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ON BELNAPIAN MODAL ALGEBRAS: REPRESENTATIONS, HOMOMORPHISMS, CONGRUENCES, AND SO ON

S. O. SPERANSKI

Abstract. We obtain a bunch of principal results on Belnapian modal algebras (henceforth called BK-lattices) — these results may serve as a semantical basis for further investigation of the lattice of extensions of Belnapian modal logic (denoted by BK here).

Keywords: Belnapian modal logic, many-valued modal logic, strong negation, Belnapian modal algebra, twist-structure, modal algebra.

1. Introduction

The system BK, the Belnapian version of the least normal modal logic K, and several important extensions of it were proposed in [18]. From the viewpoint of Kripke semantics, the fundamental difference between BK and K is the following: while two-valued (classical) valuations are associated with each possible world in Kripke models of K, four-valued valuations are employed in case of BK. For the latter, 'four-valued' means that we are working in the well-known four-valued logic suggested by Belnap and Dunn [3, 6]. Actually, BK is a conservative enrichment of K by means of the strong negation and can be viewed as the basic four-valued

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modal logic; in effect, there is an interesting analogy with considering constructive Nelson's logics N3 [11, 28] and N4 [1] (or rather, its modification $N4^{\perp}$, with the additional constant $\perp$, suggested in [14]) and as three- and four-valued versions of intuitionistic logic Int as well—cf. [18, Sections 2–3].[1]

As was also shown in [18, Section 7.1], a number of constructive non-modal logics with strong negation are faithfully embedded into BS4, the Belnapian version of the normal modal logic S4, and its explosive extension B3S4. Namely, this is true of N3 and $N4^{\perp}$ mentioned above (a suitable translation is, in fact, analogous to the famous Gödel–McKinsey–Tarski translation of Int into S4 [8, 10]), and of the connexive logic C [29] (see [30] for a modal version).[2]

An algebraic semantics for BK (and its extensions) may be described in terms of the so-called twist-structures. Initially, twist-structures over Heyting algebras appeared in [7, 27] as a representation of N-lattices [19] providing an algebraic semantics for the explosive constructive logic, N3. Furthermore, as was proved in [12], such structures can be used to characterize the paraconsistent constructive logic N4, and the abstract closure of the class of these structures forms a variety, $\mathcal{V}_{\mathsf{N4}}$ (the elements of which are called N4-lattices). On the other hand, twist-structures over modal algebras and those over topoboolean algebras characterize BK and BS4, respectively [18, 16]. It was established in [16, Sections 4–5] that the abstract closure of the collection of all twist-structures over modal algebras turns out to be a variety, $\mathcal{V}_{\mathsf{BK}}$ (consisting of BK-*lattices*, accordingly), and there exists a dual isomorphism between the lattice of subvarieties of $\mathcal{V}_{\mathsf{BK}}$ and the lattice of extensions of BK. Hence the suggested semantics is adequate (from the universal algebra perspective) for investigating BK-extensions.

With this work, we begin the systematic study of the connections between BK-lattices and modal algebras. Remark that the substantial part of the methods we employ can be viewed as a variant of the machinery used in [15]. Now, let's briefly portray the main contributions of the present paper.

As was shown earlier in [16, Section 6], there are natural invariants that uniquely determine each twist-structure over a given modal algebra. This is in parallel with the work [13] on invariants of N4-lattices, which, in turn, generalizes [23] that deals with the explosive twist-structures over Heyting algebras only. Consequently, to complete the task, in Section 3 we prove (exploiting canonical epimorphisms and embeddings from [16]) that the above result easily extends to all BK-lattices — in effect, the modification needed is almost straightforward.

Next, Section 4 deals with homomorphisms of BK-lattices. In particular, we define the so-called special ($\Box$-)filters of the first kind on BK-lattices (sffk's, for

[1] In the monograph [15], many results on the algebraic semantics for N4 (including those cited below) were extended to $N4^{\perp}$, which conservatively enriches Int by means of the strong negation.

[2] Notice, different modal enrichments of four-valued constructive logics augmented by strong negation were previously suggested, e. g., in [17, 24].

short) and prove that sffk's are exactly kernels of homomorphisms of BK-lattices.[3] Moreover, we construct an isomorphism between the lattice of sffk's on a BK-lattice $\mathfrak{A}$ and the lattice of $\Box$-filters on the underlying modal algebra $\mathfrak{A}_{\bowtie}$ (the notation will be explained below), and so the corresponding congruence lattices eventually turn out to be isomorphic as well. The latter fact allows us to obtain the following criterion: the subdirect irreducibility of a BK-lattice $\mathfrak{A}$ is equivalent the subdirect irreducibility of the underlying modal algebra $\mathfrak{A}_{\bowtie}$. A similar result for N4-lattices (and underlying Heyting algebras) was provided in [13], and even earlier it was shown for N-lattices both in [22] and (independently) in [5], where, however, the proof was based on Priestley duality. Also, we establish, inter alia, the congruence-distributivity of BK-lattices and the distributivity of the lattice of BK-extensions, as well as some criteria for the existence of homomorphisms and monomorphisms of BK-lattices (which make use of their underlying algebras).

The results of the present paper (together with those of [18, 16]) constitute the semantical basis for investigating the lattice of BK-extensions. In particular, they can be used for proving the theorems previously announced in [25, 26]. For ease of future reference, I aimed at organising the results into Propositions, Lemmas, etc. each of which possesses a relatively short proof but still has an interest in its own right, and also at giving a detailed treatment that will be accessible even to those who are not experienced in universal algebra. And for the readers already familiar with algebraic logic, it is worth mentioning that the algebraizability of BK (in the sense of [4]) was established by [16, Theorem 5.4], and some of the results may be obtained using this fact in a rather standard way — that applies to Proposition 4.2, Lemma 4.3, Remark on p. 11 and Corollary 4.5. But we will provide direct proofs for all these results, without exploiting the algebraizability of BK.

2. Preliminaries

Recall that an algebra $\mathfrak{M} = \langle M; \vee, \wedge, \neg, \Box\rangle$ is said to be a *modal algebra* iff $\langle M; \vee, \wedge, \neg\rangle$ is a Boolean algebra and, for all m_1 and m_2 in M, the operation $\Box$ satisfies the following properties: a) $\Box(m_1 \wedge m_2) = \Box m_1 \wedge \Box m_2$; b) $\Box 1 = 1$, where 1 is the greatest element of $\mathfrak{M}$ (w. r. t. the usual lattice ordering $\leqslant$ on $\mathfrak{M}$, i. e., $m_1 \leqslant m_2 \iff m_1 \wedge m_2 = m_1$). Obviously, every such $\mathfrak{M}$ also has the least element 0 (w. r. t. $\leqslant$). Define an additional unary operation $\Diamond$ by

$$\Diamond m \;:=\; \neg\Box\neg m\,, \quad \text{for each } m \in M\,.$$

Next, consider two languages

$$\mathcal{L}^m \;:=\; \{\vee, \wedge, \rightarrow, \sim, \Box, \Diamond, \bot\} \quad \text{and} \quad \mathcal{L}^m_{\neg} \;:=\; \{\vee, \wedge, \neg, \sim, \Box, \bot, \top\}\,.$$

[3]Remark: these special filters are introduced in a way similar to sffk's on N-lattices [20] and also to sffk's on N4-lattices [13].

Initially, twist-structures over modal algebras were proposed in the language $\mathcal{L}^m$, just as the logic BK itself (see [18, Sections 4–5]).

For a modal algebra $\mathfrak{M} = \langle M; \vee, \wedge, \neg, \Box \rangle$, the *full* $\mathcal{L}^m$*-twist-structure over* $\mathfrak{M}$ is the algebra $\mathfrak{M}^{\bowtie} = \langle M \times M; \vee, \wedge, \rightarrow, \sim, \Box, \Diamond, \bot \rangle$ augmented by operations

$$(m_1, m_2) \vee (m_3, m_3) \ := \ (m_1 \vee m_3, m_2 \wedge m_4)\,,$$
$$(m_1, m_2) \wedge (m_3, m_4) \ := \ (m_1 \wedge m_3, m_2 \vee m_4)\,,$$
$$(m_1, m_2) \rightarrow (m_3, m_4) \ := \ (\neg m_1 \vee m_3, m_1 \wedge m_4)\,, \quad \sim (m_1, m_2) \ := \ (m_2, m_1)\,,$$
$$\Box\,(m_1, m_2) \ := \ (\Box m_1, \Diamond m_2)\,, \quad \Diamond\,(m_1, m_2) \ := \ (\Diamond m_1, \Box m_2)\,, \quad \bot \ := \ (0, 1)\,.$$

An ($\mathcal{L}^m$-)algebra $\mathfrak{A}$ is a *twist-structure over* $\mathfrak{M}$ iff it is a subalgebra of $\mathfrak{M}^{\bowtie}$ and

$$\pi_1\,(A) := \{m_1 \mid \exists m_2\ (m_1, m_2) \in A\} \ = \ M\,.$$

Remark: due to the definition of $\sim$, the latter requirement is equivalent to

$$\pi_2\,(A) := \{m_2 \mid \exists m_1\ (m_1, m_2) \in A\} \ = \ M\,,$$

so there is no asymmetry here.[4] Denote by $S^{\bowtie}(\mathfrak{M})$ the collection of all twist-structures over $\mathfrak{M}$. Notice, while having the operation $\Diamond$ is convenient when providing the axiomatization of BK, it turns out to be, in effect, unnecessary for the semantics, since $\Diamond\,(m_1, m_2) = \sim \Box \sim (m_1, m_2)$.

Alternatively, twist-structures over modal algebras can be defined in terms of $\mathcal{L}^m_{\neg}$ (cf. [16, Section 2]): the operations from $\{\vee, \wedge, \sim, \Box, \bot\}$ are as before,

$$\top \ := \ (1, 0) \quad \text{and} \quad \neg\,(m_1, m_2) \ := \ (\neg m_1, m_1)\,.$$

Clearly, the classes of twist-structures in the languages $\mathcal{L}^m$ and $\mathcal{L}^m_{\neg}$ are inter-definable, because

$$\neg\,(m_1, m_2) \ = \ (m_1, m_2) \rightarrow \bot \quad \text{and} \quad \top \ = \ \sim \bot\,, \quad \text{while}$$
$$(m_1, m_2) \rightarrow (m_3, m_4) \ = \ \neg\,(m_1, m_2) \vee (m_3, m_4)\,.$$

We want to stick to the language of BK, that is, to $\mathcal{L}^m$, but feel free to employ expressions like $\neg\,(m_1, m_2)$ or $\top$ in our proofs in which they will be understood simply as abbreviations for the right-hand parts of the above identities.

For us, there is no need in describing the logic BK formally (as a deductive system). Still, its semantical characterization will be exploited. Given a set $\Gamma \cup \{\varphi\}$ of $\mathcal{L}^m$-formulas, $\Gamma \vDash^{\bowtie}_{\mathsf{BK}} \varphi$ means that, for any $\mathcal{L}^m$-twist-structure $\mathfrak{A}$ (over some modal algebra) and $\mathfrak{A}$-valuation v of $\mathcal{L}^m$-formulas, if $\pi_1\,(v\,(\psi)) = 1$ for all $\psi \in \Gamma$, then $\pi_1\,(v\,(\varphi)) = 1$. As was shown in [18, Theorem 8], BK is sound and (strongly) complete w. r. t. the $\mathcal{L}^m$-twist-structure semantics[5] and, in particular,

$$\mathsf{BK} \ = \ \{\varphi \mid \varnothing \vDash^{\bowtie}_{\mathsf{BK}} \varphi\}\,.$$

[4] At times, we refer to π_i's as the *projection functions*.

[5] The definition of the consequence relation for BK can be found in [18, Section 4].

Now we turn to BK-lattices. Again, the formal description of the variety of BK-lattices (denoted by $\mathcal{V}_{\mathsf{BK}}$) is not essential, but it is important that they are defined, originally, in $\mathcal{L}^m_\neg$, and the collection of all BK-lattices coincides with the abstract closure of the class of all $\mathcal{L}^m_\neg$-twist-structures [16, Section 4]. In other words, each BK-lattice is isomorphic to a twist-structure (over $\mathcal{L}^m_\neg$).

Let $\mathcal{V}'_{\mathsf{BK}}$ be the abstract closure of the class of all $\mathcal{L}^m$-twist-structures. Trivially, taking into account what was said about twist-structures in different languages, $\mathcal{V}'_{\mathsf{BK}}$ turns out to be a variety, the elements of which are in a natural one-one correspondence with those of $\mathcal{V}_{\mathsf{BK}}$.[6] E. g., given $\mathfrak{A} \in \mathcal{V}_{\mathsf{BK}}$, take the $\mathcal{L}^m$-algebra $\mathfrak{A}'$ with the same $\{\vee, \wedge, \sim, \Box, \bot\}$-reduct, and also augmented by

$$\neg a \;:=\; a \to \bot\,, \quad \Diamond a \;:=\; \sim\Box\sim a \quad \text{and} \quad \top \;:=\; \sim\bot\,,$$

then it is easy to see that $\mathfrak{A}' \in \mathcal{V}'_{\mathsf{BK}}$; and the other direction is similar. From now on, by BK-lattices we mean the elements of $\mathcal{V}'_{\mathsf{BK}}$.

For each BK-lattice $\mathfrak{A}$ with the domain A, define

$$D^{\mathfrak{A}} \;:=\; \{a \in A \mid a \to a = a\}\,.$$

which is equal to $\{a \in A \mid \neg a = \bot\} = \{a \in A \mid \neg\neg a = \top\}$, since $\mathfrak{A}$ is isomorphic to a twist-structure (over $\mathcal{L}^m$), and for the latter we have

$$(m_1, m_2) \to (m_1, m_2) \;=\; (m_1, m_2) \iff (1, m_1 \wedge m_2) \;=\; (m_1, m_2) \iff$$
$$m_1 \;=\; 1 \iff (\neg m_1, m_1) \;=\; (0, 1) \iff \neg\,(m_1, m_2) \;=\; (0, 1) \iff$$
$$(m_1, \neg m_1) \;=\; (1, 0) \iff \neg\neg\,(m_1, m_2) \;=\; (1, 0)\,.$$

Given a set $\Gamma \cup \{\varphi\}$ of $\mathcal{L}^m$-formulas, $\Gamma \vDash^*_{\mathsf{BK}} \varphi$ means that, for any BK-lattice $\mathfrak{A}$ and $\mathfrak{A}$-valuation v of $\mathcal{L}^m$-formulas, if $v\,(\psi) \in D^{\mathfrak{A}}$ for all $\psi \in \Gamma$, then $v\,(\varphi) \in D^{\mathfrak{A}}$. Thus, [16, Theorem 3.2] (via a simple reformulation) establishes the soundness and (strong) completeness of BK w. r. t. the BK-lattice semantics, whence

$$\mathsf{BK} \;=\; \{\varphi \mid \varnothing \vDash^*_{\mathsf{BK}} \varphi\}\,.$$

By a *logic* in the language $\mathcal{L}^m$ we mean a subset of $\mathcal{L}^m$-formulas which is closed under the four rules (called the *substitution rule*, *modus ponens*, and the *monotonicity rules* for $\Box$ and $\Diamond$, respectively), i. e., under

$$\frac{\varphi\,(p_1, \ldots p_n)}{\varphi\,(\psi_1, \ldots \psi_n)}\,, \quad \frac{\varphi \quad \varphi \to \psi}{\psi}\,, \quad \frac{\varphi \to \psi}{\Box\varphi \to \Box\psi}\,, \quad \frac{\varphi \to \psi}{\Diamond\varphi \to \Diamond\psi}\,.$$

Naturally, BK turns out to be closed under these rules (cf. [18]). A BK-*extension* is, of course, a logic in $\mathcal{L}^m$ that contains BK (as a subset).[7]

As notational shorthand, for any two $\mathcal{L}^m$-formulas φ and ψ, we write $\varphi \leftrightarrow \psi$ instead of $(\varphi \to \psi) \wedge (\psi \to \varphi)$, and $\varphi \Leftrightarrow \psi$ instead of $(\varphi \leftrightarrow \psi) \wedge (\sim\varphi \leftrightarrow \sim\psi)$.

[6] Just in the same way as $\mathcal{L}^m_\neg$-twist-structures correspond to $\mathcal{L}^m$-twist-structures.

[7] And any BK-extension L proves to be closed w. r. t. the semantics just described, i. e., $L = \{\varphi \mid L \vDash^{\bowtie}_{\mathsf{BK}} \varphi\} = \{\varphi \mid L \vDash^*_{\mathsf{BK}} \varphi\}$ (cf. [16, Section 5]).

There is a simple (yet useful) observation concerning BK and its extensions: let L be a BK-extension, then, for each φ,

$$(\dagger) \qquad \varphi \in L \quad \Longleftrightarrow \quad \varphi \Leftrightarrow (\varphi \to \varphi) \in L$$

(that is easily verified with the help of the foregoing semantic definition of BK).

Next, we bring several notions that are important for what follows. Let $\mathfrak{M} = \langle M; \vee, \wedge, \neg, \Box \rangle$ be a modal algebra. A non-empty subset S of M is called a *$\Box$-filter* (*$\Diamond$-ideal*) on $\mathfrak{M}$ iff S is a lattice filter (ideal) on $\langle M; \vee, \wedge \rangle$ and, for every $a \in S$, we have $\Box a \in S$ ($\Diamond a \in S$). Notice, the family of all $\Box$-filters ($\Diamond$-ideals) on $\mathfrak{M}$ forms a lattice, denoted by $\mathscr{F}^{\Box}(\mathfrak{M})$ ($\mathscr{I}^{\Diamond}(\mathfrak{M})$, respectively).

For a BK-lattice $\mathfrak{A} = \langle A; \vee, \wedge, \to, \sim, \Box, \Diamond, \bot \rangle$, the *underlying modal algebra* is

$$\mathfrak{A}_{\bowtie} \ := \ \langle A_{\bowtie}; \vee, \wedge, \neg, \Box \rangle ,$$

where $A_{\bowtie} := \{\neg\neg a \mid a \in A\}$ and the operations from $\{\vee, \wedge, \neg, \Box\}$ are induced by those of $\mathfrak{A}$ (that this definition is correct and produces a modal algebra is guaranteed by [16, Proposition 3.2]).

As usual, for an algebra $\mathfrak{X}$, $Con(\mathfrak{X})$ is the lattice of congruences on $\mathfrak{X}$.

Finally, the Gothic letters $\mathfrak{A}$ and $\mathfrak{B}$ (possibly with sub- or superscripts) henceforth designate arbitrary BK-lattices (and, particularly, twist-structures over modal algebras, which are, by default, considered in $\mathcal{L}^m$). Analogously, the letters $\mathfrak{M}$ and $\mathfrak{N}$ stand for modal algebras. For any of these, the corresponding Latin letters (say, A and B; M and N) denote the domains.

3. On representation of BK-lattices

The representation of twist-structures over modal algebras was previously described in [16, Section 6]. We show how similar results apply to BK-lattices.

Given the full twist-structure $\mathfrak{M}^{\bowtie}$ over some modal algebra $\mathfrak{M}$, each element of $S^{\bowtie}(\mathfrak{M})$ can be identified with its domain. Therefore, due to [16, Corrolary 6.2], any twist-structure $\mathfrak{A}$ over $\mathfrak{M}$ is uniquely determined by the triple $(\mathfrak{M}, \nabla(\mathfrak{A}), \Delta(\mathfrak{A}))$ (this fact is commonly written as $\mathfrak{A} = Tw(\mathfrak{M}, \nabla(\mathfrak{A}), \Delta(\mathfrak{A}))$), where

$$\nabla(\mathfrak{A}) \ := \ \{m_1 \vee m_2 \mid (m_1, m_2) \in A\} \quad \text{and} \quad \Delta(\mathfrak{A}) \ := \ \{m_1 \wedge m_2 \mid (m_1, m_2) \in A\}$$

(a $\Box$-filter and a $\Diamond$-ideal, respectively); more precisely,

$$A \ = \ \left\{(m_1, m_2) \in M^2 \mid m_1 \vee m_2 \in \nabla(\mathfrak{A}), m_1 \wedge m_2 \in \Delta(\mathfrak{A})\right\}.$$

On the other hand, having $\nabla \in \mathscr{F}^{\Box}(\mathfrak{M})$ and $\Delta \in \mathscr{I}^{\Diamond}(\mathfrak{M})$, one can construct the unique twist-structure $\mathfrak{B}$ over $\mathfrak{M}$ with

$$B \ := \ \left\{(m_1, m_2) \in M^2 \mid m_1 \vee m_2 \in \nabla, m_1 \wedge m_2 \in \Delta\right\},$$

in which case $\nabla(\mathfrak{B}) = \nabla$ and $\Delta(\mathfrak{B}) = \Delta$ hold trivially, i. e., $\mathfrak{B} = Tw(\mathfrak{M}, \nabla, \Delta)$.

Proposition 3.1. *Let $\mathfrak{A}$ be a twist-structure over a modal algebra $\mathfrak{M}$. Then the correspondence $\kappa : \neg\neg(m_1, m_2) \mapsto m_1$ is an isomorphism between $\mathfrak{A}_{\bowtie}$ and $\mathfrak{M}$.*

Proof. Since $\neg\neg(m_1, m_2) = \neg(\neg m_1, m_1) = (\neg\neg m_1, \neg m_1) = (m_1, \neg m_1)$, κ is well-defined. Furthermore, twist-operations are componentwise w. r. t. the first component and $\pi_1(A) = M$ (by definition), hence κ turns out to be an isomorphism. □

Let $\mathfrak{A} = \langle A; \vee, \wedge, \rightarrow, \sim, \Box, \Diamond, \bot \rangle$ be an arbitrary BK-lattice. The mappings $e_{\bowtie} : \langle A; \vee, \wedge, \neg, \Box \rangle \to \mathfrak{A}_{\bowtie}$ and $\iota^{\bowtie} : \mathfrak{A} \to (\mathfrak{A}_{\bowtie})^{\bowtie}$ are defined as follows: for every $a \in A$,

$$e_{\bowtie}(a) := \neg\neg a \quad \text{and} \quad \iota^{\bowtie}(a) := (e_{\bowtie}(a), e_{\bowtie}(\sim a)).$$

Note that both $e_{\bowtie}$ and $\iota_{\bowtie}$ are homomorphisms. In this context, by $e_{\bowtie}(\mathfrak{A})$ and $\iota^{\bowtie}(\mathfrak{A})$ we mean the modal algebra $\mathfrak{A}_{\bowtie}$ and the resulting twist-structure (over $\mathfrak{A}_{\bowtie}$), respectively.[8] So $e_{\bowtie}$ is a *canonical epimorphism*, while $\iota^{\bowtie}$ is a *canonical embedding* (the latter, in fact, provides an isomorphism between $\mathfrak{A}$ and $\iota^{\bowtie}(\mathfrak{A})$: see Proposition 3.2 and the proof of Proposition 3.3 in [16]). Assume

$$\nabla_l(\mathfrak{A}) := e_{\bowtie}(\{a \vee \sim a \mid a \in A\}) \quad \text{and} \quad \Delta_l(\mathfrak{A}) := e_{\bowtie}(\{a \wedge \sim a \mid a \in A\}).$$

Roughly, the intuition is that if $\mathfrak{M}$ is a modal algebra and $\mathfrak{A} \in S^{\bowtie}(\mathfrak{M})$, then it is not hard to check the equalities

$$\nabla(\mathfrak{A}) = \pi_1(\{a \vee \sim a \mid a \in A\}) \quad \text{and}$$
$$\Delta(\mathfrak{A}) = \pi_2(\{a \vee \sim a \mid a \in A\}) = \pi_1(\{a \wedge \sim a \mid a \in A\});$$

in this way, $e_{\bowtie}$ is intended to play the role of π_1 when twist-structures over modal algebras are replaced by arbitrary BK-lattices.

Proposition 3.2. *Let $\mathfrak{A}$ be a BK-lattice. Then $\nabla_l(\mathfrak{A}) \in \mathscr{F}^{\Box}(\mathfrak{A}_{\bowtie})$, $\Delta_l(\mathfrak{A}) \in \mathscr{I}^{\Diamond}(\mathfrak{A}_{\bowtie})$, and $\iota^{\bowtie}(\mathfrak{A}) = Tw(\mathfrak{A}_{\bowtie}, \nabla_l(\mathfrak{A}), \Delta_l(\mathfrak{A}))$.*

Proof. Recall that $\mathfrak{A}$ is isomorphic to the twist-structure $\mathfrak{B} := \iota^{\bowtie}(\mathfrak{A})$ (cf. [16, Proposition 3.3]). Clearly, $\mathfrak{B} \in S^{\bowtie}(\mathfrak{A}_{\bowtie})$ (by definition of $\iota^{\bowtie}$), and, therefore, $\mathfrak{B} = Tw(\mathfrak{A}_{\bowtie}, \nabla(\mathfrak{B}), \Delta(\mathfrak{B}))$. It remains to show that

$$\nabla(\mathfrak{B}) = \nabla_l(\mathfrak{A}) \quad \text{and} \quad \Delta(\mathfrak{B}) = \Delta_l(\mathfrak{A}).$$

Notice, $b \in B$ iff there exists $a \in A$ such that $b = (e_{\bowtie}(a), e_{\bowtie}(\sim a))$; in this case, we derive

$$b \vee \sim b = (e_{\bowtie}(a), e_{\bowtie}(\sim a)) \vee (e_{\bowtie}(\sim a), e_{\bowtie}(a)) =$$
$$(e_{\bowtie}(a) \vee e_{\bowtie}(\sim a), e_{\bowtie}(\sim a) \wedge e_{\bowtie}(a)) = (e_{\bowtie}(a \vee \sim a), e_{\bowtie}(a \wedge \sim a)).$$

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[8] Also, $\neg\neg a$ can be viewed as an equivalence class $[a]_{\approx} \subseteq A$, where $a_1 \approx a_2$ iff $\neg\neg a_1 = \neg\neg a_2$ (in $\mathfrak{A}$): indeed, $\mathfrak{A}_{\bowtie}$ is isomorphic to the quotient of the $\{\vee, \wedge, \neg, \Box\}$-reduct of $\mathfrak{A}$ modulo $\approx$ (which is a congruence relation on that reduct).

Thus,

$$\nabla(\mathfrak{B}) = \pi_1(\{b \vee \sim b \mid b \in B\}) = \{e_{\bowtie}(a \vee \sim a) \mid a \in A\} = \nabla_l(\mathfrak{B}),$$
$$\Delta(\mathfrak{B}) = \pi_2(\{b \vee \sim b \mid b \in B\}) = \{e_{\bowtie}(a \wedge \sim a) \mid a \in A\} = \Delta_l(\mathfrak{B}). \qquad \square$$

Hence, for every BK-lattice $\mathfrak{A}$, we have $\mathfrak{A} \cong Tw(\mathfrak{A}_{\bowtie}, \nabla_l(\mathfrak{A}), \Delta_l(\mathfrak{A}))$. Remark: if $\mathfrak{A} \in S^{\bowtie}(\mathfrak{M})$, then

$$\kappa(\nabla_l(\mathfrak{A})) = \nabla(\mathfrak{A}) \quad \text{and} \quad \kappa(\Delta_l(\mathfrak{A})) = \Delta(\mathfrak{A});$$

due to this reason, the lower index $-_l$, in $\nabla_l(\mathfrak{A})$ and $\Delta_l(\mathfrak{A})$, may be omitted.

4. Homomorphisms, congruences and special filters

For any modal algebra $\mathfrak{M}$, we have the BK-lattice $\mathfrak{M}^{\bowtie}$, namely the full twist-structure over $\mathfrak{M}$. Conversely, for each BK-lattice $\mathfrak{A}$, the (underlying) modal algebra $\mathfrak{A}_{\bowtie}$ is extracted. Now we want to extend this technique to homomorphisms.

Let $\mathfrak{M}$ and $\mathfrak{N}$ be some modal algebras, and $h : \mathfrak{M} \to \mathfrak{N}$ be a homomorphism. Then $h^{\bowtie} : \mathfrak{M}^{\bowtie} \to \mathfrak{N}^{\bowtie}$ given by

$$h^{\bowtie}(m_1, m_2) := (h(m_1), h(m_2))$$

is also a homomorphism (cf. the definition of twist-structure operations). And if $\mathfrak{A}$ and $\mathfrak{B}$ are BK-lattices, then $h_{\bowtie} : \mathfrak{A}_{\bowtie} \to \mathfrak{B}_{\bowtie}$ determined by

$$h_{\bowtie}(\neg\neg a) := h(\neg\neg a) = \neg\neg h(a) \quad (\text{i. e., } h_{\bowtie} := h\restriction_{\mathfrak{A}_{\bowtie}})$$

is a homomorphism (remember how $\mathfrak{A}_{\bowtie}$ and $\mathfrak{B}_{\bowtie}$ are constructed).

Proposition 4.1. *Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are BK-lattices, and $h^i : \mathfrak{A} \to \mathfrak{B}$, $i \in \{1, 2\}$, are homomorphisms such that $h^1_{\bowtie} = h^2_{\bowtie}$. Then $h^1 = h^2$.*

Proof. Let $g_i := \left(h^i_{\bowtie}\right)^{\bowtie}$, $i \in \{1, 2\}$. Remark that $g_1 = g_2$ (since $h^1_{\bowtie} = h^2_{\bowtie}$). For any a_1 and a_2 from $\mathfrak{A}$, we calculate

$$g_i(\neg\neg a_1, \neg\neg a_2) = \left(h^i_{\bowtie}(\neg\neg a_1), h^i_{\bowtie}(\neg\neg a_2)\right) = \left(\neg\neg h^i(a_1), \neg\neg h^i(a_2)\right).$$

and thus, for every a in $\mathfrak{A}$,

$$h^i(a) = (\iota^{\bowtie}_{\mathfrak{B}})^{-1} \circ \iota^{\bowtie}_{\mathfrak{B}}\left(h^i(a)\right) = (\iota^{\bowtie}_{\mathfrak{B}})^{-1}\left(\neg\neg h^i(a), \neg\neg\sim h^i(a)\right) =$$
$$(\iota^{\bowtie}_{\mathfrak{B}})^{-1}\left(\neg\neg h^i(a), \neg\neg h^i(\sim a)\right) = (\iota^{\bowtie}_{\mathfrak{B}})^{-1} \circ g_i(\neg\neg a, \neg\neg\sim a).$$

The latter representation readily implies $h^1 = h^2$ (in view of $g_1 = g_2$). $\square$

The next notion is analogous to the one suggested by H. Rasiowa in [20]. Let $\mathfrak{A}$ be a BK-lattice. A *special filter of the first kind* (*sffk*) on $\mathfrak{A}$ is a non-empty subset ∇ of A such that: 1) if a_1 and a_2 are in ∇, then $a_1 \wedge a_2$ is in ∇; 2) if a_1 is in ∇ and $a_1 \preceq a_2$, then a_2 is in ∇, where $a_1 \preceq a_2$ is a shorthand for $\neg\neg a_1 \leqslant \neg\neg a_2$; 3) if a is in ∇, then so is $\Box a$. The collection of all sffk's on $\mathfrak{A}$ can be naturally assigned a lattice structure, and the resulting lattice is denoted by $\mathscr{F}^1(\mathfrak{A})$. For each sffk ∇, the binary relation $\approx_\nabla$ on A is given by: $a_1 \approx_\nabla a_2$ iff $a_1 \Leftrightarrow a_2 \in \nabla$.

For a homomorphism $h : \mathfrak{A} \to \mathfrak{B}$ of BK-lattices, put $Ker\,(h) := h^{-1}(D^{\mathfrak{B}})$. It turns out that such homomorphisms are closely connected with sffk's on $\mathfrak{A}$, and that the latter are also in a one-to-one correspondence with the congruences on $\mathfrak{A}$.

Proposition 4.2. (1) *For every* BK*-lattice* $\mathfrak{A}$*,* $D^{\mathfrak{A}}$ *is an sffk on* $\mathfrak{A}$*.*

(2) *Let* $h : \mathfrak{A} \to \mathfrak{B}$ *be a homomorphism of* BK*-lattices. Then* $Ker\,(h)$ *is an sffk on* $\mathfrak{A}$ *and, for every* $a_1, a_2 \in \mathfrak{A}$*,* $h\,(a_1) = h\,(a_2)$ *iff* $a_1 \Leftrightarrow a_2 \in Ker\,(h)$*;*

(3) *Let* ∇ *be an sffk on a* BK*-lattice* $\mathfrak{A}$*. Then* $D^{\mathfrak{A}} \subseteq \nabla$*, the relation* $\approx_\nabla$ *is a congruence relation on* $\mathfrak{A}$*, and* $\nabla = Ker\,(h)$ *for a canonical epimorphism* $h : \mathfrak{A} \to \mathfrak{A}_{/\approx_\nabla}$*.*

Proof. We begin with a technical remark. As was mentioned before, $\mathfrak{A}$ is isomorphic (via $\iota^{\bowtie}$) to a twist-structure $\mathfrak{B}$ over $\mathfrak{A}_{\bowtie}$. But for this $\mathfrak{B}$, we have

$$D^{\mathfrak{B}} = \{(m_1, m_2) \in B \mid (\neg m_1, m_1) = (0, 1)\} = \{(m_1, m_2) \in B \mid m_1 = 1\} = \{(1, m_2) \in B \mid m_2 \in \mathfrak{A}_{\bowtie}\},$$

while $(m_1, m_2) \preceq (m_3, m_4)$ is equivalent to $m_1 \leqslant m_3$ (in $\mathfrak{B}$).[9] Taking into account that $\iota^{\bowtie}$ obviously preserves $\preceq$ (because $\preceq$ is definable via basic operations) and maps $D^{\mathfrak{A}}$ onto $D^{\mathfrak{B}}$, these observations easily imply

Lemma 4.3. *For every* BK*-lattice* $\mathfrak{A}$*, we have:*

(i) *if* $a_1, a_2 \in D^{\mathfrak{A}}$*, then* $a_1 \wedge a_2 \in D^{\mathfrak{A}}$*;*

(ii) *if* $a_1 \in D^{\mathfrak{A}}$*,* $a_2 \in A$ *and* $a_1 \preceq a_2$*, then* $a_2 \in D^{\mathfrak{A}}$*;*

(iii) *if* $a_1 \in D^{\mathfrak{A}}$*, then* $\Box a_1 \in D^{\mathfrak{A}}$*;*

(iv) *if* $a_1 \in A$ *and* $a_2 \in D^{\mathfrak{A}}$*, then* $a_1 \preceq a_2$*;*

(v) *for any* $a_1, a_2 \in A$*,* $a_1 = a_2$ *iff* $a_1 \Leftrightarrow a_2 \in D^{\mathfrak{A}}$*;*

(vi) *for any* $a_1, a_2 \in A$*,* $a_1 \wedge (a_1 \to a_2) \preceq a_2$*.*

Now we return to the proof of the Proposition.

$\boxed{1}$ Immediately follows from Lemma 4.3 (i–iii).

$\boxed{2}$ By the previous item, $D^{\mathfrak{B}}$ is an sffk. Therefore, since h preserves $\wedge$, $\Box$ and $\preceq$, $Ker\,(h)$ is also an sffk. Next, due to Lemma 4.3 (v), $h\,(a_1) = h\,(a_2)$ is equivalent to $h\,(a_1) \Leftrightarrow h\,(a_2) = h\,(a_1 \Leftrightarrow a_2) \in D^{\mathfrak{B}}$, i. e., $a_1 \Leftrightarrow a_2 \in Ker\,(h)$.

$\boxed{3}$ By Lemma 4.3 (iv) and the second sffk property, $D^{\mathfrak{A}} \subseteq \nabla$. Since $a \Leftrightarrow a \in D^{\mathfrak{A}}$ for any $a \in A$, $\approx_\nabla$ is reflexive. Also, from the axioms of propositional positive logic one easily derives that

$$(p \Leftrightarrow q) \to (q \Leftrightarrow p) \quad \text{and} \quad (p \Leftrightarrow q) \to ((q \Leftrightarrow r) \to (p \Leftrightarrow r)) \quad \text{are in BK}.$$

[9] In fact, for any $\{a_1, a_2\} \subseteq A$, $a_1 \leqslant a_2$ iff $a_1 \preceq a_2$ and $\sim a_2 \preceq \sim a_1$.

Due to the semantic definition of BK (namely its second version, in terms of BK-lattices), for any $\{a_1, a_2, a_3\} \subseteq \mathfrak{A}$, we have

$$(a_1 \Leftrightarrow a_2) \to (a_2 \Leftrightarrow a_1) \text{ and } (a_1 \Leftrightarrow a_2) \to ((a_2 \Leftrightarrow a_3) \to (a_1 \Leftrightarrow a_3)) \text{ are in } D^{\mathfrak{A}}.$$

Therefore, by Lemma 4.3 (vi) and the first two sffk properties,

$$a_1 \Leftrightarrow a_2 \in \nabla \quad \Longrightarrow \quad a_2 \Leftrightarrow a_1 \in \nabla\,,$$
$$a_1 \Leftrightarrow a_2 \in \nabla \quad \text{and} \quad a_2 \Leftrightarrow a_3 \in \nabla \quad \Longrightarrow \quad a_1 \Leftrightarrow a_3 \in \nabla\,.$$

Hence $\approx_\nabla$ is an equivalence relation. Moreover, it is not hard to establish (using the intended twist-structure semantics, i. e., the semantic definition of BK) that

$$(p \Leftrightarrow q) \to ((r \Leftrightarrow s) \to (p \vee r \Leftrightarrow q \vee s)) \quad \text{is in} \quad \mathsf{BK}\,,$$

and similarly for all the other non-modal connectives. Thus, by the above argument, if $a_1 \Leftrightarrow a_2 \in \nabla$ and $a_3 \Leftrightarrow a_4 \in \nabla$, then $a_1 \vee a_3 \Leftrightarrow a_2 \vee a_4 \in \nabla$ and similarly for the other non-modal connectives. Finally, if $a_1 \Leftrightarrow a_2 \in \nabla$, then $\Box\,(a_1 \Leftrightarrow a_2) \in \nabla$ due to the third sffk property, and so, employing the fact that

$$(\Box\,(p \Leftrightarrow q)) \to (\Box p \Leftrightarrow \Box q) \quad \text{is in} \quad \mathsf{BK}$$

(again, exploits the twist-structure semantics) together with Lemma 4.3 (vi), we conclude $\Box a_1 \Leftrightarrow \Box a_2 \in \nabla$. As a result, $\approx_\nabla$ is indeed a congruence relation.

What remains is to show $Ker\,(h) = \nabla$.

'$\subseteq$'. Assume $a \in Ker\,(h)$, i. e., $h\,(a) \in D^{(\mathfrak{A}/\approx_\nabla)}$. Remember, $D^{(\mathfrak{A}/\approx_\nabla)}$ coincides with $\{a \in \mathfrak{A}/_{\approx_\nabla} \mid a \to a = a\}$, hence $h\,(a \to a) = h\,(a) \to h\,(a) = h\,(a)$, namely $(a \to a) \Leftrightarrow a \in \nabla$. Since $a_1 \leqslant a_2$ implies $a_1 \preceq a_2$, $(a \to a) \Leftrightarrow a \leqslant (a \to a) \to a$, and sffk's are closed upwards under $\preceq$, we have $(a \to a) \to a \in \nabla$. Obviously, $a \to a \in D^{\mathfrak{A}} \subseteq \nabla$, so by Lemma 4.3 (vi) we get $a \in \nabla$.

'$\supseteq$'. Assume $a \in \nabla$. Since in any BK-lattice one has $(a_1 \to a_1) \to a_2 = a_2$ and $a_3 \leqslant a_1 \to a_3$, $a = (a \to a) \to a \in \nabla$ and $a \to a \preceq a \to (a \to a) \in \nabla$ (recall that $a \to a \in D^{\mathfrak{A}} \subseteq \nabla$ and $\leqslant$ is a subset of $\preceq$). On the other hand, it is easy to check that

$$a \preceq {\sim}\, a \to {\sim}\,(a \to a) \quad \text{and} \quad {\sim}\,(a \to a) \to {\sim}\, a \in D^{\mathfrak{A}}\,,$$

whence ${\sim}\, a \to {\sim}\,(a \to a) \in \nabla$ and ${\sim}\,(a \to a) \to {\sim}\, a \in \nabla$. All together this yields $(a \to a) \Leftrightarrow a \in \nabla$, i. e., $h\,(a \to a) = h\,(a)$. But $a \to a \in D^{\mathfrak{A}}$, so $h\,(a) = h\,(a \to a) \in D^{(\mathfrak{A}/\approx_\nabla)}$. Thus, $a \in Ker\,(h)$. $\Box$

Also, the collection of sffk's can be characterized in the following way.

Corollary 4.4. *Let $\mathfrak{A}$ be a BK-lattice and ∇ be a non-empty subset of A. Then ∇ is an sffk on $\mathfrak{A}$ iff, for any a_1 and a_2 in A, we have that: 1) $a_1 \in \nabla$ and $a_2 \in \nabla$ imply $a_1 \wedge a_2 \in \nabla$; 2) $a_1 \in \nabla$ implies $\Box a_1 \in \nabla$; 3) $a_1 \in \nabla$ and $a_1 \to a_2 \in \nabla$ imply $a_2 \in \nabla$.*

Proof. $\boxed{\Rightarrow}$ The first two Items are trivial, while the third follows immediately from the sffk properties and Lemma 4.3 (vi) (see the proof of the previous Proposition).

$\boxed{\Leftarrow}$ The first and the third sffk property are trivial. For the second one, notice that, by Proposition 4.2 (3), $D^{\mathfrak{A}}$ is the least sffk on $\mathfrak{A}$, and that, for any $a_1, a_2 \in \mathfrak{A}$, $a_1 \preceq a_2$ implies $a_1 \to a_2 \in D^{\mathfrak{A}}$ (this can be easily checked when passing from $\mathfrak{A}$ to the twist-structure $\iota^{\bowtie}(\mathfrak{A})$, an isomorphic copy of $\mathfrak{A}$). Now if $a_1 \in \nabla$ and $a_1 \preceq a_2$, then $a_1 \to a_2 \in D^{\mathfrak{A}} \subseteq \nabla$, whence $a_2 \in \nabla$. □

Remark. For any BK-lattice $\mathfrak{A}$ and $\theta \in Con(\mathfrak{A})$, $S = \{a \in A \mid a\theta(a \to a)\}$ is an sffk on $\mathfrak{A}$. Indeed, let $h : \mathfrak{A} \to \mathfrak{A}_{/\theta}$ be a canonical epimorphism (onto the BK-lattice $\mathfrak{A}_{/\theta}$). Obviously, $\theta = \{(a_1, a_2) \in A^2 \mid h(a_1) = h(a_2)\}$. Thus,

$$\begin{array}{ccccc} a \in Ker(h) & \iff & h(a) \in D^{(\mathfrak{A}_{/\theta})} & \iff & \\ h(a) = h(a \to a) & \iff & a\theta(a \to a) & \iff & a \in S\,, \end{array}$$

i. e., S coincides with $Ker(h)$ and, therefore, belongs to $\mathscr{F}^1(\mathfrak{A})$.

Corollary 4.5. *For every* BK*-lattice* $\mathfrak{A}$*,* $\mathscr{F}^1(\mathfrak{A}) \cong Con(\mathfrak{A})$*, and the mutually inverse isomorphisms are defined by*

$$\begin{aligned} f(\nabla) &= \approx_{\nabla}, \quad \textit{for any } \nabla \in \mathscr{F}^1(\mathfrak{A}), \quad \textit{and} \\ g(\theta) &= \{a \in A \mid a\theta(a \to a)\}, \quad \textit{for any } \theta \in Con(\mathfrak{A}). \end{aligned}$$

Proof. Clearly, $g(f(\nabla)) = \{a \in A \mid a \approx_{\nabla} (a \to a)\}$. Due to Proposition 4.2 (3), there is a homomorphism $h : \mathfrak{A} \to \mathfrak{A}'$ (of BK-lattices) such that $Ker(h) = \nabla$. Thus,

$$\begin{array}{ccccc} a \approx_{\nabla} (a \to a) & \iff & a \Leftrightarrow (a \to a) \in Ker(h) & \overset{Prop.\ 4.2\ (2)}{\iff} & \\ h(a) = h(a \to a) & \iff & h(a) \in D^{(\mathfrak{A}')} & \iff & a \in Ker(h) = \nabla\,, \end{array}$$

whence $g(f(\nabla)) = \nabla$, that is, $f \circ g = \mathrm{id}$.

It is clear that $f(g(\theta)) = \{(a_1, a_2) \in A^2 \mid (a_1 \Leftrightarrow a_2)\,\theta\,((a_1 \Leftrightarrow a_2) \to (a_1 \Leftrightarrow a_2))\}$. Obviously, $\theta = \{(a_1, a_2) \in A^2 \mid h(a_1) = h(a_2)\} =: K_h$, where $h : \mathfrak{A} \to \mathfrak{A}_{/\theta}$ is a canonical epimorphism (onto the BK-lattice $\mathfrak{A}_{/\theta}$). Then

$$(a_1 \Leftrightarrow a_2)\,\theta\,((a_1 \Leftrightarrow a_2) \to (a_1 \Leftrightarrow a_2)) \quad \iff \quad a_1 \Leftrightarrow a_2 \in Ker(h)$$

(cf. the above Remark) which is equivalent, by Proposition 4.2 (2), to $h(a_1) = h(a_2)$, i. e., $(a_1, a_2) \in \theta$, whence $f(g(\theta)) = \theta$, that is, $g \circ f = \mathrm{id}$.

Hence both f and g are injective and 'onto'. It remains to notice that f is easily seen to be a homomorphism (the verification is straightforward). □

Corollary 4.6. *Let* $h : \mathfrak{A} \to \mathfrak{B}$ *be a homomorphism of* BK*-lattices. Then* h *is injective (i. e., is a monomorphism) iff* $Ker(h) = D^{\mathfrak{A}}$*.*

Proof. $\boxed{\Rightarrow}$ If $a \notin D^{\mathfrak{A}}$, then, as we've already seen, $a \to a \neq a$, and so $h(a) \to h(a) = h(a \to a) \neq h(a)$, i.e., $h(a) \notin D^{\mathfrak{B}}$. What is means is that $Ker(h) \subseteq D^{\mathfrak{A}}$. On the other hand, due to Proposition 4.2 (2), $Ker(h)$ is an sffk and, therefore, by Proposition 4.2 (3), $D^{\mathfrak{A}} \subseteq Ker(h)$.

$\boxed{\Leftarrow}$ Conversely, if $h(a_1) = h(a_2)$, then we have $a_1 \Leftrightarrow a_2 \in Ker(h) = D^{\mathfrak{A}}$ by Proposition 4.2 (2). But the latter implies $a_1 = a_2$ in view of Lemma 4.3 (v). □

As we've shown in Proposition 4.1, any homomorphism of BK-lattices is, in a sense, uniquely determined by the corresponding homomorphism $h_{\bowtie}$ of underlying (or basic) modal algebras. In effect, there also exists a close connection between sffk's on a given BK-lattice and □-filters on the related modal algebra. To simplify the situation, we first consider the case of twist-structures.

Proposition 4.7. *Let $\mathfrak{A}$ be a twist-structure over some modal algebra $\mathfrak{M}$. Then $\mathscr{F}^{\Box}(\mathfrak{M}) \cong \mathscr{F}^{1}(\mathfrak{A})$, and the mutually inverse isomorphisms are defined by*

$$f(\nabla) = (\pi_1)^{-1}(\nabla) = \{(m_1, m_2) \in A \mid m_1 \in \nabla\}, \quad \textit{for any } \nabla \in \mathscr{F}^{\Box}(\mathfrak{M}), \quad \textit{and}$$
$$g(\nabla) = \pi_1(\nabla) = \{m_1 \in M \mid (m_1, m_2) \in \nabla\}, \quad \textit{for any } \nabla \in \mathscr{F}^{1}(\mathfrak{A}).$$

Proof. First of all, via a direct verification one proves that $f(\nabla) \in \mathscr{F}^{1}(\mathfrak{A})$ for $\nabla \in \mathscr{F}^{\Box}(\mathfrak{M})$, and $g(\nabla) \in \mathscr{F}^{\Box}(\mathfrak{M})$ for $\nabla \in \mathscr{F}^{1}(\mathfrak{A})$.

Let $\nabla \in \mathscr{F}^{\Box}(\mathfrak{M})$. Obviously, since $\pi_1(A) = M$ (due to the definition of a twist-structure over $\mathfrak{M}$), for every $m_1 \in \nabla$, there exists $m_2 \in M$ with $(m_1, m_2) \in A$. Hence $g(f(\nabla)) = \nabla$, that is, $g \circ f = \mathrm{id}$.

Now suppose $\nabla \in \mathscr{F}^{1}(\mathfrak{A})$. Clearly, if $(m_1, m_2) \in \nabla \subseteq A$, then $m_1 \in g(\nabla)$, whence $(m_1, m_2) \in f(g(\nabla))$. Conversely, if $(m_1, m_2) \in f(g(\nabla))$, then $(m_1, m_2) \in A$ and there is $(m_1, m_3) \in \nabla$ for some m_3. Since $(m_1, m_3) \preceq (m_1, m_2)$, the latter is in ∇. This implies $\nabla = f(g(\nabla))$, that is, $f \circ g = \mathrm{id}$

Thus, both f and g are injective and 'onto'. Finally, it is not hard to check that, for instance, g is a homomorphism (using the fact that the second coordinate is not essential for $\preceq$-relation). □

As we know, every BK-lattice $\mathfrak{A}$ is isomorphic to a twist-structure over $\mathfrak{A}_{\bowtie}$, namely to $\iota^{\bowtie}(\mathfrak{A})$ which is itself defined by means of the canonical epimorphism $e_{\bowtie}$ (cf. Section 3). In this context, the mapping $e_{\bowtie}$ for BK-lattices is an analog of the projection function π_1 for twist-structures.

And there is no surprise that the previous result can be easily generalized to BK-lattices (here the proof is left as an exercise).

Proposition 4.8. *Let $\mathfrak{A}$ be a* BK*-lattice. Then $\mathscr{F}^{\Box}(\mathfrak{A}_{\bowtie}) \cong \mathscr{F}^{1}(\mathfrak{A})$, and the mutually inverse isomorphisms are defined by*

$$f(\nabla) = (e_{\bowtie})^{-1}(\nabla) =: \nabla^{\bowtie}, \quad \textit{for any } \nabla \in \mathscr{F}^{\Box}(\mathfrak{A}_{\bowtie}), \quad \textit{and}$$
$$g(\nabla) = e_{\bowtie}(\nabla) =: \nabla_{\bowtie}, \quad \textit{for any } \nabla \in \mathscr{F}^{1}(\mathfrak{A}).$$

It is well-known (see, e. g., [9, Theorem 4.1.10]) that, for any modal algebra $\mathfrak{M}$, $Con\,(\mathfrak{M}) \cong \mathscr{F}^{\Box}\,(\mathfrak{M})$: given $\theta \in Con\,(\mathfrak{M})$, define

$$\Phi_\theta \;:=\; \{m \in M \mid m\theta 1\} \;\in\; \mathscr{F}^{\Box}\,(\mathfrak{M})\,;$$

conversely, if $F \in \mathscr{F}^{\Box}\,(\mathfrak{M})$, put

$$\Theta_F \;:=\; \big\{(m_1, m_2) \in M^2 \mid m_1 \leftrightarrow m_2 \in F\big\} \;\in\; Con\,(\mathfrak{M})$$

(these mappings are homomorphisms, with $\Phi_{\Theta_F} = F$ and $\Theta_{\Phi_\theta} = \theta$). Taking into account Corollary 4.5 and Proposition 4.8, what we have is

Theorem 4.9. *For every* BK*-lattice* $\mathfrak{A}$*,* $Con\,(\mathfrak{A}) \cong Con\,(\mathfrak{A}_{\bowtie})$.

As far as the property of an algebra to be subdirectly irreducible is completely determined by the structure of its congruence lattice (see [2, Proposition 8.4] for more details), we further derive

Theorem 4.10. *A* BK*-lattice* $\mathfrak{A}$ *is subdirectly irreducible iff the underlying modal algebra* $\mathfrak{A}_{\bowtie}$ *is subdirectly irreducible.*

Recall that an algebra $\mathfrak{A}$ is *congruence-distributive* iff $Con\,(\mathfrak{A})$ is a distributive lattice (cf. [2]).

Proposition 4.11. *Every* BK*-lattice is congruence-distributive.*

Proof. Readily follows from the known fact that for each algebra $\mathfrak{A}$ in a language $\{\vee, \wedge, \dots\}$, if its $\{\vee, \wedge\}$-reduct is a lattice, then $\mathfrak{A}$ is congruence-distributive □

Next, we show that the lattice of extensions of BK (denote it by $\mathcal{E}$BK) possess the important distributivity property, which is both interesting in its own right, and may serve as a tool for further investigation of BK-extensions.[10]

Theorem 4.12. *The lattice* $\mathcal{E}$BK *is distributive.*

Proof. Consider the Lindenbaum-Tarski $\mathcal{L}^m$-algebra $\mathfrak{A}_{\mathsf{BK}}$ for the logic BK modulo the strong equivalence $\Leftrightarrow$, namely its domain is

$$A_{\mathsf{BK}} \;:=\; \{[\varphi] \mid \varphi \text{ is an } \mathcal{L}^m\text{-formula}\}\,,$$

where $[\varphi] := \{\psi \mid \psi \Leftrightarrow \varphi \in \mathsf{BK}\}$, and the operations are defined in the usual way. By [16, Lemma 5.1], $\mathfrak{A}_{\mathsf{BK}}$ is a BK-lattice (and $\mathfrak{A}_{\mathsf{BK}} \Vdash \neg\varphi = \bot$ for all $\varphi \in \mathsf{BK}$).

Next, every logic L in $\mathcal{E}$BK induces a congruence on $\mathfrak{A}_{\mathsf{BK}}$ given by

$$\theta_L \;:=\; \{([\varphi], [\psi]) \mid \varphi \Leftrightarrow \psi \in L\}$$

[10] In [13], there is an error in the proof of the congruence-permutability of N4-lattices (as well as in [15], but for N4$^{\perp}$-lattices). This causes gaps in the proof of the distributivity of the lattice of N4-extensions in [13] (respectively, of N4$^{\perp}$-extensions, in [15]) that exploits the congruence-permutability. However, though for now it is not clear whether or not the congruence-permutability holds, the gaps can be easily filled in: cf. the last steps in the proof of Theorem 4.12 below.

(the verification of the congruence properties is straightforward, cf. the proof of [16, Lemma 5.1]). Clearly, the mapping $\vartheta : L \mapsto \theta_L$ preserves the $\subseteq$-ordering. Also, in view of (†) (cf. Preliminaries), it is easy to check that ϑ is injective.

Notice, any θ_L (for $L \in \mathcal{E}\mathsf{BK}$) is closed under substitutions, that is,

$$[\varphi(p_1, \dots, p_n)]\, \theta_L\, [\psi(p_1, \dots, p_n)] \quad \Longrightarrow \quad [\varphi(\chi_1, \dots, \chi_n)]\, \theta_L\, [\psi(\chi_1, \dots, \chi_n)]$$

for all $\mathcal{L}^m$-formulas $\chi_1, \dots, \chi_n$ (readily by the definition of logic).

Conversely, if $\theta \in Con(\mathfrak{A}_{\bowtie})$ is closed under substitutions (in the above sense), we define

$$L_\theta \ := \ \{\varphi \mid [\varphi]\, \theta\, [\varphi \to \varphi]\}\,.$$

Let us show that L_θ is indeed a BK-extension. To simplify the argument, put

$$F_\theta \ := \ \{[\varphi] \mid [\varphi]\, \theta\, [\varphi \to \varphi]\} = \{[\varphi] \mid \varphi \in L_\theta\}\,,$$

which is an sffk on $\mathfrak{A}_{\bowtie}$ by Corollary 4.5. Due to (†), $\varphi \in \mathsf{BK}$ implies $[\varphi] = [\varphi \to \varphi]$; thus, we have $\mathsf{BK} \subseteq L_\theta$. Moreover, L_θ is obviously closed under substitutions (since θ is so). In case of *modus ponens*, assume φ and $\varphi \to \psi$ are in L_θ, then $[\varphi]$ and $[\varphi \to \psi] = [\varphi] \to [\psi]$ are in F_θ, whence, by Corollary 4.4 (3), $[\psi] \in F_\theta$, i. e., $\psi \in L_\theta$. In case of monotonicity rules, suppose $\varphi \to \psi \in L_\theta$ and, therefore, $[\varphi \to \psi] \in F_\theta$. Hence $\Box\, [\varphi \to \psi] = [\Box\, (\varphi \to \psi)] \in F_\theta$. Next,

$$\Box\,(p \to q) \to (\Box p \to \Box q) \quad \text{and} \quad \Box\,(p \to q) \to (\Diamond p \to \Diamond q) \quad \text{are in } \mathsf{K}\,,$$

and so in BK (together with all their substitution instances). Since $\mathsf{BK} \subseteq L_\theta$,

$$[\Box\,(\varphi \to \psi)] \to [\Box\varphi \to \Box\psi] \quad \text{and} \quad [\Box\,(\varphi \to \psi)] \to [\Diamond\varphi \to \Diamond\psi] \quad \text{are in } F_\theta\,.$$

Then, by applying Corollary 4.4 (3), we get that both $[\Box\varphi \to \Box\psi]$ and $[\Diamond\varphi \to \Diamond\psi]$ belong to F_θ, i. e., $\Box\varphi \to \Box\psi$ and $\Diamond\varphi \to \Diamond\psi$ are in L_θ.

In addition, $\theta = \theta_{L_\theta}$, because

$$[\varphi]\, \theta\, [\psi] \quad \overset{Cor.\ 4.5}{\Longleftrightarrow} \quad [\varphi] \Leftrightarrow [\psi] = [\varphi \Leftrightarrow \psi] \in F_\theta \quad \overset{Cor.\ 4.5}{\Longleftrightarrow}$$
$$[\varphi \Leftrightarrow \psi]\, \theta\, [(\varphi \Leftrightarrow \psi) \to (\varphi \Leftrightarrow \psi)] \quad \Longleftrightarrow \quad \varphi \Leftrightarrow \psi \in L_\theta \quad \Longleftrightarrow \quad [\varphi]\, \theta_{L_\theta}\, [\psi]\,.$$

Accordingly, the full image $\vartheta(\mathcal{E}\mathsf{BK})$ consists of exactly the congruences that are closed under substitutions. Now, given $\{L_1, L_2\} \subseteq \mathcal{E}\mathsf{BK}$, we calculate $\theta_{L_1} \wedge \theta_{L_2}$ and $\theta_{L_1} \vee \theta_{L_2}$ (in the context of $Con(\mathfrak{A}_{\mathsf{BK}})$). Trivially, $\theta_{L_1} \wedge \theta_{L_2} = \theta_{L_1} \cap \theta_{L_2}$ is closed under substitutions. And for $\theta_{L_1} \vee \theta_{L_2}$, it is straightforward that

$$([\varphi], [\psi]) \in \theta_{L_1} \vee \theta_{L_2} \quad \Longleftrightarrow \quad \exists n\, \exists\, \{\chi_i\}_{i=0}^{n} \text{ s. t. } [\varphi] = [\chi_0]\,, \ [\psi] = [\chi_n] \text{ and}$$
$$(([\chi_i], [\chi_{i+1}]) \in \theta_{L_1} \text{ or } ([\chi_i], [\chi_{i+1}]) \in \theta_{L_2}) \text{ for all } i = 0, \dots, n-1\,,$$

which relation is clearly closed under substitutions. Consequently, each of these has the form θ_L for a suitable $L \in \mathcal{E}\mathsf{BK}$.

Thus, $\vartheta(\mathcal{E}\mathsf{BK})$ proves to be a sublattice of the lattice $Con(\mathfrak{A}_{\mathsf{BK}})$ which is, in turn, distributive due to Proposition 4.11. On the other hand, $\mathcal{E}\mathsf{BK} \cong \vartheta(\mathcal{E}\mathsf{BK})$ (as orders, via ϑ, and so as lattices too), whence the result follows. □

Now we establish some criteria for the existence of homomorphisms and monomorphisms from one BK-lattice into another. To do this, we exploit several facts on the interrelations between homomorphisms h, working with BK-lattices, and the corresponding $h_{\bowtie}$, working with modal algebras. Notice, if f is a homomorphism of BK-lattices and g is a homomorphism of modal algebras, then $Ker(f)$ is an sffk (by Proposition 4.2 (2)), while $Ker(g) := g^{-1}(1)$ is easily seen to be a $\Box$-filter. Thus, we are free to use the notation from Proposition 4.8.

Proposition 4.13. *Let $h : \mathfrak{A} \to \mathfrak{B}$ be a homomorphism of* BK*-lattices. Then*

(1) $Ker(h_{\bowtie}) = (Ker(h))_{\bowtie}$ *and* $Ker(h) = (Ker(h_{\bowtie}))^{\bowtie}$*;*
(2) *h is injective (i. e., is a monomorphism) iff $h_{\bowtie}$ is injective.*

Proof. [1] By definition, $(Ker(h))_{\bowtie} = \{e_{\bowtie}(a) \mid a \in Ker(h)\} = \{\neg\neg a \mid h(a) \in D^{\mathfrak{B}}\}$. At the same time,

$$h(a) \in D^{\mathfrak{B}} \iff \neg h(a) = \bot \iff \neg\neg h(a) = \top$$

(the latter is because $\neg\neg\neg a = \sim\sim\neg a = \neg a$, in every BK-lattice). But $\top$ in $\mathfrak{B}$ coincides with 1 in $\mathfrak{B}_{\bowtie}$, so $h(a) \in D^{\mathfrak{B}}$ turns out to be equivalent to $h_{\bowtie}(\neg\neg a) = \neg\neg h(a) = 1_{\mathfrak{B}_{\bowtie}}$, i. e., $\neg\neg a \in Ker(h_{\bowtie})$. Thus, $(Ker(h))_{\bowtie} = Ker(h_{\bowtie})$. The second equality is an immediate consequence of the former and Proposition 4.8.

[2] By Corollary 4.6, h is a monomorphism iff $Ker(h) = D^{\mathfrak{A}}$. Then, taking into account that $(D^{\mathfrak{A}})_{\bowtie} = \{\neg\neg a \mid a \in D^{\mathfrak{A}}\} = \{\top_{\mathfrak{A}}\} = \{1_{\mathfrak{A}_{\bowtie}}\}$, $D^{\mathfrak{A}} = \{1_{\mathfrak{A}_{\bowtie}}\}^{\bowtie}$ and the previous Item, we arrive at

$$Ker(h) = D^{\mathfrak{A}} \iff Ker(h_{\bowtie}) = (Ker(h))_{\bowtie} = (D^{\mathfrak{A}})_{\bowtie} = \{1_{\mathfrak{A}_{\bowtie}}\},$$

i. e., $h_{\bowtie}$ is a monomorphism (this fact is proved in a standard way). □

Proposition 4.14. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be* BK*-lattices. Then there exists a homomorphism (monomorphism) $f : \mathfrak{A} \to \mathfrak{B}$ iff there exists a homomorphism (monomorphism) $g : \mathfrak{A}_{\bowtie} \to \mathfrak{B}_{\bowtie}$ with $g(\nabla_l(\mathfrak{A})) \subseteq \nabla_l(\mathfrak{B})$ and $g(\Delta_l(\mathfrak{A})) \subseteq \Delta_l(\mathfrak{B})$.*

Proof. First of all, recall that

$$\nabla_l(\mathfrak{A}) = \{e_{\bowtie}(a \vee \sim a) \mid a \in A\} \quad \text{and} \quad \Delta_l(\mathfrak{A}) = \{e_{\bowtie}(a \wedge \sim a) \mid a \in A\}$$

(and the same is for $\mathfrak{B}$ in place of $\mathfrak{A}$).

[⇒] In case $f : \mathfrak{A} \to \mathfrak{B}$ is a homomorphism, we take $g := f_{\bowtie}$. Then, since any element of $\nabla_l(\mathfrak{A})$ has the form $e_{\bowtie}(a \vee \sim a) = \neg\neg(a \vee \sim a)$ for some $a \in A$, and similarly for $\nabla_l(\mathfrak{B})$, we have

$$g(\neg\neg(a \vee \sim a)) = \neg\neg f(a \vee \sim a) = \neg\neg(f(a) \vee \sim f(a)) \in \nabla_l(\mathfrak{A}),$$

that is, $g(\nabla_l(\mathfrak{A})) \subseteq \nabla_l(\mathfrak{B})$. Analogously, one proves the inclusion $g(\Delta_l(\mathfrak{A})) \subseteq \Delta_l(\mathfrak{B})$. Now if f is also injective (i. e., is a monomorphism), then $g = f_{\bowtie}$ is a monomorphism by Proposition 4.13 (2).

$\boxed{\Leftarrow}$ Assume $g : \mathfrak{A}_{\bowtie} \to \mathfrak{B}_{\bowtie}$ is a homomorphism such that $g(\nabla_l(\mathfrak{A})) \subseteq \nabla_l(\mathfrak{B})$ and $g(\Delta_l(\mathfrak{A})) \subseteq \Delta_l(\mathfrak{B})$. Consider three more homomorphisms, namely

$$g^{\bowtie} : (\mathfrak{A}_{\bowtie})^{\bowtie} \to (\mathfrak{B}_{\bowtie})^{\bowtie}, \quad \iota_{\mathfrak{A}}^{\bowtie} : \mathfrak{A} \to (\mathfrak{A}_{\bowtie})^{\bowtie} \quad \text{and} \quad \iota_{\mathfrak{B}}^{\bowtie} : \mathfrak{B} \to (\mathfrak{B}_{\bowtie})^{\bowtie}$$

(note that the last two are injective). Let us show $g^{\bowtie} \circ \iota_{\mathfrak{A}}^{\bowtie}(A) \subseteq \iota_{\mathfrak{B}}^{\bowtie}(B)$. Indeed, due to Proposition 3.2, $\iota_{\mathfrak{A}}^{\bowtie}(\mathfrak{A}) = Tw(\mathfrak{A}_{\bowtie}, \nabla_l(\mathfrak{A}), \Delta_l(\mathfrak{A}))$, so if $(m_1, m_2) \in \iota_{\mathfrak{A}}^{\bowtie}(\mathfrak{A})$, then $m_1 \vee m_2 \in \nabla_l(\mathfrak{A})$ and $m_1 \vee m_2 \in \Delta_l(\mathfrak{A})$. Combining these two with the assumptions about g, we obtain

$$\begin{aligned} g(m_1) \vee g(m_2) &= g(m_1 \vee m_2) \in \nabla_l(\mathfrak{B}), \\ g(m_1) \wedge g(m_2) &= g(m_1 \wedge m_2) \in \Delta_l(\mathfrak{B}). \end{aligned}$$

By Proposition 3.2 again, one gets $\iota_{\mathfrak{B}}^{\bowtie}(\mathfrak{B}) = Tw(\mathfrak{B}_{\bowtie}, \nabla_l(\mathfrak{B}), \Delta_l(\mathfrak{B}))$, whence $g^{\bowtie}(m_1, m_2) = (g(m_1), g(m_2)) \in \iota_{\mathfrak{A}}^{\bowtie}(\mathfrak{A})$. What it means is that the mapping $f := (\iota_{\mathfrak{A}}^{\bowtie})^{-1} \circ g^{\bowtie} \circ \iota_{\mathfrak{A}}^{\bowtie}$ is well-defined, and it certainly turns out to be a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. Finally, if g is injective, then so is $g^{\bowtie}$ and, therefore, the above f will be a monomorphism. □

We conclude with a useful result on the interrelations between quotients of modal algebras and those of twist-structures.

Given $\theta \in Con(\mathfrak{M})$, $\nabla \in \mathscr{F}^{\Box}(\mathfrak{M})$ and $\Delta \in \mathscr{I}^{\Diamond}(\mathfrak{M})$, it is easy to check that $\nabla_{/\theta} := \{m_{/\theta} \mid m \in \nabla\} \in \mathscr{F}^{\Box}(\mathfrak{M}_{/\theta})$ and $\Delta_{/\theta} := \{m_{/\theta} \mid m \in \Delta\} \in \mathscr{I}^{\Diamond}(\mathfrak{M}_{/\theta})$. Hence one is free to introduce $Tw(\mathfrak{M}_{/\theta}, \nabla_{/\theta}, \Delta_{/\theta})$.

On the other hand, for $\mathfrak{A} \in S^{\bowtie}(\mathfrak{M})$ (and θ as above), the preimage $\pi_{\mathfrak{A}}^{-1}(F)$, where $F = \Phi_\theta$, is an sffk on $\mathfrak{A}$, due to Proposition 4.7. Let us temporarily denote the corresponding congruence on $\mathfrak{A}$ (see Corollary 4.5) by $\theta^{\bowtie}$.

Proposition 4.15. *Assume $\mathfrak{M}$ is a modal algebra, θ is a congruence on $\mathfrak{M}$, and $\mathfrak{A} = Tw(\mathfrak{M}, \nabla, \Delta)$. Then $\mathfrak{A}_{/(\theta^{\bowtie})} \cong Tw(\mathfrak{M}_{/\theta}, \nabla_{/\theta}, \Delta_{/\theta})$.*

Proof. Define $h : \mathfrak{A}_{/(\theta^{\bowtie})} \to (\mathfrak{M}_{/\theta})^{\bowtie}$ as follows: for each $(m_1, m_2) \in A$,

$$h\left((m_1, m_2)_{/(\theta^{\bowtie})}\right) := \left(m_{1/\theta}, m_{2/\theta}\right).$$

This definition is correct and the resulting h is injective, since, taking F to be Φ_θ,

$$(m_1, m_2)_{/(\theta^{\bowtie})} = (m_3, m_4)_{/(\theta^{\bowtie})} \iff (m_1, m_2) \Leftrightarrow (m_3, m_4) \in \pi_{\mathfrak{A}}^{-1}(F)$$

(cf. Corollary 4.5), and it is not hard to verify that the latter is equivalent to $(m_1 \leftrightarrow m_3) \wedge (m_2 \leftrightarrow m_4) \in F$, i. e., both (m_1, m_3) and (m_2, m_4) belong to $\Theta_F = \theta$, that is, $\left(m_{1/\theta}, m_{2/\theta}\right) = \left(m_{3/\theta}, m_{4/\theta}\right)$.

Next, it is straightforward that h is a homomorphism (and, therefore, a monomorphism). For instance,

$$h\left((m_1, m_2)_{/(\theta^{\bowtie})} \wedge (m_1, m_2)_{/(\theta^{\bowtie})}\right) \;=\; h\left(((m_1, m_2) \wedge (m_1, m_2))_{/(\theta^{\bowtie})}\right) \;=$$
$$h\left((m_1 \wedge m_3, m_2 \vee m_4)_{/(\theta^{\bowtie})}\right) \;=\; \left((m_1 \wedge m_3)_{/\theta}, (m_2 \vee m_4)_{/\theta}\right) \;=$$
$$\left(m_{1/\theta}, m_{2/\theta}\right) \wedge \left(m_{3/\theta}, m_{4/\theta}\right) \;=\; h\left((m_1, m_2)_{/(\theta^{\bowtie})}\right) \wedge h\left((m_3, m_4)_{/(\theta^{\bowtie})}\right),$$

and similarly for all the other connectives (modal and non-modal ones).

To ensure that h provides the desired isomorphism, it remains to establish

$$\mathfrak{B} \;:=\; h\left(\mathfrak{A}_{/(\theta^{\bowtie})}\right) \;=\; Tw\left(\mathfrak{M}_{/\theta}, \nabla_{/\theta}, \Delta_{/\theta}\right),$$

or, alternatively, that $\nabla(\mathfrak{B}) = \nabla_{/\theta}$ and $\Delta(\mathfrak{B}) = \Delta_{/\theta}$. Obviously, we have $B = \left\{\left(m_{1/\theta}, m_{2/\theta}\right) \mid (m_1, m_2) \in A\right\}$, whence

$$\nabla(\mathfrak{B}) \;=\; \left\{m_{1/\theta} \vee m_{2/\theta} \mid (m_1, m_2) \in A\right\} \;=\; \left\{(m_1 \vee m_2)_{/\theta} \mid (m_1, m_2) \in A\right\},$$

which trivially coincides with $\nabla_{/\theta}$. Analogously, the equality between $\Delta(\mathfrak{B})$ and $\Delta_{/\theta}$ can be proved. □

In conclusion, I would like to thank the anonymous referee for useful remarks and, particularly, for drawing my attention to the paper [21], which is closely related to the present investigation. In effect, it seems that many of the above results can be generalised to the modal logic of [21] supplied with the respective algebraic semantics — and this may be an interesting direction for future research.

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Stanislav O. Speranski
Sobolev Institute of Mathematics,
4 Koptyug ave., 630090, Novosibirsk, Russia
Novosibirsk State University,
2 Pirogova st., 630090, Novosibirsk, Russia
E-mail address: katze.tail@gmail.com