S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 10, стр. 454–463 (2013) УДК 514.132
MSC 51M10

ON SOME ASPECTS OF A HYPERBOLIC TANGENTIAL QUADRILATERAL

M.P. LIMONOV

Abstract. Previously Martin Josefsson obtained certain properties of a tangential quadrilateral in terms of distances from the four vertices to the points of tangency. We consider the hyperbolic analogue of these properties. For the sake of clarity, most assertions are followed by the Euclidean case.

Keywords: hyperbolic tangential quadrilateral, tangent lengths.

1. Introduction

The properties of a tangential quadrilateral may be described in terms of distances from the four vertices to the points of tangency. In his paper [1], Martin Josefsson has obtained some of them. He has considered tangential quadrilaterals in Euclidean geometry. But are there any analogues of these properties in hyperbolic geometry? An answer to the question is given in the present article. For illustration of results the Poincaré disc model is used. The reader may find information about hyperbolic geometry in [2].

Definition 1. A *hyperbolic tangential quadrilateral* is a hyperbolic quadrilateral with an incircle, i.e., a circle tangent to its four sides. We will call the distances from the four vertices to the points of tangency the *tangent lengths*, and denote these by e, f, g and h, as indicated in Figure 1.

For equations to be of a compact form, we make use of the following functions

$$\begin{aligned} P(x_1, x_2, \ldots, x_n) &= \tanh x_1 \tanh x_2 \ldots \tanh x_n, \\ S(x_1, x_2, \ldots, x_n) &= \tanh x_1 + \tanh x_2 + \ldots + \tanh x_n. \end{aligned} \tag{1}$$

Limonov, M.P., On some aspects of a hyperbolic tangential quadrilateral.

The work was supported by the Russian Foundation for Basic Research (grant 12-01-00210).
Received December, 22, 2012, published June, 3, 2013.

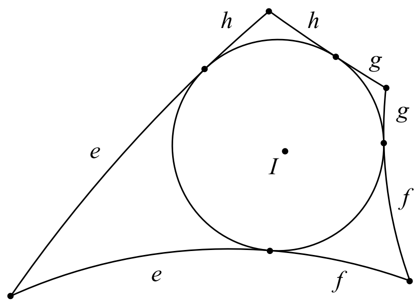

FIGURE 1. The tangent lengths

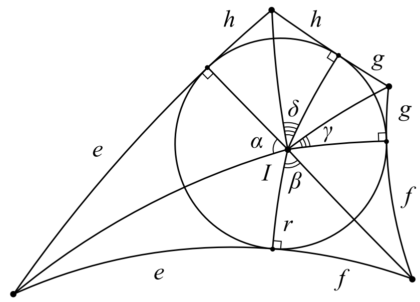

FIGURE 2

2. The inradius of the hyperbolic tangential quadrilateral

The following theorem gives the formula for the squared hyperbolic sine of the inradius of a hyperbolic tangential quadrilateral in terms of the tangent lengths.

Theorem 1. *The radius r of the incircle in a hyperbolic tangential quadrilateral in terms of the tangent lengths is given by the following formula*

$$\sinh^2 r = \frac{P(e,f,g)+P(f,g,h)+P(g,h,e)+P(h,e,f)}{S(e,f,g,h)}. \tag{2}$$

Remark 1. In the Euclidean case (see [1], p. 119) we have

$$r^2 = \frac{efg+fgh+ghe+hef}{e+f+g+h}.$$

Proof. If I is the incenter and angles $\alpha,\beta,\gamma,\delta$ are defined as in Figure 2, by Theorem 7.11.2 (see [2], p. 147) we have

$$\tan\alpha = \frac{\tanh e}{\sinh r}, \quad \tan\beta = \frac{\tanh f}{\sinh r}, \quad \tan\gamma = \frac{\tanh g}{\sinh r}, \quad \tan\delta = \frac{\tanh h}{\sinh r}. \tag{3}$$

We know that $\alpha+\beta+\gamma+\delta=\pi$, so $\tan(\alpha+\beta+\gamma+\delta)=0$. The formula of angle addition for the tangent is given by

$$T(\alpha+\beta+\gamma+\delta) = \frac{T\alpha+T\beta+T\gamma+T\delta-T\alpha T\beta T\delta-T\alpha T\beta T\gamma-T\alpha T\gamma T\delta-T\beta T\gamma T\delta}{1-T\alpha T\beta-T\alpha T\delta-T\beta T\delta-T\alpha T\gamma-T\beta T\gamma-T\gamma T\delta+T\alpha T\beta T\gamma T\delta},$$

where for the formula to be of a compact form T stands for the tangent function. Because of $\tan(\alpha+\beta+\gamma+\delta)=0$ we get

$$T\alpha+T\beta+T\gamma+T\delta-T\alpha\, T\beta\, T\delta-T\alpha\, T\beta\, T\gamma-T\alpha\, T\gamma\, T\delta-T\beta\, T\gamma\, T\delta=0.$$

Substituting (3) into the latter equation gives

$$\frac{S(e,f,g,h)}{\sinh r} = \frac{P(e,f,g)+P(e,f,h)+P(e,g,h)+P(f,g,h)}{\sinh^3 r}.$$

Solving it with respect to $\sinh^2 r$, we get (2). $\square$

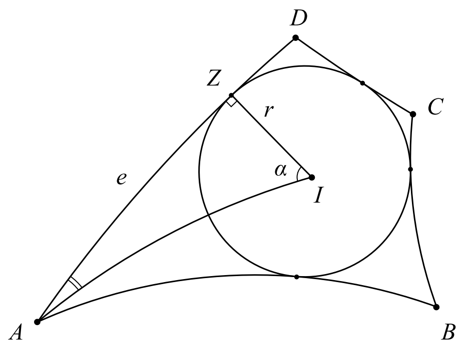

Figure 3

3. The angles of the hyperbolic tangential quadrilateral

By using the previous theorem we may derive the formulas for the angles of the hyperbolic tangential quadrilateral in terms of the tangent lengths.

Theorem 2. *If e, f, g and h are the tangent lengths in a hyperbolic tangential quadrilateral $ABCD$, then its angles satisfy*

$$\begin{aligned}
\sin^2\frac{A}{2} &= \frac{P(e,f,g)+P(f,g,h)+P(g,h,e)+P(h,e,f)}{S(e,f)\,S(e,g)\,S(e,h)\ \cosh^2 e},\\
\sin^2\frac{B}{2} &= \frac{P(e,f,g)+P(f,g,h)+P(g,h,e)+P(h,e,f)}{S(f,e)\,S(f,g)\,S(f,h)\ \cosh^2 f},\\
\sin^2\frac{C}{2} &= \frac{P(e,f,g)+P(f,g,h)+P(g,h,e)+P(h,e,f)}{S(g,e)\,S(g,f)\,S(g,h)\ \cosh^2 g},\\
\sin^2\frac{D}{2} &= \frac{P(e,f,g)+P(f,g,h)+P(g,h,e)+P(h,e,f)}{S(h,e)\,S(h,f)\,S(h,g)\ \cosh^2 h}.
\end{aligned}\tag{4}$$

Remark 2. In the Euclidean case (see [1], p. 126) we have

$$\begin{aligned}
\sin^2\frac{A}{2} &= \frac{efg+fgh+ghe+hef}{(e+f)(e+g)(e+h)},\\
\sin^2\frac{B}{2} &= \frac{efg+fgh+ghe+hef}{(f+e)(f+g)(f+h)},\\
\sin^2\frac{C}{2} &= \frac{efg+fgh+ghe+hef}{(g+e)(g+f)(g+h)},\\
\sin^2\frac{D}{2} &= \frac{efg+fgh+ghe+hef}{(h+e)(h+f)(h+g)}.
\end{aligned}$$

Proof. If I is the incenter and angle α is defined as in Figure 3, by Theorem 7.11.3 (see [2], p. 147) we have

$$\sin^2\frac{A}{2} = \Big(\frac{\cos\alpha}{\cosh e}\Big)^2 = \frac{1}{\big(1+\tan^2\alpha\big)\cosh^2 e}.$$

Further, by (3) we get

$$\sin^2\frac{A}{2}=\frac{1}{\left(1+\dfrac{\tanh^2 e}{\sinh^2 r}\right)\cosh^2 e}.$$

Substituting (2) into the latter equation gives

$$\begin{aligned}\sin^2\frac{A}{2}=&\frac{1}{\left(1+\dfrac{S(e,f,g,h)\ \tanh^2 e}{P(e,f,g)+P(e,f,h)+P(e,g,h)+P(f,g,h)}\right)\cosh^2 e}\\ =&\frac{P(e,f,g)+P(e,f,h)+P(e,g,h)+P(f,g,h)}{\big(P(e,f,g)+P(e,f,h)+P(e,g,h)+P(f,g,h)+S(e,f,g,h)\tanh^2 e\big)\cosh^2 e}.\end{aligned}$$

Making use of the reduction

$$P(e,f,g)+P(e,f,h)+P(e,g,h)+P(f,g,h)+S(e,f,g,h)\ \tanh^2 e=S(e,f)\,S(e,g)\,S(e,h),$$

finally, we get the first equation in (4). The formulas for the remaining angles are derived the same way, or we get them at once using symmetry. □

4. The diagonals

Now we are able to establish the formula for the diagonals of the hyperbolic tangential quadrilateral in terms of the tangent lengths.

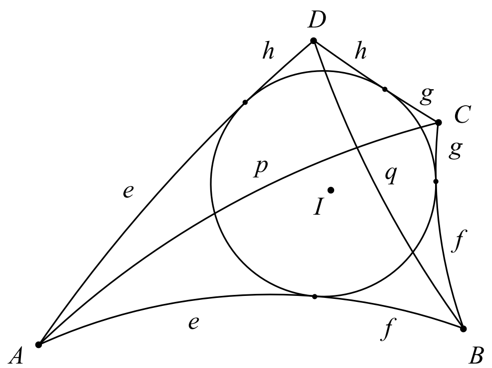

Figure 4. The diagonals p and q

Theorem 3. *If e, f, g and h are the tangent lengths in a hyperbolic tangential quadrilateral $ABCD$, then its diagonals $AC = p$ and $BD = q$ satisfy*

$$\begin{aligned}\sinh^2\frac{p}{2}&=\frac{1}{\sinh(f+h)}\left(\sinh\frac{e+g+2f}{2}\sinh h+\sinh\frac{e+g+2h}{2}\sinh f\right)\sinh\frac{e+g}{2},\\ \sinh^2\frac{q}{2}&=\frac{1}{\sinh(e+g)}\left(\sinh\frac{f+h+2g}{2}\sinh e+\sinh\frac{f+h+2e}{2}\sinh g\right)\sinh\frac{f+h}{2}.\end{aligned}\tag{5}$$

Remark 3. In the Euclidean case (see [1], p. 120) we have

$$\begin{aligned}p^2&=\frac{e+g}{f+h}((e+g)(f+h)+4fh),\\ q^2&=\frac{f+h}{e+g}((f+h)(e+g)+4eg).\end{aligned}$$

Proof. Let us consider triangle ACD (see Figure 4). By The Cosine Rule I (see [2], p. 148), we have

$$\begin{aligned}\cosh p &= \cosh(e+h)\cosh(h+g) - \sinh(e+h)\sinh(h+g)\cos D\\ &= \cosh(e+h)\cosh(h+g) - \sinh(e+h)\sinh(h+g)\left(1 - 2\sin^2\frac{D}{2}\right).\end{aligned}$$

Since

$$\sinh^2\frac{p}{2} = \frac{\cosh p - 1}{2}$$

by the last equation in (4) we obtain

$$\begin{aligned}\sinh^2\frac{p}{2} =& \frac{1}{2}\cosh(e+h)\cosh(h+g) - \frac{1}{2}\sinh(e+h)\sinh(h+g)\\ &\times\left(1 - 2\frac{P(e,f,g)+P(f,g,h)+P(g,h,e)+P(h,e,f)}{S(h,e)\,S(h,f)\,S(h,g)\,\cosh^2 h}\right) - \frac{1}{2}.\end{aligned}$$

We use the identities

$$\cosh(e+h)\cosh(h+g) - \sinh(e+h)\sinh(h+g) = \cosh(e-g)$$

and $\cosh(e-g) - 1 = 2\sinh^2\frac{e-g}{2}$ to get

$$\begin{aligned}\sinh^2\frac{p}{2} =& \sinh^2\frac{e-g}{2} + \sinh(e+h)\sinh(h+g)\\ &\times\frac{P(e,f,g)+P(f,g,h)+P(g,h,e)+P(h,e,f)}{S(h,e)\,S(h,f)\,S(h,g)\,\cosh^2 h}.\end{aligned}$$

Now by making use of

$$\begin{aligned}P(e,f,g)+P(f,g,h)+P(g,h,e)+P(h,e,f) =& \frac{1}{4\cosh e\cosh f\cosh g\cosh h}\\ &\times(\sinh(e-f-g-h) - \sinh(e+f+g-h) - \sinh(e+f-g+h)\\ &- \sinh(e-f+g+h) + 2\sinh(e+f+g+h)),\end{aligned}$$

$$S(h,e)\,S(h,f)\,S(h,g)\,\cosh^2 h = \frac{\sinh(e+h)\sinh(f+h)\sinh(g+h)}{4\cosh e\cosh f\cosh g\cosh h}$$

and

$$\sinh^2\frac{e-g}{2}\sinh(f+h) = \frac{1}{4}(\sinh(e+f-g+h) - \sinh(e-f-g-h) - 2\sinh(f+h))$$

we have

$$\begin{aligned}\sinh^2\frac{p}{2} =& \frac{1}{4\sinh(f+h)}(2\sinh(e+f+g+h)\\ &- \sinh(e+f+g-h) - 2\sinh(f+h) - \sinh(e-f+g+h)).\end{aligned}$$

By taking into account

$$\begin{aligned}4\sinh f\sinh\frac{e+g+2h}{2}\sinh\frac{e+g}{2} =& \sinh(e+f+g+h) - \sinh(f+h)\\ &- \sinh(e-f+g+h) - \sinh(f-h)\end{aligned}$$

and

$$\begin{aligned}4\sinh h\sinh\frac{e+2f+g}{2}\sinh\frac{e+g}{2} =& \sinh(e+f+g+h) - \sinh(f+h)\\ &- \sinh(e+f+g-h) + \sinh(f-h)\end{aligned}$$

we get the first formula in the theorem. The second one is established by considering triangle ABD. □

5. The area of the hyperbolic tangential quadrilateral

The next theorem gives a formula for the sine squared of the area divided by four of a hyperbolic tangential quadrilateral in terms of the tangent lengths.

Theorem 4. *If e, f, g and h are the tangent lengths in a tangential quadrilateral, then its area $\mathcal{A}$ is given by*

$$\sin^2\frac{\mathcal{A}}{4}=\frac{1}{8}\sinh\frac{e+f+g+h}{2}\left(P(e,f,g)+P(f,g,h)+P(g,h,e)+P(h,e,f)\right)J(e,f,g,h),$$

where

$$J(e,f,g,h)=\frac{\cosh e\,\cosh f\,\cosh g\,\cosh h}{\cosh\frac{e+f}{2}\,\cosh\frac{e+g}{2}\,\cosh\frac{f+g}{2}\,\cosh\frac{e+h}{2}\,\cosh\frac{f+h}{2}\,\cosh\frac{g+h}{2}}$$

is close to 1 *for sufficiently small tangential quadrilaterals.*

Remark 4. In the Euclidean case (see [1], p. 119) we have

$$\left(\frac{\mathcal{A}}{4}\right)^2=\frac{1}{16}(e+f+g+h)(efg+fgh+ghe+hef).$$

Proof. Let us consider the following formula which is a hyperbolic analogue of the spherical formula established in [3], p. 46 (see [4], p. 16 for an independent proof).

$$\sin^2\frac{\mathcal{A}}{4}=\frac{\left(\sinh\frac{p}{2}\sinh\frac{q}{2}\right)^2-\left(\cosh\frac{a}{2}\cosh\frac{c}{2}-\cosh\frac{b}{2}\cosh\frac{d}{2}\right)^2}{4\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}\cosh\frac{d}{2}},\tag{6}$$

It expresses the area $\mathcal{A}$ of a hyperbolic quadrilateral in terms of diagonals p,q and sides a,b,c,d. By taking into account

$$a=e+f,\quad b=f+g,\quad c=g+h,\quad d=h+e\tag{7}$$

we rewrite (6) as

$$\sin^2\frac{\mathcal{A}}{4}=\frac{\sinh^2\frac{p}{2}\sinh^2\frac{q}{2}-\left(\cosh\frac{e+f}{2}\cosh\frac{g+h}{2}-\cosh\frac{f+g}{2}\cosh\frac{h+e}{2}\right)^2}{4\cosh\frac{e+f}{2}\cosh\frac{f+g}{2}\cosh\frac{g+h}{2}\cosh\frac{h+e}{2}}.$$

Substituting (5) into the latter equation after straightforward calculations by Mathematica we establish the theorem. □

6. The tangency chords and related questions

Definition 2. If the incircle in a hyperbolic tangential quadrilateral $ABCD$ is tangent to the sides AB, BC, CD and DA at W, X, Y and Z respectively, then the segments WY and XZ are called the *tangency chords*.

Theorem 5. *The lengths k and l of the tangency chords WY and XZ in a hyperbolic tangential quadrilateral are respectively*

$$\sinh^2\frac{k}{2}=\frac{\left(P(e,f,g)+P(f,g,h)+P(g,h,e)+P(h,e,f)\right)^2}{S(e,f)\,S(g,h)\,S(e,g)\,S(f,h)},$$

$$\sinh^2\frac{l}{2}=\frac{\left(P(e,f,g)+P(f,g,h)+P(g,h,e)+P(h,e,f)\right)^2}{S(f,g)\,S(h,e)\,S(e,g)\,S(f,h)}.$$

Remark 5. In the Euclidean case (see [1], p. 120) we have

$$\left(\frac{k}{2}\right)^2 = \frac{(efg+fgh+ghe+hef)^2}{(f+h)(e+g)(e+f)(g+h)},$$
$$\left(\frac{l}{2}\right)^2 = \frac{(efg+fgh+ghe+hef)^2}{(f+h)(e+g)(f+g)(h+e)}.$$

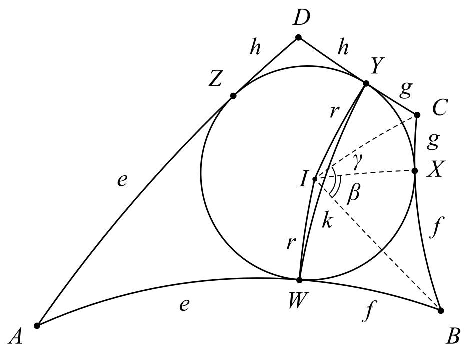

FIGURE 5. Tangent chord k

Proof. If I is the incenter and angles β and γ are defined as in Figure 5, by The Cosine Rule I (see [2], p. 148) in triangle WYI we get

$$\begin{aligned}\cosh k =& \cosh^2 r - \sinh^2 r \cos(2\beta + 2\gamma)\\ =&1 + \sinh^2 r - \sinh^2 r \frac{1-\tan^2(\alpha+\beta)}{1+\tan^2(\alpha+\beta)}\\ =&1 + \sinh^2 r - \sinh^2 r \frac{1-\left(\frac{\tan\beta+\tan\gamma}{1-\tan\beta\tan\gamma}\right)^2}{1+\left(\frac{\tan\beta+\tan\gamma}{1-\tan\beta\tan\gamma}\right)^2}.\end{aligned}$$

Now we can make use of (2) and (3). That establishes the first equation of the theorem. The formula for l can either be derived the same way, or we can use the symmetry in the hyperbolic tangential quadrilateral and need only to make the change $f \leftrightarrow h$ in the formula for k. □

Corollary 1. *In a hyperbolic tangential quadrilateral with sides a, b, c and d, the quotient of the tangency chords satisfy*

$$\left(\frac{\sinh\frac{k}{2}}{\sinh\frac{l}{2}}\right)^2 = \frac{\sinh b\,\sinh d}{\sinh a\,\sinh c}.$$

Remark 6. In the Euclidean case (see [1], p. 122) we have

$$\left(\frac{k}{l}\right)^2 = \frac{b\,d}{a\,c}.$$

Proof. Taking the quotient of $\sinh^2 \frac{k}{2}$ and $\sinh^2 \frac{l}{2}$ from Theorem 5, after simplification we get

$$\left(\frac{\sinh \frac{k}{2}}{\sinh \frac{l}{2}}\right)^2 = \frac{\sinh(f+g)\,\sinh(e+h)}{\sinh(e+f)\,\sinh(g+h)}.$$

Since $f+g=b,\ e+h=d,\ e+f=a,\ g+h=c$ the result follows. □

Definition 3. A kite is a quadrilateral with two pairs of equal adjacent sides.

Corollary 2. *The tangency chords in a hyperbolic tangential quadrilateral are of equal length if and only if it is a kite.*

Proof. If the hyperbolic quadrilateral is a kite it directly follows that the tangency chords are of equal length because of the mirror symmetry in the longest diagonal (see Figure 6).

Conversely, if the tangency chords are of equal length in a hyperbolic tangential quadrilateral, from Corollary 1 we get $\sinh a \, \sinh c = \sinh b \, \sinh d$. Substituting a, b, c and d with (7), after simplification we get

$$\sinh(e-g)\sinh(f-h) = 0.$$

Either $\sinh(e-g) = 0$ or $\sinh(f-h) = 0$. In the first case, $e = g$ and therefore $a = b,\ c = d$. In the second case, $f = h$ and therefore $a = d,\ b = c$. In both cases two pairs of adjacent sides are equal, so the quadrilateral is a kite. □

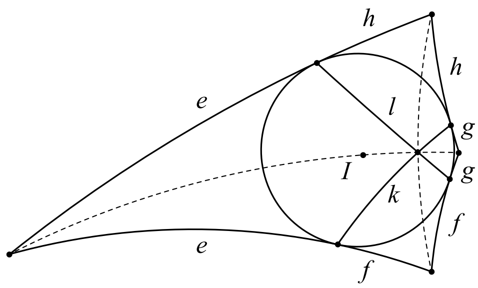

FIGURE 6. The tangency chords in a kite

7. The area of a bicentric quadrilateral

Definition 4. A *hyperbolic bicentric quadrilateral* is a hyperbolic tangential quadrilateral that has a circumcircle.

Theorem 6. *A hyperbolic tangential quadrilateral has a circumcircle if and only if its tangent lengths e, f, g and h satisfy the equation*

$$\tanh\frac{e}{2}\tanh\frac{g}{2} = \tanh\frac{f}{2}\tanh\frac{h}{2}. \tag{8}$$

Remark 7. In the Euclidean case (see [6]) we have

$$e\,g = f\,h.$$

Remark 8. There is a different statement which is equivalent to Theorem 6. Namely, let A, B, C, D be the angles of a hyperbolic tangential quadrilateral. Then it has a circumcircle if and only if the equation $A-B+C-D=0$ takes place (for more information see [7]).

Proof. According to an analogue of Ptolemy,??s theorem in the hyperbolic geometry (see [8], p. 817), a convex hyperbolic quadrilateral with side lengths a, b, c, d and diagonal lengths p and q has a circumcircle if and only if the equation

$$\sinh\frac{p}{2}\sinh\frac{q}{2}=\sinh\frac{a}{2}\sinh\frac{c}{2}+\sinh\frac{d}{2}\sinh\frac{b}{2} \tag{9}$$

holds. We are dealing with a hyperbolic tangential quadrilateral, so, making use of (7) and (5), after simplification we may rewrite (9) as

$$\frac{\cosh\frac{e+f+g+h}{2}\left(\cosh\frac{e}{2}\cosh\frac{f}{2}\cosh\frac{g}{2}\cosh\frac{h}{2}\right)^2}{\frac{1}{4}\cosh\frac{e+g}{2}\cosh\frac{f+h}{2}}\left(P\left(\frac{e}{2},\frac{g}{2}\right)-P\left(\frac{f}{2},\frac{h}{2}\right)\right)^2=0, \tag{10}$$

where P is the function defined in (1).

Let (8) hold for a hyperbolic tangential quadrilateral. Then, by the analogue of Ptolemy,??s theorem, a hyperbolic tangential quadrilateral has a circumcircle.

Conversely, if a hyperbolic tangential quadrilateral has a circumcircle, since the fraction on the left side of the equation does not vanish, by making use of (10) we immediately finish the proof. □

Theorem 7. *The area $\mathcal{A}$ of a hyperbolic bicentric quadrilateral with tangent lengths e, f, g and h is given by the formula*

$$\sin^2\frac{\mathcal{A}}{4}=\tanh\frac{e+f}{2}\tanh\frac{f+g}{2}\tanh\frac{g+h}{2}\tanh\frac{h+e}{2}.$$

Remark 9. In the Euclidean case (see [1], p. 127) we have

$$\mathcal{A}^2=(e+f)(f+g)(g+h)(h+e).$$

Proof. Let $ABCD$ be a hyperbolic tangential quadrilateral. Then, the following formula (see [4], p. 18)

$$\sin^2\frac{\mathcal{A}}{4}=P\left(\frac{a}{2},\frac{b}{2},\frac{c}{2},\frac{d}{2}\right)-P\left(\frac{a}{2},\frac{b}{2},\frac{c}{2},\frac{d}{2}\right)\sin^2\frac{A-B+C-D}{4}, \tag{11}$$

where function P is defined in (1), expresses the area of a hyperbolic tangential quadrilateral in terms of side lengths a, b, c and d and angles A, B, C and D. Further, the formula (see [4], p. 17)

$$\sin^2\frac{A-B+C-D}{4}=\frac{\left(\sinh\frac{a}{2}\sinh\frac{c}{2}+\sinh\frac{b}{2}\sinh\frac{d}{2}\right)^2-\left(\sinh\frac{p}{2}\sinh\frac{q}{2}\right)^2}{4\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}\sinh\frac{d}{2}}$$

holds for diagonals p and q of a hyperbolic tangential quadrilateral. Substituting the latter equation into (11) and making use of (5) and (7), after calculation by the

Mathematica we get

$$\begin{aligned}\sin^2\frac{\mathcal{A}}{4} =&\tanh\frac{e+f}{2}\tanh\frac{f+g}{2}\tanh\frac{g+h}{2}\tanh\frac{h+e}{2}\\ &-\left(\tanh\frac{e}{2}\tanh\frac{g}{2}-\tanh\frac{f}{2}\tanh\frac{h}{2}\right)^2\\ &\times\frac{\cosh\frac{e+f+g+h}{2}\cosh^2\frac{e}{2}\cosh^2\frac{f}{2}\cosh^2\frac{g}{2}\cosh^2\frac{h}{2}}{\cosh\frac{e+f}{2}\cosh\frac{e+g}{2}\cosh\frac{f+g}{2}\cosh\frac{e+h}{2}\cosh\frac{f+h}{2}\cosh\frac{g+h}{2}}.\end{aligned}\tag{12}$$

We have

$$\sin^2\frac{\mathcal{A}}{4} = \tanh\frac{e+f}{2}\tanh\frac{f+g}{2}\tanh\frac{g+h}{2}\tanh\frac{h+e}{2}$$

if and only if

$$\tanh\frac{e}{2}\tanh\frac{g}{2} = \tanh\frac{f}{2}\tanh\frac{h}{2},$$

which according to Theorem 6 is a characterization for a hyperbolic tangential quadrilateral to be cyclic, i.e., bicentric. □

Remark 10. The equation (12) is a hyperbolic analogue of the Euclidean Bretschneider's formula. It expresses the area of a hyperbolic tangential quadrilateral in terms of the tangent lengths.

8. Acknowledgments

The author is grateful to his mentor, Professor A.D. Mednykh, for his suggestions and editing of the article. Also, the author would like to thank the referee for helpful comments and suggestions.

References

[1] M. Josefsson, *Calculations Concerning the Tangent Lengths and Tangency Chords of a Tangential Quadrilateral*, Forum Geometricorum, **10** (2010), 119–130. MR2726487

[2] A.F. Beardon, *The geometry of discrete groups*, New York, Heidelberg, Berlin: Springer-Verlag, 1983. MR0698777

[3] W.J. M'Clelland, *A Treatise on Spherical Trigonometry with application to Spherical Geometry and Numerous Examples. Part II*, London: Macmillian and Co. 1886.

[4] G.A. Bajgonakova, A.D. Mednykh, *On the Bretschneider's formula for a hyperbolic tangential quadrilateral*, Matematicheskie Zametki YAGU, **19**:2 (2012), 12–20. (Russian). Zbl pre06135472

[5] A.D. Mednykh, *Brahmagupta formula for cyclic quadrilaterals in the hyperbolic plane*, Siberian Electronic Mathematical Reports, **9** (2012), 247–255. MR2954694

[6] M. Hajja, *A condition for a circumscriptible quadrilateral to be cyclic*, Forum Geometricorum, **8** (2008), 103–106. MR2429397

[7] K. Bezdek, *Ein elementarer Beweis für die isoperimetrische Ungleichung in der euklidischen und hyperbolischen Ebene*, Ann. Univ. Sci. Budap. Rolando Eötvös, Sect. Math., **27** (1984), 107–112. MR0823098

[8] J.E. Valentine, *An analogue of Ptolemy's theorem and its converse in hyperbolic geometry*, Pacific J. Math., **34** (1970), 817–825. MR0270261

Maxim Petrovich Limonov
Sobolev Institute of Mathematics,
pr. Koptyuga, 4,
630090, Novosibirsk, Russia
E-mail address: volsterm@gmail.com