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ON p-COMPLEMENTS OF FINITE GROUPS

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Abstract. A subgroup H of a finite group G is called a p-complement for a prime p, if the order of H is not divided by p and the index $|G : H|$ is a power of p. We give examples of a finite group that possesses two nonisomorphic p-complements and of a finite group in which all p-complements are isomorphic but not conjugate in the automorphism group.

Keywords: finite group, p-complement.

Introduction

In the paper the term "group" always means "finite group", p is a prime number.

A subgroup H of a group G is called a *p-complement*, if there exists a Sylow p-subgroup P of G such that $H \cap P = 1$ and $G = HP$.

The existence of a p-complement yields important information about structure of the groups. For example, the well-known Chunikhin — Hall theorem [1–3], see also [4, Theorems 3.13 and 3.15], states that a finite group is soluble if and only if it possesses a p-complement for every prime number p. Also all p-complements are conjugate in this case [4, Theorem 3.14].

In an insoluble group, p-complements can be not conjugate. For example, in group $\mathrm{GL}_3(2)$ of order $168 = 2^3 \cdot 3 \cdot 7$ the stabilizer H of a line and the stabilizer K of a plane of the natural module are nonconjugate 7-complements (for more details see section 1). On the other hand, subgroups H and K are conjugate in the automorphism group of $\mathrm{GL}_3(2)$.

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V.D. Mazurov noted that the following questions are of some interest in connection with the Chunikhin — Hall theorem.

(1) Are any two p-complements of a finite group isomorphic?
(2) Are any two p-complements of a finite group conjugate in the automorphism group?

Note two corollaries of the classification of finite simple groups, giving affirmative answers to the questions (1) and (2) in some important cases.

Proposition 1. *Let G be a finite simple group and p be a prime number. If H and K are p-complements of G, then $K = H^{\varphi}$ for some $\varphi \in \mathrm{Aut}(G)$.*

Proposition 2. *In a finite group, 2-complements are conjugate.*

The validity of Proposition 1 follows from [5, Corollary 5.3] and Lemma 1 (see below). Proposition 2 is proved in [6, Corollary 4.5].

In the paper we show that there exist groups with nonisomorphic p-complements and with isomorphic p-complements which are not conjugate in the automorphism group. Thus the answers to the questions (1) and (2) are negative in general.

Nevertheless, Proposition 2 implies that there exist primes p such that in every finite group any two p-complements are conjugate (in particular, they are isomorphic and conjugate in the automorphism group). It would be interesting to find all prime numbers p such that in every finite group G

(1) any two p-complements are conjugate,
(2) any two p-complements are conjugate in the automorphism group,
(3) any two p-complements are isomorphic.

The following lemma is well-known and is a simple corollary of elementary facts from linear algebra.

Lemma 1. *Let V be a vector space of finite dimension over a finite field F and $G = \mathrm{GL}(V)$. Let U and W be subspaces of V. Put*

$$H = \{g \in G \mid U^g = U\} \text{ and } K = \{g \in G \mid W^g = W\}.$$

Then

(1) *the subgroups H and K are conjugate in G if and only if $\dim U = \dim W$;*
(2) *if $\dim U + \dim W = \dim V$, then the subgroups H and K are conjugate in $\mathrm{Aut}(G)$.*

In particular, if $\dim V > 2$, then stabilizers of line and hyperplane are not conjugate in G, but they are conjugate in $\mathrm{Aut}(G)$.

1. Example of nonisomorphic p-complements

Let V be a three-dimensional vector space over the field $\mathbb{F}_2$ and $G = \mathrm{GL}(V) \simeq \mathrm{GL}_3(2)$. Let U and W be subspaces of V of dimensions 1 and 2 respectively. Put

$$H = \{g \in G \mid U^g = U\} \text{ and } K = \{g \in G \mid W^g = W\}.$$

Denote by G^* the natural semidirect product of V and G. Put $H^* = \langle V, H\rangle$ and $K^* = \langle V, K\rangle$. Let us show that the subgroups H^* and K^* are nonisomorphic 7-complements of G^*.

We have $|G| = 168 = 2^3 \cdot 7 \cdot 3$. Obviously, matrices of elements of the groups H and K have the following form in bases of V containing bases of U and W, respectively:

$$\left(\begin{array}{cc|c} \multicolumn{2}{c|}{A} & \\ \hline * & * & 1 \end{array}\right) \text{ and } \left(\begin{array}{c|cc} 1 & & \\ \hline * & \multicolumn{2}{c}{A} \\ * & & \end{array}\right),$$

where $A \in \mathrm{GL}_2(2)$. Thus $|H| = |K| = 2^2 \cdot |\mathrm{GL}_2(2)| = 2^3 \cdot 3$ and $|G : H| = |G : K| = 7$. Hence $|H^*| = |K^*| = 2^6 \cdot 3$ and $|G^* : H^*| = |G^* : K^*| = 7$. Therefore, H^* and K^* are 7-complements of G^*. We are going to show that H^* and K^* are nonisomorphic.

Note that U contains a unique nonzero vector u. Hence $u^h = u$ for every $h \in H$ and we have $u \in Z(H^*)$. Thus the center of H^* is nontrivial and it suffices to show that $Z(K^*) = 1$.

Every element of K^* not lying in V acts nontrivially on the subgroup V of K^* and do not lie in $Z(K^*)$. If there is a nonzero vector v of V lying in the center of K^*, then one-dimensional subspace $\langle v \rangle$ is invariant under the action of the subgroup K of K^* and, therefore, K is a subgroup of H up to conjugacy in G. Since orders of H and K coincide, this implies the conjugacy of H and K contradicting Lemma 1. Thus $Z(K^*) = 1$ and the subgroups H^* and K^* of G^* are nonisomorphic.

2. Example of isomorphic p-complements which are not conjugate in the automorphism group

Let G, H and K be the same groups as in the previous section. Consider the group

$$X = \underbrace{G \times G \times \cdots \times G}_{7 \text{ times}}$$

and its automorphism $\varphi : (g_1, g_2, \ldots, g_7) \mapsto (g_2, \ldots, g_7, g_1)$. Denote by G^* the natural semidirect product of X and $\langle \varphi \rangle$. We have $G^* \simeq \mathrm{GL}_3(2) \wr \mathbb{Z}_7$. Let H^* and K^* be the subgroups of X, defined as follows:

$$H^* = \underbrace{H \times H \times \cdots \times H}_{7 \text{ times}},$$

$$K^* = K \times \underbrace{H \times \cdots \times H}_{6 \text{ times}}.$$

Since H and K are isomorphic by Lemma 1, H^* and K^* are also isomorphic. Also we have $|H^*| = |K^*| = (2^3 \cdot 3)^7$ and $|G^* : H^*| = |G^* : K^*| = 7^8$, i.e. H^* and K^* are 7-complements of G^*.

Let us find the number of subgroups conjugate to H^* in G^*. Since H is not normal in G and its index is a prime number, we have $H = N_G(H)$. Thus G contains $|G : N_G(H)| = 7$ subgroups conjugate to H. We conclude that X contains exactly 7^7 subgroups conjugate to H^*. Every such subgroup has the form

$$(H^*)^x = H^{g_1} \times H^{g_2} \times \cdots \times H^{g_7},$$

where $g_1, g_2, \ldots, g_7 \in G$ and $x = (g_1, g_2, \ldots, g_7) \in X$. We have

$$(H^*)^{x\varphi} = (H^{g_1} \times H^{g_2} \times \cdots \times H^{g_7})^{\varphi} = H^{g_2} \times \cdots \times H^{g_7} \times H^{g_1} = (H^*)^y,$$

where $y = x^{\varphi} = (g_2, \ldots, g_7, g_1) \in X$. Therefore, the conjugacy class of H^* in X is invariant under the action of φ, thus there are 7^7 subgroups of G^* conjugate to H^*.

Let us find the number of subgroups conjugate to K^* in G^*. In the same way as before we conclude that X contains 7^7 subgroups conjugate to K^* and every such subgroup has the form

$$(K^*)^x = K^{g_1} \times H^{g_2} \times \cdots \times H^{g_7},$$

where $g_1, g_2, \ldots, g_7 \in G$ and $x = (g_1, g_2, \ldots, g_7)$. But in this case the image of K^* under the action of φ equals the subgroup

$$(K^*)^\varphi = (K \times H \times \cdots \times H)^\varphi = H \times \cdots \times H \times K$$

that is not conjugate to K^* in X. Thus G^* contains more than 7^7 subgroups conjugate to K^* (it is not difficult to see that there are 7^8 such subgroups).

So we conclude that the subgroups H^* and K^* are not conjugate in the automorphism group of G^*.

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