| \le C_0 n m^{\beta-3/2}$ and

$$\Delta^{(1,2)}_n := \sum_{m=n}^{\infty} \big(\mu_{n+m} - \mu_m\big)^2 \le C_0^2 n^2 \sum_{m=n}^{\infty} m^{2\beta-3} = n^2 O\big(n^{2\beta-2}\big). \tag{32}$$

Compare definitions (14), (30), (31) and (32) to obtain

$$\Delta^{(3)}_n = 0 \quad \text{and} \quad \Delta^{(1)}_n = \Delta^{(1,1)}_n + \Delta^{(1,2)}_n = O\big(n^{2\gamma}\big) + O\big(n^{2\beta}\big).$$

The last formula coincides with the result announced in the lemma.

Proof of Corollary 3. It follows immediately from the previous two lemmas.

Corollary 4 is obvious by definitions of symbols $O(\cdot)$ and $o(\cdot)$.

5. Proof of functional convergence

We start by noticing that for each fixed t the variable $Z_{n,H}(t)$ is a linear combination of countably many independent random variables. This motivates the following lemma, which is also of independent interest.

Lemma 7. *Let $\{b_{ni}, n \in \mathbb{N}, i \in \mathbb{Z}\}$ be a doubly indexed sequence of real numbers, and $\{\zeta_{ni}, n \in \mathbb{N}, i \in \mathbb{Z}\}$ a doubly indexed sequence of random variables such that*

L1. $\lim_{n\to\infty} \sum_{i\in\mathbb{Z}} b_{ni}^2 = 1$.
L2. $\lim_{n\to\infty} \sup_{i\in\mathbb{Z}} |b_{ni}| = 0$.
L3. *For all n, $\{\zeta_{ni}, i \in \mathbb{Z}\}$ are i.i.d. with* $\mathbf{E}\zeta_{n0} = 0$ *and* $\mathbf{E}\zeta_{n0}^2 = 1$.
L4. $\lim_{K\to\infty} \sup_{n\geq 1} \mathbf{E}\big\{\zeta_{n0}^2, |\zeta_{n0}| > K\big\} = 0$.

Define the weighted sums

$$z_n = \sum_{i\in\mathbb{Z}} b_{ni}\zeta_{ni}.$$

Then z_n converges weakly, as $n\to\infty$, to the standard normal distribution $\mathcal{N}(0,1)$.

Proof. Owing to condition **L1** we can select a sequence $\{k_n\}$ of positive integers, increasing to infinity, so that $\widetilde{\sigma}_n^2 := \sum_{i=-k_n}^{k_n} b_{ni}^2 \to 1$, as $n\to\infty$. Define the weighted partial sums $\tilde{z}_n := \sum_{i=-k_n}^{k_n} b_{ni}\zeta_{ni}$. It is easy to see that

$$\widetilde{\sigma}_n^2 = \mathbf{E}\tilde{z}_n^2 \to 1 \quad \text{and} \quad \mathbf{E}(z_n - \tilde{z}_n)^2 = \sum_{|i|>k_n} b_{ni}^2 \to 0, \quad \text{as} \quad n\to\infty. \tag{33}$$

Next, we consider the triangular array $\{b_{ni}\zeta_{ni}, i = -k_n, \dots, k_n, n \in \mathbb{N}\}$ and check that it satisfies Lindeberg's conditions for central limit theorem (see Billingsley [3, Theorem 7.2.]). Namely, we claim that, for any $\varepsilon > 0$,

$$\mathbf{E} \sum_{i=-k_n}^{k_n} b_{ni}^2\zeta_{ni}^2 \mathbf{1}\big\{|b_{ni}\zeta_{ni}| > \varepsilon\widetilde{\sigma}_n\big\} \to 0 \ , \quad \text{as} \quad n\to\infty \ . \tag{34}$$

We estimate the left hand side as follows:

$$\mathbf{E} \sum_{i=-k_n}^{k_n} b_{ni}^2\zeta_{ni}^2 \mathbf{1}\big\{|b_{ni}\zeta_{ni}| > \varepsilon\widetilde{\sigma}_n\big\} \leq \sum_{i=-k_n}^{k_n} b_{ni}^2 \mathbf{E}\big\{\zeta_{n0}^2, |\zeta_{n0}| > \varepsilon\widetilde{\sigma}_n / \sup_i |b_{ni}|\big\} \ .$$

We observe that condition **L2** and the above discussion imply that $\widetilde{\sigma}_n / \sup_i |b_{ni}| \to \infty$, as $n\to\infty$, and use the uniform integrability condition **L4** to obtain

$$\mathbf{E}\big\{\zeta_{n0}^2, |\zeta_{n0}| > \varepsilon\widetilde{\sigma}_n / \sup_i |b_{ni}|\big\} \to 0, \quad \text{as} \quad n\to\infty$$

and hence the claim (34) follows.

This shows that $\tilde{z}_n/\widetilde{\sigma}_n$ converges weakly to $\mathcal{N}(0,1)$ and hence $\tilde{z}_n$ also does the same. To finish the proof note that (33) implies that $z_n - \tilde{z}_n$ converges to 0 in probability and hence z_n converges weakly to $\mathcal{N}(0,1)$. □

For $k \in \mathbb{Z}$ and $n \in \mathbb{N}$ define

$$A_{k,n}(t) = n^{-H} \sum_{j=-k+1}^{-k+[nt]} a_j \quad \text{and} \quad \rho_n(t) = \sup_{k\in\mathbb{Z}} \big|A_{k,n}(t)\big|. \tag{35}$$

It follows from (3) and (7) that

$$Z_{n,H}(t)=\sum_{k\in\mathbb{Z}}A_{k,n}(t)\xi_k \quad \text{and} \quad n^{-2H}V_n^2=\sum_{k\in\mathbb{Z}}A_{k,n}^2(1). \tag{36}$$

To prove Theorem 1 we need several auxiliary lemmas.

Lemma 8. *Suppose that conditions* (6) *hold. Then for all* $t\geq 0$

$$\rho_n^2(t)\leq tn^{1-2H}\sum_{j\in Z}a_j^2\to 0, \qquad as \qquad n\to\infty.$$

This fact follows immediately from the next inequalities

$$A_{k,n}^2(t)\leq n^{-2H}\Big(\sum_{j=-k+1}^{-k+[nt]}|a_j|\Big)^2\leq n^{-2H}[nt]\sum_{j=-k+1}^{-k+[nt]}a_j^2\leq n^{1-2H}t\sum_{j\in\mathbb{Z}}a_j^2$$

since $2H>1$.

Lemma 9. *Assume that (ii) holds. Then for all fixed* $t\geq\tau\geq 0$

$$\mathbf{E}Z_{n,H}(\tau)Z_{n,H}(t)\to\mathbf{E}B_H(\tau)B_H(t) \qquad as \qquad n\to\infty.$$

Proof. It follows from (35) and (36) that

$$\begin{aligned}\mathbf{E}(Z_{n,H}(t)-Z_{n,H}(\tau))^2&=\sum_{k\in\mathbb{Z}}(A_{k,n}(t)-A_{k,n}(\tau))^2\\ &=\sum_{j\in\mathbb{Z}}A_{j,[nt]-[n\tau]}^2(1)=\frac{([nt]-[n\tau])^{2H}}{n^{2H}}\frac{V_{[nt]-[n\tau]}}{([nt]-[n\tau])^{2H}}.\end{aligned} \tag{37}$$

Hence, by (ii)

$$\mathbf{E}(Z_{n,H}(t)-Z_{n,H}(\tau))^2\to L(t-\tau)^2=\mathbf{E}(B_H(t)-B_H(\tau))^2, \qquad \text{as} \qquad n\to\infty,$$

and we may write

$$\begin{aligned}&2\mathbf{E}Z_{n,H}(t)Z_{n,H}(\tau)=\mathbf{E}Z_{n,H}^2(t)+\mathbf{E}Z_{n,H}^2(\tau)-\mathbf{E}(Z_{n,H}(t)-Z_{n,H}(\tau))^2\\ &\to\mathbf{E}B_H^2(t)+\mathbf{E}B_H^2(\tau)-\mathbf{E}(B_H(t)-B_H(\tau))^2=2\mathbf{E}B_H(t)B_H(\tau).\end{aligned}$$

Thus, the desired convergence is proved. □

Lemma 10. *Assume that (ii) holds. Then for some* $K<\infty$

$$\mathbf{E}(Z_{n,H}(t)-Z_{n,H}(\tau))^2\leq K(t-\tau)^{2H} \qquad if \qquad [nt]>[n\tau].$$

This inequality follows immediately from (37) with

$$K \;=\; 2^{2H}\sup_{m\geq 1}V_m/m^H<\infty.$$

We now use these lemmas to give a simple proof of the first theorem.

Proof of Theorem 1. Without loss of generality, consider $L=1$. Assume that (ii) and (6) hold. Let us first show that the finite dimensional distributions of $Z_{n,H}$ converge to those of an FBM$(H,1)$ process B_H, i.e,

$$(Z_{n,H}(t_1),\dots,Z_{n,H}(t_\ell))\Rightarrow(B_H(t_1),\dots,B_H(t_\ell)), \qquad \text{as} \qquad n\to\infty,$$

for any finite tuple $0 \le t_1 < \cdots < t_\ell$, $\ell \in \mathbb{N}$. To prove this, we use the Cramér-Wold device (see Billingsley [3, Theorem 7.7.]) and show that

$$z_n := \sum_{i=1}^{\ell} c_i Z_{n,H}(t_i) = \sum_k \sum_{i=1}^{\ell} c_i A_{k,n}(t_i)\xi_k \Rightarrow \sum_{i=1}^{\ell} c_i B_H(t_i) \tag{38}$$

for any constants $c_1, \dots, c_\ell$.

To show (38) we use Lemma 7 with $\{\zeta_{ni} = \xi_i, n \in \mathbb{N}, i \in \mathbb{Z}\}$. It is clear that **L3**, and **L4** hold. Condition **L2** follows immediately from Lemma 8, since

$$\sup_k \Big| \sum_{i=1}^{\ell} c_i A_{k,n}(t) \Big| \le \sum_{i=1}^{\ell} |c_i| \rho_n(t) \to 0, \quad \text{as} \quad n\to\infty.$$

To check **L1** we write

$$\mathbf{E} z_n^2 = \sum_{i=1}^{\ell} \sum_{j=1}^{\ell} c_i c_j \mathbf{E} Z_{n,H}(t_i) Z_{n,H}(t_j)$$

$$\to \sum_{i=1}^{\ell} \sum_{j=1}^{\ell} c_i c_j \mathbf{E} B_H(t_i) B_H(t_j) = \mathbf{E}\Big(\sum_{i=1}^{\ell} c_i B_H(t_i) \Big)^2,$$

as $n\to\infty$. Thus, all conditions of Lemma 7 hold and z_n converges weakly to $\sum_{i=1}^{\ell} c_i B_H(t_i)$.

We now examine tightness of $\{Z_{n,H}\}$. First of all we prove that

$$\mathbf{E}\big(\big| Z_{n,H}(t_2) - Z_{n,H}(t_1) \big| \cdot \big| Z_{n,H}(t_3) - Z_{n,H}(t_2) \big| \big) \le K(t_3 - t_1)^{2H} \tag{39}$$

for all $t_1 < t_2 < t_3$. If $[nt_1] = [nt_2]$ or $[nt_2] = [nt_3]$ then (39) is true because its left hand side is equal to zero. If $[nt_1] < [nt_2] < [nt_3]$ then, by Lemma 10,

$$\begin{aligned} &\mathbf{E}\big(\big| Z_{n,H}(t_2) - Z_{n,H}(t_1) \big| \cdot \big| Z_{n,H}(t_3) - Z_{n,H}(t_2) \big| \big) \\ &\le \ \big(\mathbf{E}\big(Z_{n,H}(t_2) - Z_{n,H}(t_1) \big)^2 \big)^{1/2} \big(\mathbf{E}\big(Z_{n,H}(t_3) - Z_{n,H}(t_2) \big)^2 \big)^{1/2} \\ &\le \ K(t_2 - t_1)^H (t_3 - t_2)^H \le K(t_3 - t_1)^{2H}. \end{aligned}$$

Thus, inequality (39) is proved for all $t_1 < t_2 < t_3$.

But this inequality, together with the convergence of finite dimensional distributions, gives the weak convergence of the distributions of $\{Z_{n,H}\}$ (see Billingsley [3, Theorem 15.6]) in $D[0,T]$ with the J_1–Skorohod topology for all $T < \infty$. However, in our case the limiting process is a continuous one. Hence, (see Pollard [11, p.137] and the discussion in Billingsley [3, Section 18]), the weak convergence in the J_1–Skorohod topology actually implies the weak convergence under the uniform topology of $D[0,T]$ for all $T < \infty$. Thus, we proved the weak convergence in the space $D[0,\infty)$ with the topology endowed by the metric from (4).

We finally show that (i) implies (ii). Suppose $Z_{n,H} \Rightarrow B_H$. Then $Z_{n,H}(1)$ converges weakly to $\mathcal{N}(0,1)$. Also, $L_n := \mathbf{E} Z_{n,H}^2(1) < \infty$ by the square integrability of $\{a_k\}$. Let $\hat{L}$ be a limit point of $\{L_n, n \ge 1\}$, and assume it is finite (provided such a point exists). Then, we can find a subsequence

$\{L_{n_j}, j \geq 1\} \subseteq \{L_n, n \geq 1\}$ with $L_{n_j} \to \hat{L}$, as $j \to \infty$. By (5), Lemma 7 holds for Z_{n_j} and so $Z_{n_j}(1) \Rightarrow N(0, \hat{L})$. But since $Z_{n,H}(1) \Rightarrow \mathcal{N}(0,1)$, we have $\hat{L} = 1$.

To rule out the possibility that there is a limit point of $\{L_n, n \geq 1\}$ at infinity, assume that $L_{n_j} \to \infty$, as $j \to \infty$. Still, by (5), Lemma 7 applies to the sequence $\{L_{n_j}^{-1/2} Z_{n_j}(1),\ j \geq 1\}$, and gives $L_{n_j}^{-1/2} Z_{n_j}(1) \Rightarrow \mathcal{N}(0,1)$, which contradicts with the assumptions $L_{n_j} \to \infty$ and $Z_{n_j}(1) \Rightarrow N(0,1)$.

Hence the only limit point of $\{L_n, n \geq 1\}$ is $\hat{L} = 1$, and so (ii) holds. □

Proof of Theorem 2. The proof of Theorem 2 is a standard modification of the proof of Theorem 1. The only additional item we need is to show that if (5) with $L > 0$ holds then $g(n) \to \infty$ as $n \to \infty$. To prove this, note that in (5) the limiting process is continuous. Hence it follows from (5) that random variable $X_1/g(n) = Z_n(1/n) - Z_n(0)$ converges to 0 in probability. This fact implies that $g(n) \to \infty$ unless X_1 is 0 with probability 1. Of course, this is not possible, because it contradicts (5) with $L > 0$. □

Proof of Corollary 2. We can prove this assertion, as well as compute L_p, by using statement (ii) of Theorem 1. In the considered case

$$V_n^2 = \sum_{k=0}^{n-1} A_k^2 + V_n' \qquad \text{with} \qquad V_n' := \sum_{k=0}^{\infty} \big(A_{k+n} - A_k\big)^2. \tag{40}$$

Using (11) for the first term we have

$$A_k \equiv \sum_{m=0}^{k-1} a_m \sim \int_0^k p x^{p-1} dx \sim k^p, \quad \text{as} \quad k \to \infty. \tag{41}$$

Hence,

$$\sum_{m=0}^{n-1} A_k^2 \sim \int_0^n x^{2p} dx \sim \frac{n^{2p+1}}{2p+1}, \quad \text{as} \quad n \to \infty. \tag{42}$$

Introduce now notations

$$f(x) := (x+1)^p - x^p, \qquad f_n(x) := (x+n)^p - x^p \equiv n^p f(x/n). \tag{43}$$

It is easy to see that $f_n(x)$ is a decreasing function on the interval $[0, \infty)$, whereas the function x^{2p} increase. Hence, for all $k \geq 0$ and $m \geq 1$ we have

$$f_n^2(k) > \int_k^{k+1} f_n^2(x) dx \equiv n^{2p+1} \int_{k/n}^{(k+1)/n} f^2(x) dx > f_n^2(k+1). \tag{44}$$

It follows from (11) that

$$(1 - \varepsilon_N) f(m-1) \leq a_m \leq (1 + \varepsilon_N) f(m-1) \qquad \forall m > N, \tag{45}$$

with $\varepsilon_N \to 0$ if $N \to \infty$. Thus, (45) holds if we choose

$$N := [\sqrt{n}] \to \infty, \qquad N/n \to 0, \qquad \varepsilon_N \to 0 \quad \text{as} \quad n \to \infty.$$

But (43) and (45) yield

$$(1 - \varepsilon_N) f_n(k) \leq A_{k+n} - A_k \leq (1 + \varepsilon_N) f_n(k) \qquad \forall k \geq N, \tag{46}$$

Now, using relations (43), (44) and (46) we obtain

$$\begin{aligned} V_n'' := \sum_{k=N}^{\infty} \big(A_{k+n} - A_k\big)^2 &\le (1+\varepsilon_N)^2 \int_{N-1}^{\infty} f_n^2(x)dx \\ &\le (1+\varepsilon_N)^2 n^{2p+1} \int_0^{\infty} f^2(x)dx. \end{aligned} \tag{47}$$

On the other hand

$$\begin{aligned} V_n'' &\ge (1-\varepsilon_N)^2 \int_N^{\infty} f_n^2(x)dx \\ &= (1-\varepsilon_N)^2 n^{2p+1} \int_{N/n}^{\infty} f^2(x)dx \sim n^{2p+1} \int_0^{\infty} f^2(x)dx. \end{aligned} \tag{48}$$

It follows immediately from (40) and (41) that

$$0 \le V_n' - V_n'' \equiv \sum_{k=o}^{N-1} \big(A_{k+n} - A_k\big)^2 = N \cdot O\big((n+N)^{2p}\big) = o(n^{2p+1}).$$

Hence, by (47) and (48),

$$V_n' \sim V_n'' \sim n^{2p+1} \int_0^{\infty} f^2(x)dx. \tag{49}$$

But (42) and (49) yield

$$V_n^2/n^{2p+1} \to \frac{1}{2p+1} + \int_0^{\infty} f^2(x)dx \equiv L_p. \tag{50}$$

The latter equality in (50) follows from the definitions (12) and (43).

Thus, condition (ii) of Theorem 1 holds with $L = L_p$. □

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