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A CHARACTERIZATION OF THE SIMPLE SPORADIC GROUPS

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ABSTRACT. Let G be a finite group, $n_p(G)$ be the number of Sylow $p-$subgroup of G and $t(2,G)$ be the maximal number of vertices in cocliques of the prime graph of G containing 2. In this paper we prove that if G is a centerless group with $t(2,G) \geq 2$ and $n_p(G)=n_p(S)$ for every prime $p \in \pi(G)$, where S is the sporadic simple groups, then $S \leq G \leq \mathrm{Aut}(S)$.

Keywords: Finite Group, simple group, Sylow subgroup.

1. Introduction

Let G be a finite group, let $\pi(G)$ be the set of prime divisors of its order, and let $\pi_e(G)$ be the set of the element orders of G. We construct the *prime graph* of G, which is denoted by $GK(G)$, as follows: the vertex set is $\pi(G)$ and two distinct vertices p and p' are joined by an edge if and only if G has an element of order pp' (we write $p \sim p'$). Gruenberg and Kegel introduced this graph (which is also called the Gruenberg–Kegel graph) in the mid–1970s and gave a characterization of finite groups with disconnected prime graph (we denote the number of connected components of $GK(G)$ by $s(G)$). This deep result, together with classification of finite simple groups with $s(G) > 1$, obtained by Williams and Kondratiev (see [9, 4]), implied a series of important corollaries. Denote by $t(G)$ the maximal number of primes in $\pi(G)$ pairwise nonadjacent in $GK(G)$. In other words, $t(G)$ is the maximal number of vertices in the cocliques, i.e., the independent sets of $GK(G)$. This number is usually called the independence number of G. We denote by $t(r,G)$ the maximal number of vertices in cocliques of $GK(G)$ containing r. We call this the r-independence number. In [6] there was given a characterization of finite groups G with $t(G) \geq 3$ and $t(2,G) \geq 2$, and in [5] it was proved that all

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finite non-abelian simple groups except the alternating permutation groups satisfy the condition $t(2, G) \geq 2$.

Throughout this paper, we denote by $n_p(G)$ the number of Sylow $p-$subgroup of G, that is, $n_p(G) = |\mathrm{Syl}_p(G)|$, all other notations are standard and we refer to [8], for example.

In [1, 2] it is proved that if G is a finite centerless group and $n_p(G) = n_p(L_2(q))$ for every prime $p \in \pi(G)$, where $5 \leq q \leq 25$ and is prime power, then $L_2(q) \leq G \leq \mathrm{Aut}(L_2(q))$. Also in [3] it is proved that if G is a finite centerless group and $n_p(G) = n_p(M)$, where M denotes either of the Mathieu groups M_{11} or M_{12} for every prime $p \in \pi(G)$, then $M \leq G \leq \mathrm{Aut}(M)$. We note that if S is one of the sporadic simple group except M_{12}, M_{22}, J_2, J_3, HS, Suz, McL, He, $O^{'}N$, $F_{i_{22}}$, $Fi_{24}^{'}$ and HN, then $\mathrm{Aut}(S) = S$.

Suppose G_1 and G_2 are two finite groups. If $n_p(G_1) = n_p(G_2)$ for every prime p, then we say G_1 and G_2 are Sylow equivalent. In this paper, we prove that a finite centerless group G with $t(2, G) \geq 2$ is Sylow equivalent with S, where S is the sporadic simple groups. Also if S is one of the groups: M_{12}, M_{22}, J_2, J_3, HS, Suz, McL, He, $O^{'}N$, $F_{i_{22}}$, $Fi_{24}^{'}$ or HN, then $S \leq G \leq \mathrm{Aut}(S)$. In fact the main theorem of our paper is as follows:

Main Theorem : *Let G be a finite centerless group with $t(2, G) \geq 2$ such that $n_p(G) = n_p(S)$ for every prime $p \in \pi(G)$, where S is the sporadic simple groups, then $S \leq G \leq Aut(S)$.*

2. Preliminary Results

In this section we bring some preliminary lemmas to be used in the proof of the main theorem.

Lemma 2.1. [6] *Let G be a finite group satisfying the two conditions:*
(a) there exist three primes in $\pi(G)$ pairwise nonadjacent in $GK(G)$; i.e., $t(G) \geq 3$;
(b) there exists an odd prime in $\pi(G)$ nonadjacent in $GK(G)$ to the prime 2; i.e., $t(2, G) \geq 2$. Then there is a finite non-abelian simple group S such that $S \leq \overline{G} = G/K \leq Aut(S)$ for the maximal normal soluble subgroup K of G. Furthermore, $t(S) \geq t(G) - 1$, and one of the following statements holds:
(1) $S \simeq Alt_7$ or $L_2(q)$ for some odd q, and $t(S) = t(2, S) = 3$.
(2) For every prime $p \in \pi(G)$ nonadjacent to 2 in $GK(G)$ a Sylow p-subgroup of G is isomorphic to a Sylow p-subgroup of S. In particular, $t(2, S) \geq t(2, G)$.

Remark 2.2. *Note that condition (a) in Lemma 2.1, implies an insolubility of G (see Proposition 1 in* [6]*), and so by the Feit–Thompson theorem it is not necessary to assume in the hypotheses of the theorem that G is of even order. Moreover, it turns out that condition (a) can be replaced by a weaker condition of insolubility of G without any modification in the claim of the theorem (see Proposition 2 in* [6]*).*

Lemma 2.3. [10] *Let G be a finite group and M be a normal subgroup of G. Then both the Sylow $p-$number $n_p(M)$ and the Sylow $p-$number $n_p(G/M)$ of the quotient G/M divide the Sylow $p-$number $n_p(G)$ of G and moreover $n_p(M)$ $n_p(G/M) \mid n_p(G)$.*

Lemma 2.4. [11, Theorem 9.3.1] *Let G be a finite soluble group and $|G| = m \cdot n$,* *where* $m = p_1^{\alpha_1} ... p_r^{\alpha_r}$, $(m, n) = 1$. *Let* $\pi = \{p_1, ..., p_r\}$ *and* h_m *be the number of* $\pi-$*Hall subgroups of* G. *Then* $h_m = q_1^{\beta_1} ... q_s^{\beta_s}$ *satisfies the following conditions for all* $i \in \{1, 2, ..., s\}$:

(1) $q_i^{\beta_i} \equiv 1$ *(mod* p_j*), for some* p_j.
(2) *The order of some chief factor of* G *is divisible by* $q_i^{\beta_i}$.

3. Proof of the Main Theorem

By [3] it is proved that if G is a finite centerless group and $n_p(G) = n_p(M)$, where M denotes either of the Mathieu groups M_{11} or M_{12} for every prime $p \in \pi(G)$, then $M \leq G \leq \mathrm{Aut}(M)$. Now let G be a finite centerless group with $t(2, G) \geq 2$ such that $n_p(G)$=$n_p(S)$ for every prime $p \in \pi(G)$, where $S \neq M_{11}$ and M_{12} is the sporadic simple groups. The following Lemmas reduce the problem to a study of groups with Sylow equivalent with S.

Lemma 3.1. *There is a finite non-abelian simple group* L *such that* $L \leq \overline{G} = G/K \leq Aut(L)$ *for the maximal normal soluble subgroup* K *of* G.
Proof. First we prove that G is insoluble group. Let G be soluble group. By assumption $n_p(G) = n_p(S)$ for every prime $p \in \pi(G)$. On the other hand, by Lemma 2.4 if $n_p(G) = q_1^{\beta_1} ... q_s^{\beta_s}$ then, $q_i^{\beta_i} \equiv 1$ (mod p). For every S, it is easy to check that this gives a contradiction. So G is not soluble group. Therefore by using Remark 2.2, there is a finite non-abelian simple group L such that $L \leq \overline{G} = G/K \leq \mathrm{Aut}(L)$ for the maximal normal soluble subgroup K of G. □

Lemma 3.2. *L is isomorphic to S.*
Proof. Let r be the greatest prime in $\pi(G)$. First we show that if $G/K = L$, then $r \in \pi(L)$ and L is isomorphic to S. Let $r \notin \pi(L)$. Then $r \in \pi(K)$ and the order of a Sylow r-subgroup in G and K are equal. As K is normal in G thus the number of Sylow r-subgroups of G and K are equal. Therefore $n_r(G) = n_r(K)$. Since K is soluble group it is easy to check that this gives a contradiction by Lemma 2.4.

Let $G/K = L$. By [7] (Table 1), we can find all simple groups L such that $r \in \pi(L)$. On the other hand, by Lemma 2.3, $n_p(L) \mid n_p(G)$ for every prime p. Thus for every S, it is easy to find all simple groups L such that $n_p(L) \mid n_p(S)$ for every prime p.

Also if $G/K = L$, then $\pi(G) = \pi(L)$. If there exists a prime $t \in \pi(G)$ such that $t \notin \pi(L)$, then $n_t(G) = n_t(K)$ and we can get a contradiction with solubility of K. Thus for every S it is easy to check, except S there is not simple group L such that $n_p(L) \mid n_p(S)$ for every prime p and $\pi(G) = \pi(L)$. Therefore L is isomorphic to S.

Arguing as above if $L \lneqq \overline{G} = G/K \leq \mathrm{Aut}(L)$, then we can prove $r \in \pi(G/K)$ and $\pi(G) = \pi(G/K)$. If $r \in \pi(L)$, then similar to the above discussion L is isomorphic to S. But if $r \notin \pi(L)$, then there exists $s \in \pi(G) \setminus \{r\}$ such that s is the greatest prime in $\pi(L)$. We can find all simple groups L such that $s \in \pi(L)$ by [7] (Table 1). It is easy to check that there is not simple group L such that $r \in \pi(\mathrm{Aut}(L))$, $\pi(L) \subseteq \pi(G) \setminus \{r\}$, $n_p(L) \mid n_p(S)$ for every prime p and s is the greatest prime in $\pi(L)$. □

Lemma 3.3. *If* $G/K = S$, *then* $K = 1$ *and* $G \cong S$.
Proof. By Lemma 2.3, $n_p(K)n_p(S) \mid n_p(G)$ for every prime p. Hence $n_p(K) = 1$

for every prime p, and so K is nilpotent subgroup of G. Let Q be a Sylow $q-$subgroup of K, since K is nilpotent, Q is normal in G. If $P \in \mathrm{Syl}_p(G)$, then Q normalizes P and so if $p \neq q$, then $P \leq N_G(Q) = G$. Also we note that KP/K is a Sylow p-subgroup of G/K. On the other hand, if $R/K = N_{G/K}(\ KP/K)$, then $R = N_G(P)K$. We know that $n_p(G) = n_p(G/K)$, so $|G : R| = |G : N_G(P)|$. Thus $R = N_G(P)$ and therefore $K \leq N_G(P)$. Because K is nilpotent, so P normalizes Q and $Q \leq N_G(P)$. Since $P \leq N_G(Q)$ and $Q \leq N_G(P)$, this implies that $[P, Q] \leq P$ and $[P, Q] \leq Q$. Then $[P, Q] \leq P \cap Q = 1$, so $P \leq C_G(Q)$ and $Q \leq C_G(P)$. In other words, P and Q centralize each other.

Let $C = C_G(Q)$, then C contains a full Sylow $p-$subgroup of G for all primes p different from q, and thus $|G : C|$ is a power of q. Now let T be a Sylow $q-$subgroup of G. Then $G = CT$. Also if $Q > 1$, then $C_Q(T)$ is nontrivial, and $C_Q(T) \leq Z(G)$. Since by assumption $Z(G) = 1$, it follows that $Q = 1$. Since q is arbitrary, $K = 1$. Therefore $G \cong S$. □

Lemma 3.4. *If S is one of the groups: M_{12}, M_{22}, J_2, J_3, HS, Suz, McL, He, $O^{'}N$, $F_{i_{22}}$, $Fi_{24}^{'}$ or HN, then $S \leq G \leq Aut(S)$.*

Proof. By Lemma 3.3, if $G/K = S$, then $K = 1$ and $G \cong S$. Let $S \lneqq \overline{G} = G/K \leq \mathrm{Aut}(S)$. Since $n_p(S) \mid n_p(\mathrm{Aut}(S))$ for every p, similar to the proof of Lemma 3.3, K is nilpotent subgroup of G, so $K = 1$ and $G \cong \mathrm{Aut}(S)$. Therefore $S \leq G \leq \mathrm{Aut}(S)$. □

The proof of the main theorem is now complete.

As an example, we illustrate this method for the Mathieu group of degree 24. First we prove that G is insoluble group. Let G be soluble group. By assumption $n_3(G) = n_3(M_{22}) = 6160 = 2^4 \cdot 5 \cdot 7 \cdot 11$. By Lemma 2.4, $5 \equiv 1 \pmod 3$, which is a contradiction. So G is not soluble group. Therefore by Remark 2.2, there is a finite non-abelian simple group L such that $L \leq \overline{G} = G/K \leq \mathrm{Aut}(L)$ for the maximal normal soluble subgroup K of G. Now let $G/K = L$, we show that $11 \in \pi(L)$ and L is isomorphic to M_{22}. If $G/K = L$ and $11 \notin \pi(L)$, then $11 \in \pi(K)$ and the order of a Sylow 11-subgroup in G and K are equal. As K is normal in G thus the number of Sylow 11-subgroups of G and K are equal. Thus the number of Sylow 11-subgroups of K is $8064 = 2^4 \cdot 3^2 \cdot 7$. Since K is soluble group, $7 \equiv 1 \pmod{11}$ by Lemma 2.4, a contradiction.

Suppose that $G/K = L$. By [7] (Table 1) L is isomorphic to one of the groups: $L_2(11)$, M_{11}, M_{12}, $U_5(2)$, A_{11}, McL, HS, A_{12}, $U_6(2)$ or M_{22}. We know that $n_p(L) \mid n_p(G)$ for every prime p. It is easy to check that L is not isomorphic to M_{12}, $U_5(2)$, A_{11}, McL, HS, A_{12} or $U_6(2)$. If L is isomorphic to $L_2(11)$ and M_{11}, then $7 \notin \pi(L)$ and similar to the above discussion we get a contradiction. Therefore L is isomorphic to M_{22}. By Lemma 2.3, $n_p(K)n_p(S) \mid n_p(G)$ for every prime p. Hence $n_p(K) = 1$ for every prime p, and so K is nilpotent subgroup of G. Similar to the proof of Lemma 3.3, $K = 1$. Therefore $G \cong M_{22}$.

Arguing as above if $L \lneqq \overline{G} = G/K \leq \mathrm{Aut}(L)$, then $11 \in \pi(G/K)$ and $\pi(G) = \pi(G/K)$. If $11 \in \pi(L)$, then similar to the above discussion L is isomorphic to M_{22} and similar to the proof of Lemma 3.4, $M_{22} \leq G \leq \mathrm{Aut}(M_{22})$. But if $11 \notin \pi(L)$, then 7 or 5 is the greatest prime in $\pi(L)$. By [7] (Table 1) we can find all simple groups L such that 7 or $5 \in \pi(L)$, $\pi(L) \subseteq \pi(G) \backslash \{11\}$ and $n_p(L) \mid n_p(S)$ for every prime p. It is easy to check that $11 \notin \pi(\mathrm{Aut}(L))$, a contradiction.

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