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SPECIAL CASE OF THE CAHN-HILLIARD EQUATION

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Abstract. A qualitative behaviour of the Cauchy problem solution for the Cahn-Hilliard kind equation is analyzed. The sufficient condition of the global solution existence and its collapse for a finite time for the periodic function has been formulated. The examples of the stationary, self-similar and collapsing solutions are constructed.

Keywords: Cahn-Hilliard Equation, Cauchy problem, Lyapunov functional, similarity solutions.

1. Introduction

In this paper, we study the qualitative properties of the solutions of Cahn-Hilliard kind equation

$$u_t + \Delta^2 u + \Delta(u^2 - \beta u) = 0; \quad u = u_0(x, y),\ t = 0, \tag{1}$$

where β is a constant.

The equation (1) has arisen from analysis of the thermocapillary flow in the thin layer of viscous liquid with a free surface at non-monotonic dependence of the surface tension σ on the temperature θ [1]. The dependence of interfacial tension of liquids on temperature is very important in applications such as thermocapillary or Marangoni convection. It is known that for a pure liquid the surface tension is a monotonically decreasing function of temperature. This dependence is typical for a large class of fluids like water, silicone oil, water-benzene solutions, etc. It is also known that the surface tension coefficient of the melts of some alloys increases together with the temperature. Anomalous Marangoni effect in two-layer

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system was investigated on the basis of the full Navier-Stokes equations in [2]. However, some solutions, molten steel and alloys have a surface tension which is not nonmonotonic function with a minimum at some temperature θ_* [3]. The interfacial tension dependence on the temperature is approximated by a quadratic function. This dependence is well described by the relation $\sigma = \sigma_0 + \sigma_\theta(\theta - \theta_*)^2$ with appropriate positive constants σ_0, σ_θ, θ_*. The influence of the surface tension minimum on thermocapillary flows in such solutions has been investigated both experimentally and theoretically by means of numerical and analytical methods.

The equation (1) arises also in the study of the phase dynamics in the reaction-diffusion systems, the nonlinear cross-field instability in a weakly ionized plasma [4, 5], morphological instability of a planar crystal-melt interface [6]. Equation (1) is close to the Sivashinsky equation [7] that governs the weakly nonlinear evolution of the long-scale morphological instability in the modeling of an alloy solidification problem.

The classical Cahn-Hilliard equation contains the term u^3 and has both stable and unstable stationary solutions. Previously, the initial boundary value problem for the family of fourth order equations, including Cahn-Hilliard equation, in a bounded domain with the Dirichlet condition on a boundary was considered in [8]. The global existence and blow up in the one-dimensional Cahn-Hilliard equation were proved in [9]. The blow up problem in the Cahn-Hilliard type equations in both one and two dimensions was discussed in [10]. The asymptotic behavior of classes of global and blow up solutions of a semilinear parabolic equation of the "limit" Cahn-Hilliard kind with bounded integrable initial data have been studied in [11]. An asymptotics for collapsing solution in 1D space periodic case has been derived in [12].

The goal of the present work is the investigation of the Benard-Marangoni problem with the condition $\theta = \theta_*$ on the free surface in the thin layer approximation. Here θ_* is the equilibrium temperature at the nondeformable free surface.

2. Statement of the problem

It is supposed that the characteristic disturbance amplitude of the free surface $u(x, y, t)$ is much less than the average layer thickness h. In this case the evolution of the non-dimensional deviation of a free boundary from a horizontal equilibrium state can be described in terms of Cauchy problem solutions for the equation (1), and $\beta = \rho g h^2/\sigma_0$ is the Bond number, ρ is the liquid density, g is the magnitude of the gravity acceleration, $\beta \geq 0$. The function $u_0(x, y)$ is assumed to be periodic function on both variables or rapidly decreasing at $x, y \to \infty$.

The "mass" conservation law takes place for Cauchy problem (1)

$$\iint\limits_{\mathbb{R}^2} u\,dx\,dy = \iint\limits_{\mathbb{R}^2} u_0 dxdy = c, \tag{2}$$

quotes are caused by that the value c can be negative.

Let $u_0 \in H_0^2(\Pi)$ and $||u_0||_{H^2} \leq \epsilon$, where $H_0^2(\Pi)$ is the subspace of Sobolev space formed by the periodic functions, $\Pi = \{x, y : 0 < x < 2\pi, 0 < y < 2\pi\kappa^{-1}\}$, $\kappa \geq 1$, $\epsilon = \epsilon(\beta, \kappa)$ is a sufficient small positive number. If $\beta > -1$, Cauchy problem (1) has a unique generalized solution $u(x, y, t) \in L^2(0, \infty; H_0^2(\Pi))$. There exist constants $\gamma \in (0, 1 + \beta)$ and $C > 0$ independent of t such that the estimate $e^{\gamma t}||u||_{L^2} \leq C\epsilon$ is true for any fixed $t > 0$. One should note that the condition of smallness of $||u_0||_{H^2}$ is essential for the global existence of the problem solution (1). Solutions having a "large" initial norm can be destroyed in finite time.

Space-periodic solutions of Cauchy problem and rapidly decreasing solutions at infinity are studied.

3. Lyapunov functional

Let consider Eq. (1). The following identity is valid

$$\frac{dS(u)}{dt} = \iint\limits_{\Pi} |\nabla(\Delta u + (u - \beta/2)^2)|^2 \, dx \, dy. \tag{3}$$

Here $S(u)$ is the Lyapunov functional that is defined by the equality

$$S(u) = \iint\limits_{\Pi} \left(\frac{1}{3} \left(u - \frac{\beta}{2} \right)^3 - \frac{1}{2} |\nabla u|^2 \right) dx \, dy, \tag{4}$$

where u is an arbitrary solution of (1), which satisfies the condition of periodicity and has the zero mean value on Π. The first variation of $S(u)$ is

$$\delta S = \iint\limits_{\Pi} (\Delta u + u^2 - \beta u) \delta u \, dx \, dy. \tag{5}$$

It allows one to write the equation (1) in the form

$$u_t = \mathrm{grad}_{H^{-2}} S(u). \tag{6}$$

The gradient form (6) of the equation (1) is useful for the investigation of qualitative properties of solutions of the Cauchy problem (1).

Each stationary space periodic solution u_s of equation (1) is the extremal point for the functional $S(u)$. According to (5), the second variation of this functional has the form

$$\delta^2 S(u_s) = \iint\limits_{\Pi} (-|\nabla \delta u|^2 + (2u_s - \beta)(\delta u)^2) \, dx \, dy. \tag{7}$$

It follows from (7) that the function S has no local minima. This functional is not bounded below or above according to definition (4). The inequality $2u_s < 1 + \beta$ is the sufficient condition for a stability of the stationary solution u_s. In this case, the functional $S(u)$ has local maximum at the point u_s. If last inequality is not fulfilled at some points of the domain Π, the form $\delta^2 S(u_s)$ is indefinite, and the instability of the stationary solution is expected.

4. Stationary solutions

Equation (1) has numerous stationary solutions u_s (see also [6]). A special place is occupied by solutions, which depend only on one variable. In this case the equation is integrated in the form of the elliptic functions, the corresponding periodic solutions are the well-known cnoidal waves. Another interesting set of stationary solutions consists of solutions rapidly decreasing when one or both variables tend to infinity. The Korteweg and de Vries soliton, $u_s = 3\beta/(2\cosh^2(x\sqrt{\beta}/2))$, is the example of such solution. The existence of axially symmetric soliton, $u_s = g(r)$, where $r =$

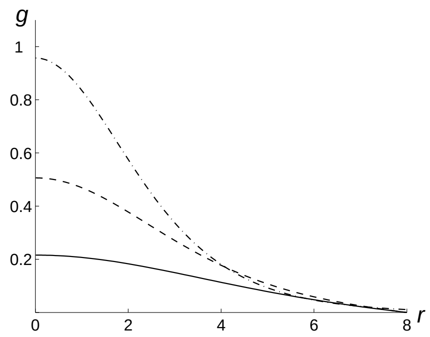

FIG 1. Axially symmetric soliton for various values of the parameter β: $\beta = 0.05$ (solid line), $\beta = 0.2$ (dashed line), $\beta = 0.4$ (dash-dotted line).

$\sqrt{x^2+y^2}$ was proved in [13]. The function g is the solution of the boundary value problem

$$\frac{d^2g}{dr^2} + \frac{1}{r}\frac{dg}{dr} + g^2 - \beta g = 0,\ r > 0, \tag{8}$$

$$|g| < \infty \text{ at } r \to 0,\ g \to 0 \text{ at } r \to \infty. \tag{9}$$

The profiles of the solution (8), (9) are illustrated in Fig. 1 for various values of the parameter β.

Note that there is no nontrivial solutions in the form of travelling waves, $u = q(x-ct)$, which are defined and bounded at all values of its arguments. The periodic travelling waves do not exist also.

In the periodic problem depending on the initial data, there were either a stabilization to the stationary solution (in particular, to the trivial solution), or a collapse in finite time (see Section 6). Besides the trivial solutions, there are the periodic solutions that can be obtained by a branching from the trivial solution. Among these solutions, there are the stable solutions as the following example shows. Let us consider the trivial steady-state solution of the Eq. (1) $u = \beta$. We obtain two families 2π-periodic stationary solutions corresponding to the value $\beta = 1+\epsilon$, where $\epsilon > 0$. The solution at $\epsilon \to 0$ can be presented in the form

$$u = 1 + \epsilon^{1/2}k\cos 2x + \epsilon\left(-\frac{k^2}{2} + \frac{k^2}{6}\cos 2x\right) + \epsilon^{3/2}\frac{k^3}{48}\cos 3x + O(\epsilon^2),$$

where $k = \pm\sqrt{6/5}$. These solutions exist in a supercritical domain, and they are expected to be stable.

5. Self-similar solutions

The equation (1) is reduced to the case $\beta = 0$ by substitution $u = v+\beta/2$. In this case the equation (1) has self-similar solutions. We consider the Cauchy problem

$$v_t + \Delta^2 v + \Delta(v^2) = 0; \quad v = v_0(x,y),\ t = 0. \tag{10}$$

Equation (10) is invariant with respect to scaling transformation $\bar{x} = ax$, $\bar{y} = ay$, $\bar{t} = a^4 t$, $\bar{v} = a^{-2}v$ ($a = const$), and this equation is also invariant with respect to the rotation in (x, y)-plane. This makes it possible to look for the self-similar solutions in the form

$$v = t^{-1/2} f(\xi), \quad \xi = t^{-1/4}(x^2 + y^2)^{1/2}.$$

These solutions satisfy the mass conservation law (2) which takes place for the equation (10). To determine $f(\xi)$, we have the fourth-order ordinary differential equation that can be allowed one to decrease of equation order

$$[\xi^{-1}(\xi f')']' - \frac{1}{4}\xi f + 2ff' = A\xi^{-1}, \tag{11}$$

where A is the arbitrary constant. A two-dimensional analog of the equation (11) has been considered in [12]. The self-similar solutions of the Cahn-Hilliard type equation (10) with the replacement of v^2 by v^3 were investigated in [11] for the plane case also. The equation (11) has regular solutions at $\xi \to 0$ only for $A = 0$. The considered equation has two singular points: regular point $\xi = 0$ and irregular point $\xi = \infty$. We seek non-trivial solutions that are defined for all $\xi > 0$, regular at $\xi \to 0$, and rapidly decreasing at $\xi \to \infty$. Such solutions form the one-parameter family with the parameter c, where

$$c = \int_0^\infty \xi f \, d\xi. \tag{12}$$

The function $f = f(\xi)$ satisfies the problem:

$$[\xi^{-1}(\xi f')']' - \frac{1}{4}\xi f + 2ff' = 0, \tag{13}$$

$$|f| < \infty \text{ at } \xi \to 0, \quad f \to 0 \text{ at } \xi \to \infty, \tag{14}$$

and the condition (12). The function $f(\xi)$ is the solution of the Cauchy problem with the initial function $v_0 = 2\pi c\delta(x)\delta(y)$, where δ is the delta function. Preliminary analysis of the problem (13), (14) was performed in [14].

Equations (13), (14), (12) have been solved using the Runge-Kutta method in conjunction with the shooting technique. For that the relation $2f''(0) + f^2(0) = -c/4$ was derived. Analytical and numerical researches show that axially symmetric self-similar solutions exist at small values of $|c|$, and they do not exist for large and positive c because of indefinitely increasing derivative $d\lambda/dc$ at approach to a critical value $c_* \simeq 0.8155$, here $\lambda = f(0)$. The curve λ as a double-valued function of the parameter c is presented in Fig. 2. The stability of the lower branch of curve Γ can be justified at least for small $|c|$ following arguments [11]. On the analogy with [11], it is natural to suppose that the upper branch of curve Γ is seemed to be unstable. Figure 3(a) displays the dependence $f(\xi)$ for a critical value $c = c_*$. For negative values of c, there were found two branches of the self-similar solutions with various qualitative behaviors. Figures 3(b), 3(c) illustrate the function $f(\xi)$ for the same value $c = -6$, and various values of λ. In Fig. 3(b), $\lambda = 5.019$ for the upper solution; in Fig. 3(c), $\lambda = -1.122$ for the down solution. The function $f(\xi)$ for $c = 0$ and $\lambda = 2.057$ is depicted in Fig. 3(d).

The self-similar solutions of the plane problem satisfying the conservation law exist only for $c = 0$. The solutions of the 1D analogue of (10) has the form

$$v = t^{-1/2}\phi(\eta), \quad \eta = xt^{-1/4}.$$

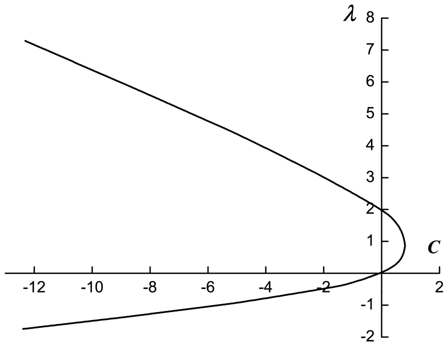

FIG 2. Curve Γ is a double-valued function $\lambda = f(0)$ of the parameter c.

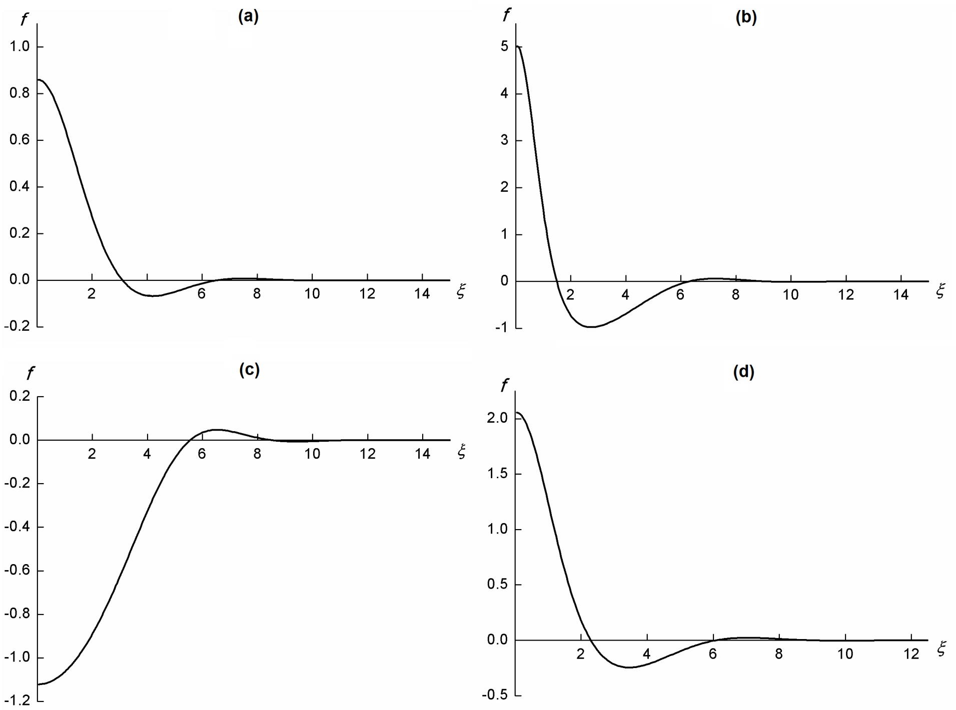

FIG 3. Self-similar solution of the axially symmetric problem for (a) $c = c_* \approx 0.8155$, $\lambda = 0.860$; (b) $c = -6$, $\lambda = 5.019$; (c) $c = -6$, $\lambda = -1.122$; (d) $c = 0$, $\lambda = 2.057$.

It is interesting to note that these solutions are not compatible with the mass conservation law. The function $\phi(\eta)$ satisfies the ordinary differential equation

$$\phi'''' + (\phi^2)'' - \frac{\eta}{4}\phi' - \frac{1}{2}\phi = 0, \tag{15}$$

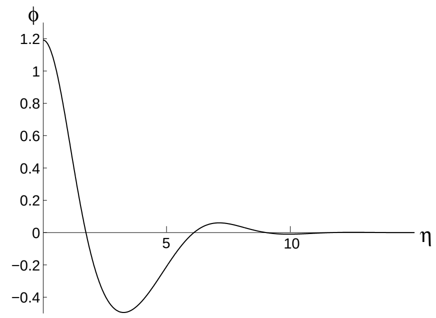

FIG 4. Self-similar solution of the 1D problem.

which is integrated using the boundary conditions

$$\phi'(0) = 0, \quad \phi \to 0 \text{ at } \eta \to \infty. \tag{16}$$

In this case

$$\int_{-\infty}^{\infty} \phi \, d\eta = 0.$$

Figure 4 shows the results of numerical simulation of the boundary-value problem (15), (16). One can see that the function ϕ rapidly tends to zero at infinity.

6. Collapsing solutions

The behavior of the Cauchy problem solutions (1) is following: either $u \to u_s$ when $t \to \infty$, where u_s is some stationary solution, or its solution is destroyed in finite or infinite time [13]. We formulate below sufficient conditions of collapse existence.

PROPOSITION. *Let initial function $u_0 \in H_0^2(\Pi)$ in the condition (1) satisfy the inequality*

$$\iint\limits_{\Pi} \left(\frac{u_0^3}{3} - \frac{|\nabla u_0|^2}{2} \right) dx\, dy > \frac{6}{5}(1+\beta^2) \iint\limits_{\Pi} [(-\Delta)^{-1/2} u_0]^2 dx\, dy. \tag{17}$$

There exists such $t^* > 0$ that for solution u of the Cauchy problem (1) we have

$$||(-\Delta)^{-1/2} u||_{L^2} \to \infty \quad \textit{when} \quad t \to t^* - 0.$$

It should be emphasized that the inequality (17) cannot be fulfilled for "small" values of u_0, and also for odd function u_0. Solutions having a "large" initial norm can be destroyed in finite time. To illustrate this proposition, let us consider the boundary-value problem (1) for the function $u(x,t)$ with 2π-periodic initial function $u_0(x)$. The collapse takes place for a simple example of an initial function $u_0(x) = a_1 \cos x + a_2 \cos 2x$ of the Cauchy problem (1) with $\beta = 0$, where constants a_1 and a_2 satisfy inequalities $|a_1| > 2$, $a_1^2 - (a_1^4 - 16)^{1/2} < 2a_2 < a_1^2 + (a_1^4 - 16)^{1/2}$. The problem was solved using finite difference method. The calculations were made for one thousand and two thousand nodes. The difference does not exceed 10^{-3}. The results of the numerical simulation are presented in Fig. 5 for the initial data $a_1 = 2.3$, $a_2 = 2$. Figure 5(a) demonstrates the behaviour of the function u in

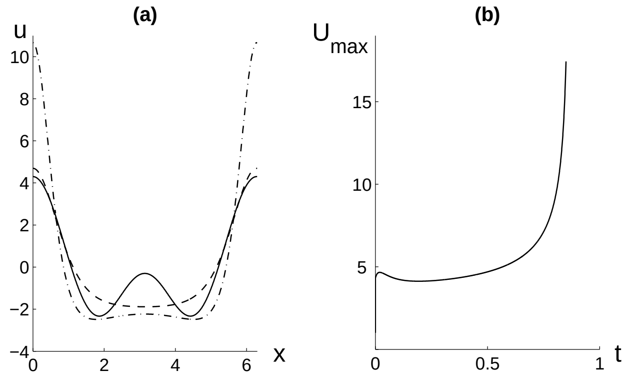

FIG 5. (a) Collapsing solution of the periodic problem (1) for $\beta = 0$ and $t = 0$ (solid line), $t = 0.4$ (dashed line), $t = 0.82$ (dash-dotted line). (b) The maximum amplitude $U_{max}(t)$.

different time point. The maximum amplitude of the solution $U_{max}(t)$ is shown in Fig. 5(b). One can see that the solution destroys in finite time.

Let us consider now the problem (1), when u_0 is an even aperiodic and rapidly decreasing at infinity function. The approximate solution is looked for in the form

$$u^{(N)}(x,t) = \sum_{n=0}^{N} u_n(t) C_n(x), \tag{18}$$

where the basis functions $C_n(x)$ are [15]

$$C_n(x) = \sqrt{\frac{2}{\pi}} \frac{\sum_{k=1}^{n+1} (-1)^{n+k+1} \binom{2n+1}{2k-2} x^{2k-2}}{(x^2+1)^{n+1}}, \quad n = 0, 1, 2, \ldots.$$

The functions $C_n(x)$ form the complete orthonormal family in the space $L^2(-\infty, \infty)$ with the weight 1.

Equations set of functions $u_n(t)$ in (18) is integrated for the various values of the initial data

$$u_n(0) = q_n, \quad n = 0, \ldots, N. \tag{19}$$

Numerical calculations show that the solution of the non-periodic problem (18), (19) collapses in finite time. Figure 6 illustrates the function $u^{(8)}(x,t)$ for $\beta = 0.1$ and $t = 0$, $t = 10$, $t = 18$ and $t = 18.45$. This function is depicted in Fig. 7 for the same β and various values of x. The initial data were selected as following $q_0 = 1, q_1 = 1, q_2 = -7/5, q_3 = 3/7$, $q_i = 0$, $i = 4, \ldots, 8$.

Unfortunately, we failed to prove that the solution with "small" initial norm is stabilized to the stationary solution. However, the numerical calculations allow to hope that our assumption is correct. As the Fig. 8 indicates, the solution is stabilized to the soliton. The figure presents the function $u^{(8)}(x,t)$ for $\beta = 0.1$ and $q_0 = 2$, $q_1 = -4$, $q_2 = 2$, $q_i = 0$, $i = 3, \ldots, 8$, in various time moments. Eight basis functions, used in the calculations, are enough to present the solution with a satisfactory accuracy.

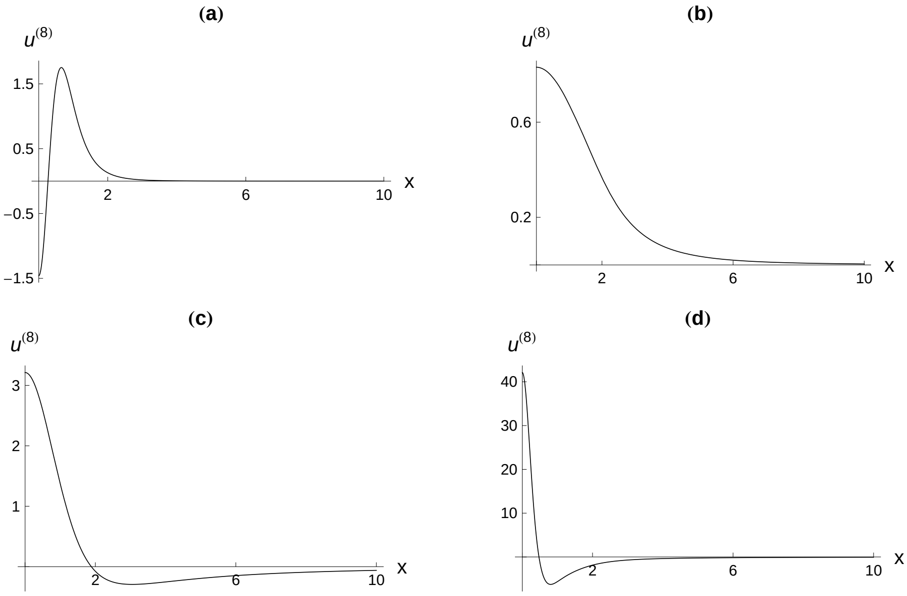

FIG 6. Collapsing solution of the 1D non-periodic problem (1) for $\beta = 0.1$ and (a) $t = 0$, (b) $t = 10$, (c) $t = 18$, (d) $t = 18.45$.

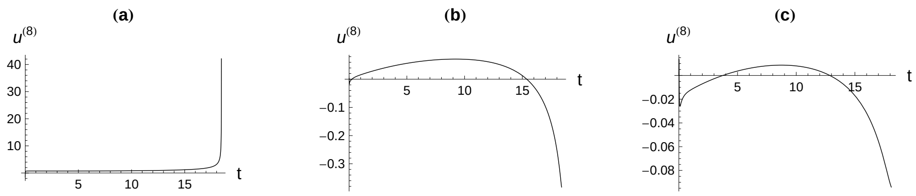

FIG 7. Collapsing solution of the 1D non-periodic problem (1) for $\beta = 0.1$ and (a) $x = 0$, (b) $x = 4$, (c) $x = 8$.

7. Conclusions

Space-periodic solutions of the Cauchy problem and rapidly decreasing solutions at infinity are studied. The sufficient instability condition of the equilibrium has been obtained in the framework of the long-wave approximation. The sufficient condition of the global solution existence of problem (1) and its collapse for a finite time for the periodic function has been formulated. The behaviour of the Cauchy problem solutions (1) is following: either $u \to u_s$, when $t \to \infty$, where u_s is some stationary solution, or its solution is destroyed in finite or infinite time. The Korteweg and de Vries solitons, axially symmetric solitons and cnoidal waves are stationary solutions of the problem. Besides, there is the one-parameter family of self-similar axially symmetric solutions of the modified equation (10), which are compatible with the mass conservation law for this equation.

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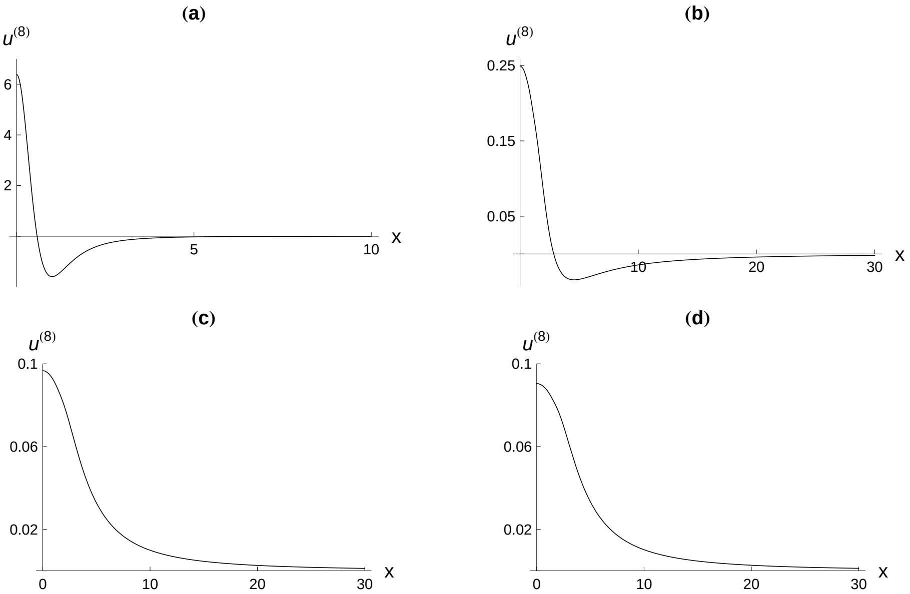

FIG 8. Solution of the 1D non-periodic problem (1) for $\beta = 0.1$ and (a) $t = 0$, (b) $t = 5$, (c) $t = 15$, (d) $t = 20$.

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