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ON TRANSPARENT BOUNDARY CONDITIONS FOR THE HIGH-ORDER HEAT EQUATION

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Abstract. In this paper we develop an artificial initial boundary value problem for the high-order heat equation in a bounded domain Ω. It is found an unique classical solution of this problem in an explicit form and shown that the solution of the artificial initial boundary value problem is equal to the solution of the infinite problem (Cauchy problem) in Ω.

Keywords: transparent boundary conditions, an artificial initial boundary value problem, a high-order parabolic equation.

1. Introduction

When computing the solution of a partial differential equation in an unbounded domain, one often introduces artificial boundaries. In order to limit the computational cost, these boundaries must be chosen not too far from the domain of interest. Therefore, the boundary conditions must be good approximations to the so-called transparent boundary condition (i.e., such that the solution of the problem in the bounded domain is equal to the solution in the original domain).

One of the numerical methods for the solution of problems in unbounded domains is the Dirichlet-to-Neumann (DtN) Finite Elements Method. Its name comes from the fact that it involves the nonlocal Dirichlet-to-Neumann (DtN) map on an artificial boundary which encloses the computational domain. Originally the method has been developed for the solution of linear elliptic problems, such as wave scattering problems governed by Helmholtz equation or by the equations of time-harmonic elasticity. Recently, the method has been extended in a number of directions, and further analyzed and improved, by the D. Givoli [1] and others. In

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[1]-[5] it can be find more references to various numerical methods associated with solving similar problems.

This question is of crucial interest in such different areas as geophysics, plasma physics, fluid dynamics (see [6] - [9]).

The need for practical transparent (artificial) boundary conditions combining efficiency and simplicity is evident. Such conditions must satisfy several criteria: (i) The resulting initial boundary value problem should be unique and stable; (ii) the solution to the initial boundary value problem should coincide or closely approximate the solution of the infinite problem on the definitional domain of the boundary value problem; and (iii) the conditions must allow for an analytical solution or an efficient numerical implementation.

In this paper we consider artificial boundary conditions for the high-order Cauchy problem for the heat equation. The conditions satisfy the above-mentioned criteria (i), (ii) and (iii). Similar results were taken for the Laplace equation in [10] and for high-order Laplace equation (polyharmonic equations) in [11, 12]. And a transfer of the Sommerfeld radiation condition to a boundary of bounded domains for the Helmholtz operator is studied in work [13].

The transparent boundary condition is usually an integral relation in time and space between u and its normal derivative on the boundary, which makes it impractical from a numerical point of view. Alternatively, the requirement for boundary conditions can be avoided when the solution of a partial differential equation is approximated in the form of convolution in space and time with the fundamental solution. An efficient approximation of this type for the heat equation is proposed in [14].

2. Main results

In what follows, we shall need in definitions of the Holder spaces (see, e.g., [15], pp. 33-34 and 117-118).

On a cylindrical domain $\Omega \equiv Q \times (0,T)$, where $Q \subset \mathbb{R}^n, n \in \mathbb{N}$ is a simply-connected bounded domain with a sufficiently smooth boundary ∂Q, we consider the following heat potential

$$u(x,t) = \int_0^t \int_Q \varepsilon_{m,n}(x-\xi, t-\tau) f(\xi,\tau) d\xi d\tau, \tag{1}$$

where $\varepsilon_{m,n}(x,t) = \frac{\theta(t)t^{m-1}}{(2\sqrt{\pi t})^n} \exp(-\frac{|x|^2}{4t}), m \in \mathbb{N}$ is a fundamental solution of the Cauchy problem for the high-order heat equation, i.e. this solves the following high-order heat equation

$$\diamondsuit_{x,t}^m \varepsilon_{m,n}(x-\xi,t-\tau) = (\frac{\partial}{\partial t} - \Delta_x)^m \varepsilon_{m,n}(x-\xi,t-\tau) = 0$$

and its adjoint

$$(\diamondsuit_{\xi,\tau}^+)^m \varepsilon_{m,n}(x-\xi,t-\tau) = (-\frac{\partial}{\partial \tau} - \Delta_\xi)^m \varepsilon_{m,n}(x-\xi,t-\tau) = 0$$

for all $t > \tau$ and $\xi, x \in R^n$.

Indeed, it's not difficult to check that for $m > 1$

$$\diamondsuit_{x,t} \varepsilon_{m,n}(x-\xi,t-\tau) = \varepsilon_{m-1,n}(x-\xi,t-\tau)$$

and

$$\diamondsuit^{+}_{\xi,\tau}\varepsilon_{m,n}(x-\xi,t-\tau)=\varepsilon_{m-1,n}(x-\xi,t-\tau).$$

From [17, chapter 1] for $x\in Q$ the condition

$$\lim_{\tau\to t}\int_Q(\diamondsuit^{+}_{\xi,\tau})^{m-1}\varepsilon_{m,n}(x-\xi,t-\tau)f(\xi,\tau)d\xi=f(x,t)$$

is valid.

It is easy to see that if $f(x,t)\in C^{\gamma,\frac{\gamma}{2}}_{x,t}(\overline{\Omega})$ then $u(x,t)\in C^{2m+\gamma,m+\frac{\gamma}{2}}_{x,t}(\overline{\Omega})$, where $0<\gamma<1$ and

$$\diamondsuit^m u(x,t)=f(x,t),(x,t)\in\Omega, \tag{2}$$

with initial conditions

$$\frac{\partial^i u(x,t)}{\partial t^i}\mid_{t=0}=0, i=\overline{0,m-1}. \tag{3}$$

We note that if the function $u(x,t)\in C^{2m+\gamma,m+\frac{\gamma}{2}}_{x,t}(\overline{\Omega})$ satisfies the conditions (3) then it will satisfy to the following conditions too

$$\diamondsuit^i u(x,t)\mid_{t=0}=0, i=\overline{0,m-1}.$$

Consider the problem (2)-(3) with non-local type boundary conditions

$$\begin{aligned} I^k_u(x,t)\mid_{x\in\partial Q}\equiv -\frac{\diamondsuit^k u(x,t)}{2} \\ +\sum_{i=0}^{m-k-1}\Big[\int_0^t\int_{\partial Q}\frac{\partial(\diamondsuit^{+}_{\xi,\tau})^{i+k}\varepsilon_{m,n}(x-\xi,t-\tau)}{\partial n_\xi}\diamondsuit^{m-i-1}u(\xi,\tau)dS_\xi d\tau- \\ -\int_0^t\int_{\partial Q}\frac{\partial\diamondsuit^{m-i-1}u(\xi,\tau)}{\partial n_\xi}(\diamondsuit^{+}_{\xi,\tau})^{i+k}\varepsilon_{m,n}(x-\xi,t-\tau)dS_\xi d\tau\Big]=0, \\ k=\overline{0,m-1},(x,t)\in\partial Q\times(0,T), \end{aligned} \tag{4}$$

where $\frac{\partial}{\partial n_\xi}$ denotes the exterior normal derivative on the boundary ∂Q.

Theorem 1. *The heat potential for the high-order heat equation (2) is an unique classical solution of the non-local type initial boundary value problem (2)-(4).*

Proof. The following equalities are valid

$$(\diamondsuit^{+}_{\xi,\tau})^m\varepsilon_{m,n}(x-\xi,t-\tau)=(\diamondsuit^{+}_{\xi,\tau})^{m-1}\varepsilon_{m-1,n}(x-\xi,t-\tau)=...$$
$$=\diamondsuit^{+}_{\xi,\tau}\varepsilon_{1,n}(x-\xi,t-\tau)=0,$$

for all $t>\tau$ and $\xi,x\in R^n$,

$$\lim_{\tau\to t}(\diamondsuit^{+}_{\xi,\tau})^k\varepsilon_{m,n}(x-\xi,t-\tau)=0,$$

for $k<m-1$. Therefore

$$\lim_{\alpha\to 0+0}\int_Q(\diamondsuit^{+}_{\xi,\tau})^k\varepsilon_{m,n}(x-\xi,\alpha)u(\xi,t-\alpha)d\xi=0$$

for $k<m-1$ and

$$\int_0^t\int_Q(\diamondsuit^{+}_{\xi,\tau})^m\varepsilon_{m,n}(x-\xi,t-\tau)u(\xi,\tau)d\xi d\tau=$$

$$= \lim_{\alpha\to 0+0}\int_0^{t-\alpha}\int_Q (\diamondsuit^+_{\xi,\tau})^m \varepsilon_{m,n}(x-\xi,t-\tau)u(\xi,\tau)d\xi d\tau =$$

$$= \lim_{\alpha\to 0+0}\int_0^{t-\alpha}\int_Q 0u(\xi,\tau)d\xi d\tau = 0.$$

We assume that $u(x,t) \in C^{2m+\gamma, m+\frac{\gamma}{2}}_{x,t}(\overline{\Omega})$. Let us denote $\diamondsuit^0 \equiv I$, where I is identity operator. A direct calculation shows that

$$u(x,t) = \lim_{\alpha\to 0+0}\int_0^{t-\alpha}\int_Q \varepsilon_{m,n}(x-\xi,t-\tau)f(\xi,\tau)d\xi d\tau =$$

$$= \lim_{\alpha\to 0+0}\int_0^{t-\alpha}\int_Q \varepsilon_{m,n}(x-\xi,t-\tau)\diamondsuit(\diamondsuit^{m-1}u(\xi,\tau))d\xi d\tau =$$

$$= \lim_{\alpha\to 0+0}[\int_Q \varepsilon_{m,n}(x-\xi,t-\tau)\diamondsuit^{m-1}u(\xi,\tau)\mid_0^{t-\alpha} d\xi -$$

$$-\int_0^{t-\alpha}\int_Q \frac{\partial\varepsilon_{m,n}(x-\xi,t-\tau)}{\partial\tau}\diamondsuit^{m-1}u(\xi,\tau)d\xi d\tau -$$

$$-\int_0^{t-\alpha}\int_Q \varepsilon_{m,n}(x-\xi,t-\tau)\Delta_\xi(\diamondsuit^{m-1}u(\xi,\tau))d\xi d\tau] =$$

$$= \lim_{\alpha\to 0+0}[\int_0^{t-\alpha}\int_Q \diamondsuit^+_{\xi,\tau}\varepsilon_{m,n}(x-\xi,t-\tau)\diamondsuit^{m-1}u(\xi,\tau)d\xi d\tau +$$

$$+\int_0^{t-\alpha}\int_{\partial Q} \frac{\partial\varepsilon_{m,n}(x-\xi,t-\tau)}{\partial n_\xi}\diamondsuit^{m-1}u(\xi,\tau)dS_\xi d\tau$$

$$-\int_0^{t-\alpha}\int_{\partial Q} \frac{\partial\diamondsuit^{m-1}u(\xi,\tau)}{\partial n_\xi}\varepsilon_{m,n}(x-\xi,t-\tau)dS_\xi d\tau] = ...$$

$$= \lim_{\alpha\to 0+0}[\int_Q (\diamondsuit^+)^{m-1}\varepsilon_{m,n}(x-\xi,t-\tau)u(\xi,\tau)\mid_0^{t-\alpha} d\xi +$$

$$+\int_0^{t-\alpha}\int_Q (\diamondsuit^+_{\xi,\tau})^m\varepsilon_{m,n}(x-\xi,t-\tau)u(\xi,\tau)d\xi d\tau] +$$

$$+\sum_{i=0}^{m-1}[\int_0^t\int_{\partial Q} \frac{\partial(\diamondsuit^+_{\xi,\tau})^i\varepsilon_{m,n}(x-\xi,t-\tau)}{\partial n_\xi}\diamondsuit^{m-i-1}u(\xi,\tau)dS_\xi d\tau -$$

$$-\int_0^t\int_{\partial Q} \frac{\partial\diamondsuit^{m-i-1}u(\xi,\tau)}{\partial n_\xi}(\diamondsuit^+_{\xi,\tau})^i\varepsilon_{m,n}(x-\xi,t-\tau)dS_\xi d\tau] =$$

$$= u(x,t) + \sum_{i=0}^{m-1}[\int_0^t\int_{\partial Q} \frac{\partial(\diamondsuit^+_{\xi,\tau})^i\varepsilon_{m,n}(x-\xi,t-\tau)}{\partial n_\xi}\diamondsuit^{m-i-1}u(\xi,\tau)dS_\xi d\tau -$$

$$-\int_0^t\int_{\partial Q} \frac{\partial\diamondsuit^{m-i-1}u(\xi,\tau)}{\partial n_\xi}(\diamondsuit^+_{\xi,\tau})^i\varepsilon_{m,n}(x-\xi,t-\tau)dS_\xi d\tau]$$

for any $(x,t) \in \Omega$.

Here we get

$$\sum_{i=0}^{m-1}[\int_0^t\int_{\partial Q}\frac{\partial(\diamondsuit^+_{\xi,\tau})^i\varepsilon_{m,n}(x-\xi,t-\tau)}{\partial n_\xi}\diamondsuit^{m-i-1}u(\xi,\tau)dS_\xi d\tau- \tag{5}$$
$$-\int_0^t\int_{\partial Q}\frac{\partial\diamondsuit^{m-i-1}u(\xi,\tau)}{\partial n_\xi}(\diamondsuit^+_{\xi,\tau})^i\varepsilon_{m,n}(x-\xi,t-\tau)dS_\xi d\tau]=0,\forall(x,t)\in\Omega.$$

Applying properties of the double layer and single layer potentials (see [16] or [18]) to (5) as $(x,t)\to\partial Q\times(0,T)$, we obtain

$$-\frac{u(x,t)}{2}+\sum_{i=0}^{m-1}[\int_0^t\int_{\partial Q}\frac{\partial(\diamondsuit^+_{\xi,\tau})^i\varepsilon_{m,n}(x-\xi,t-\tau)}{\partial n_\xi}\diamondsuit^{m-i-1}u(\xi,\tau)dS_\xi d\tau-$$
$$-\int_0^t\int_{\partial Q}\frac{\partial\diamondsuit^{m-i-1}u(\xi,\tau)}{\partial n_\xi}(\diamondsuit^+_{\xi,\tau})^i\varepsilon_{m,n}(x-\xi,t-\tau)dS_\xi d\tau]=0,\forall(x,t)\in\partial Q\times(0,T).$$

Applying the differential expression $\diamondsuit^k_{x,t},\ k=\overline{1,m-1}$ to $u(x,t)$, we get

$$\diamondsuit^k_{x,t}u(x,t)=\diamondsuit^k_{x,t}\left(\lim_{\alpha\to0+0}\int_0^{t-\alpha}\int_Q\varepsilon_{m,n}(x-\xi,t-\tau)f(\xi,\tau)d\xi d\tau\right)=$$
$$=\diamondsuit^{k-1}_{x,t}\Big(\lim_{\alpha\to0+0}\int_Q\varepsilon_{m,n}(x-\xi,\alpha)f(\xi,t-\alpha)d\xi+$$
$$+\lim_{\alpha\to0+0}\int_0^{t-\alpha}\int_Q\varepsilon_{m-1,n}(x-\xi,t-\tau)f(\xi,\tau)d\xi d\tau\Big)=$$
$$=\diamondsuit^{k-1}_{x,t}\left(\lim_{\alpha\to0+0}\int_0^{t-\alpha}\int_Q\varepsilon_{m-1,n}(x-\xi,t-\tau)f(\xi,\tau)d\xi d\tau\right)=...$$
$$=\lim_{\alpha\to0+0}\int_0^{t-\alpha}\int_Q\varepsilon_{m-k,n}(x-\xi,t-\tau)f(\xi,\tau)d\xi d\tau$$

and after similar calculations as above, we get the following boundary conditions for each k

$$I^k_u(x,t)\equiv-\frac{\diamondsuit^k u(x,t)}{2}+$$
$$+\sum_{i=0}^{m-k-1}[\int_0^t\int_{\partial Q}\frac{\partial(\diamondsuit^+_{\xi,\tau})^{i+k}\varepsilon_{m,n}(x-\xi,t-\tau)}{\partial n_\xi}\diamondsuit^{m-i-1}u(\xi,\tau)dS_\xi d\tau-$$
$$-\int_0^t\int_{\partial Q}\frac{\partial\diamondsuit^{m-i-1}u(\xi,\tau)}{\partial n_\xi}(\diamondsuit^+_{\xi,\tau})^{i+k}\varepsilon_{m,n}(x-\xi,t-\tau)dS_\xi d\tau]=0$$
$$\forall(x,t)\in\partial Q\times(0,T),k=\overline{0,m-1}.$$

Thus, the heat potential (1) satisfies the boundary conditions (4).

Conversely, if a function $u_1(x,t)\in C^{2m+\gamma,m+\frac{\gamma}{2}}_{x,t}(\overline{\Omega})$ satisfies equation (2), the initial condition (3) and boundary conditions (4) then $u_1(x,t)=u(x,t)$, where $u(x,t)$ is the heat potential (1). If this is not so, then a function $v(x,t)=u_1(x,t)-u(x,t)$ satisfies the homogeneous equation

$$\diamondsuit^m v(x,t)=0,(x,t)\in\Omega, \tag{6}$$

with the initial conditions

$$\diamondsuit^k v(x,t)\mid_{t=0}=0, x\in\Omega, k=\overline{0,m-1}, \tag{7}$$

and the boundary conditions

$$I_v^k(x,t)=0, \forall (x,t)\in\partial Q\times(0,T), k=\overline{0,m-1}. \tag{8}$$

Where we use the following notations

$$I_v^k(x,t)\equiv\sum_{i=0}^{m-k-1}[\int_0^t\int_{\partial Q}\frac{\partial(\diamondsuit^+_{\xi,\tau})^{i+k}\varepsilon_{m,n}(x-\xi,t-\tau)}{\partial n_\xi}\diamondsuit^{m-i-1}v(\xi,\tau)dS_\xi d\tau- \tag{9}$$

$$-\int_0^t\int_{\partial Q}\frac{\partial\diamondsuit^{m-i-1}v(\xi,\tau)}{\partial n_\xi}(\diamondsuit^+_{\xi,\tau})^{i+k}\varepsilon_{m,n}(x-\xi,t-\tau)dS_\xi d\tau]=0,$$

$$\forall(x,t)\in\Omega, k=\overline{0,m-1}.$$

On the other hand, by using (6) and (7), we obtain following equalities

$$0=\int_0^t\int_Q\varepsilon_{m,n}(x-\xi,t-\tau)\cdot 0d\xi d\tau=$$

$$=\int_0^t\int_Q\varepsilon_{m,n}(x-\xi,t-\tau)\diamondsuit^{m-k}(\diamondsuit^k v(\xi,\tau))d\xi d\tau=$$

$$=\diamondsuit^k v(x,t)+I_v^k(x,t), \forall(x,t)\in\Omega, k=\overline{0,m-1}.$$

Applying properties of the double layer potential (see [16], [17] or [18]) to (9) as $(x,t)\to\partial Q\times(0,T)$, we have

$$\diamondsuit^k v(x,t)\mid_{\partial Q\times(0,T)}=-I_v^k(x,t)\mid_{\partial Q\times(0,T)}=0, k=\overline{0,m-1}.$$

I.e. the problem (6)-(8) is equivalent to

$$\diamondsuit^m v(x,t)=0, (x,t)\in\Omega, \tag{10}$$

with the initial conditions

$$\diamondsuit^k v(x,t)\mid_{t=0}=0, x\in Q, k=\overline{0,m-1}, \tag{11}$$

and the boundary conditions

$$\diamondsuit^k v(x,t)\mid_{\partial Q\times(0,T)}=0, k=\overline{0,m-1}. \tag{12}$$

From (10)-(12) with $k=m-1$ using the Maximum Principle we get

$$\diamondsuit^{m-1}v(x,t)=0, (x,t)\in\Omega.$$

And repeating the similar calculations we have

$$\diamondsuit^i v(x,t)=0, \forall(x,t)\in\bar{\Omega}, i=\overline{0,m-1},$$

i.e., $v(x,t)=u_1(x,t)-u(x,t)$ and $u_1(x,t)=u(x,t)$.

This completes the proof of Theorem 1.

Remark 1. *The heat kernel, i.e. the fundamental solution of the high-order heat equation $\varepsilon_{m,n}(x,t)$ is the Green function for the non-local type boundary value problem (2)-(4).*

Let us consider the heat equation

$$\Diamond u(x,t) = f(x,t), (x,t) \in \Omega, \tag{13}$$

with initial condition

$$u(x,0) = 0 \tag{14}$$

and with inhomogeneous non-local type boundary condition

$$\begin{aligned} & -\frac{u(x,t)}{2} + \int_0^t \int_{\partial Q} \frac{\partial \varepsilon_{1,n}(x-\xi,t-\tau)}{\partial n_\xi} u(\xi,\tau) dS_\xi d\tau - \\ & - \int_0^t \int_{\partial Q} \frac{\partial u(\xi,\tau)}{\partial n_\xi} \varepsilon_{1,n}(x-\xi,t-\tau) dS_\xi d\tau = \varphi(x,t), \quad (x,t) \in \partial Q \times (0,T), \end{aligned} \tag{15}$$

where $\varphi(x,t) \in C_{x,t}^{2+\gamma,1+\frac{\gamma}{2}}(\overline{\partial Q} \times [0,T])$, $\varphi(x,0) = 0, x \in \partial Q$ and $\dfrac{\partial}{\partial n_\xi}$ denotes the exterior normal derivative on the boundary ∂Q.

Remark 2. *In case $\varphi(x,t) \equiv 0$, from the theorem 1 follows that an unique solution of the problem (13)-(15) is given by potential*

$$u(x,t) = \int_0^t \int_Q \varepsilon_{1,n}(x-\xi,t-\tau) f(\xi,\tau) d\xi d\tau. \tag{16}$$

Lemma 1. *The function*

$$u(x,t) = -\int_0^t \int_{\partial Q} \frac{\partial G(x,\xi,t,\tau)}{\partial n_\xi} \varphi(\xi,\tau) d\xi d\tau \tag{17}$$

is a unique solution of the following problem

$$\Diamond u(x,t) = 0, (x,t) \in \Omega, \tag{18}$$

with initial condition

$$u(x,0) = 0 \tag{19}$$

and with inhomogeneous non-local type boundary condition

$$\begin{aligned} & -\frac{u(x,t)}{2} + \int_0^t \int_{\partial Q} \frac{\partial \varepsilon_{1,n}(x-\xi,t-\tau)}{\partial n_\xi} u(\xi,\tau) dS_\xi d\tau - \\ & - \int_0^t \int_{\partial Q} \frac{\partial u(\xi,\tau)}{\partial n_\xi} \varepsilon_{,n}(x-\xi,t-\tau) dS_\xi d\tau = \varphi(x,t), \quad (x,t) \in \partial Q \times (0,T), \end{aligned} \tag{20}$$

where $\dfrac{\partial}{\partial n_\xi}$ denotes the exterior normal derivative on the boundary ∂Q and $G(x,\xi,t,\tau)$ is the Green function of the problem (13)-(14) with the Dirichlet boundary condition.

Proof.

$$0 = \int_0^t \int_Q \varepsilon_{1,n}(x-\xi,t-\tau) 0 d\xi d\tau = \int_0^t \int_Q \varepsilon_{1,n}(x-\xi,t-\tau) \Diamond u(\xi,\tau) d\xi d\tau =$$

$$= u(x,t) + \int_0^t \int_Q \Diamond_{\xi,\tau}^{+} \varepsilon_{1,n}(x-\xi,t-\tau) u(\xi,\tau) d\xi d\tau +$$

$$+\int_0^t\int_{\partial Q}\frac{\partial\varepsilon_{1,n}(x-\xi,t-\tau)}{\partial n_\xi}u(\xi,\tau)dS_\xi d\tau-\int_0^t\int_{\partial Q}\varepsilon_{1,n}(x-\xi,t-\tau)\frac{\partial u(\xi,\tau)}{\partial n_\xi}dS_\xi d\tau=$$

$$=u(x,t)+\int_0^t\int_{\partial Q}\frac{\partial\varepsilon_{1,n}(x-\xi,t-\tau)}{\partial n_\xi}u(\xi,\tau)dS_\xi d\tau-$$

$$-\int_0^t\int_{\partial Q}\varepsilon_{1,n}(x-\xi,t-\tau)\frac{\partial u(\xi,\tau)}{\partial n_\xi}dS_\xi d\tau$$

for all $(x,t)\in Q\times(0,T)$. Hence,

$$-u(x,t)=\int_0^t\int_{\partial Q}\frac{\partial\varepsilon_{1,n}(x-\xi,t-\tau)}{\partial n_\xi}u(\xi,\tau)dS_\xi d\tau-$$

$$-\int_0^t\int_{\partial Q}\varepsilon_{1,n}(x-\xi,t-\tau)\frac{\partial u(\xi,\tau)}{\partial n_\xi}dS_\xi d\tau$$

for all $(x,t)\in Q\times(0,T)$. And

$$-u(x,t)=-\frac{u(x,t)}{2}+\int_0^t\int_{\partial Q}\frac{\partial\varepsilon_{1,n}(x-\xi,t-\tau)}{\partial n_\xi}u(\xi,\tau)dS_\xi d\tau-$$

$$-\int_0^t\int_{\partial Q}\varepsilon_{1,n}(x-\xi,t-\tau)\frac{\partial u(\xi,\tau)}{\partial n_\xi}dS_\xi d\tau$$

as $(x,t)\to\partial Q\times(0,T)$. Since,

$$-u(x,t)=-\frac{u(x,t)}{2}+\int_0^t\int_{\partial Q}\frac{\partial\varepsilon_{1,n}(x-\xi,t-\tau)}{\partial n_\xi}u(\xi,\tau)dS_\xi d\tau-$$

$$-\int_0^t\int_{\partial Q}\varepsilon_{1,n}(x-\xi,t-\tau)\frac{\partial u(\xi,\tau)}{\partial n_\xi}dS_\xi d\tau=\varphi(x,t)$$

for all $(x,t)\in\partial Q\times(0,T)$. So, problem (18)-(20) equivalent to the problem for homogeneous heat equation (18) with initial condition (19) and with the following Dirichlet boundary condition

$$u(x,t)=-\varphi(x,t)$$

for all $(x,t)\in\partial Q\times(0,T)$. The lemma is proved.

From the Lemma 1 and the Remark 2 follows statement of the following theorem.

Theorem 2. *The solution of the problem (13)-(15) is given by the following formula* (21)

$$u(x,t)=\int_0^t\int_Q\varepsilon_{1,n}(x-\xi,t-\tau)f(\xi,\tau)d\xi d\tau-\int_0^t\int_{\partial Q}\frac{\partial G(x,\xi,t,\tau)}{\partial n_\xi}\varphi(\xi,\tau)d\xi d\tau,$$

where $G(x,\xi,t,\tau)$ is the Green function of the problem (13)-(14) with the Dirichlet boundary condition.

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