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APPLICATIONS OF (PROXIMAL) TAIMANOV THEOREM

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Abstract. Let $P^*(X)$ be the algebra of bounded, real-valued proximally continuous functions on an EF-proximity space (X, δ), where X is a dense subspace of a Tychonoff topological space S. Mattson obtained several conditions which are equivalent to the following property: every member of $P^*(X)$ has a continuous extension to S. In this paper, we generalize the above problem to L-proximity via proximal Taimanov theorem when S is a T_1 space.

Keywords: Taimanov Theorem, EF-proximity, L-proximity, extension of continuous functions, bunch, Wallman topology.

1. Introduction

A continuous extension of continuous functions from dense subspaces is an important topic in topology/analysis and has a vast literature. Taimanov [11] proved the following valuable result: "*Let S be a T_1-space, X a dense subspace of S, and Y a compact Hausdorff space. A continuous function f on X to Y admits a continuous extension over S if and only if for all disjoint closed subsets A, B of Y, the relation $(f^{-1}(A))^- \cap (f^{-1}(B))^- = \emptyset$. From this result, a theorem of Smirnov* [10] *is easily proved, as well as a theorem of Vulih* [12]. *A final corollary is a special case of a theorem of Katětov* [3]" [2].

Proximal and nearness extensions of Taimanov theorem [1], [8] generalize many special results showing thereby the beauty and importance of Taimanov theorem.

Let us see how Taimanov theorem is connected to proximity. Define fine Leader–Lodato or L-proximity δ_0 on S and its subspace proximity δ on subsets A, B of X by:

$A \,\delta\, B$ in X if and only if closures of A, B in S intersect.

Since Y is compact Hausdorff the fine proximity η_0 on Y is EF or Efremovič.

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[11] can now be expressed as:

(TT) Taimanov Theorem.

Let S be a T_1-space, X a dense subspace of S, and Y a compact Hausdorff space. Let X have L-proximity δ, which is the subspace proximity induced by δ_0 on S. A continuous function f on X to Y admits a continuous extension over S if and only if $f : (X, \delta) \to (Y, \eta_0)$ is proximally continuous.

By replacing the condition of compactness on Y by Tychonoff, we get the

(PTT) Proximal Taimanov Theorem. [1]

Let S be a T_1-space, X a dense subspace of S, and Y a Tychonoff space with EF-proximity η. Let X have an L-proximity δ induced by fine L-proximity δ_0 on S. Then a continuous function f on X to Y admits a continuous extension over S to the Smirnov compactification Y^* of Y if and only if $f : (X, \delta) \to (Y, \eta)$ is proximally continuous.

Above result includes, as special cases, almost all results in extension of continuous functions from dense subspaces [8].

2. Preliminaries

An L-proximity δ on a nonempty set X is defined as follows. For subsets A, B, C of X and $x, y \in X$ we have:

(a) $A \,\delta\, B \to B \,\delta\, A$, (symmetry)
(b) $A \,\delta\, B \to A \neq \emptyset$ and $B \neq \emptyset$,
(c) $A \cap B \neq \emptyset \to A \,\delta\, B$,
(d) $A \,\delta\, (B \cup C) \Leftrightarrow A \,\delta\, B$ or $A \,\delta\, C$, (union axiom)
(e) $A \,\delta\, B$ and $\{b\} \,\delta\, C$ for each $b \in B \to A \,\delta\, C$, ($L$-axiom)
(f) $\{x\} \,\delta\, \{y\} \to x = y$.

Every T_1-space X has a compatible *fine* L-proximity δ_0, defined by

$$A \,\delta_0\, B \Leftrightarrow \mathrm{cl}\, A \cap \mathrm{cl}\, B \neq \emptyset.$$

That is $A \,\delta_0\, B \to A \,\delta\, B$ for any compatible L-proximity δ. Further in EF-proximity, (e) is replaced by a stronger condition [9]:

(g) $A \underline{\delta} B \Rightarrow$ there is a $C \subset X$ such that $A \underline{\delta} C$ and $(X - C) \underline{\delta} B$.

3. Extension of functions

Let $P^*(X)$ be the algebra of bounded, real-valued proximally continuous functions on an L-proximity space (X, δ), where X is a dense subspace of a T_1 topological space S. Let δ be induced by fine L-proximity δ_0 on S. If $f \in P^*(X)$, then the closure of $f(X)$, being bounded, is compact in $\mathbb{R}$. Hence by Taimanov theorem (TT), f has an extension $F \in P^*(S)$. It is easy to see that the result follows even if S has a proximity α which induces proximity on X finer than its proximity δ. Hence we have the following result:

(3.1) Theorem.

Let $P^*(X)$ be the algebra of bounded, real-valued proximally continuous functions on an L-proximity space (X, δ), where X is a dense subspace of a T_1 topological space S which has a compatible L-proximity α. Then the following are equivalent:

(i) every $f \in P^*(X)$ has an extension $F \in P^*(S)$;
(ii) α induces a finer proximity than δ on X;

(iii) $A \underline{\delta} B$ *in* X *implies closures of* A, B *in* S *are disjoint.*

Now we generalize Mattson's result. Let $P(X)$ be the algebra of real-valued proximally continuous functions on an L-proximity space (X, δ), where X is a dense subspace of a T_1 topological space S. Let δ be induced by fine L-proximity δ_0 on S. Then by proximal Taimanov theorem (PTT), each $f \in P(X)$, has an extension $F : P(S) \to \mathbb{R}^*$, the Stone-Čech compactification of $\mathbb{R}$. As in (3.1) the result follows even if S has a proximity α which induces proximity on X finer than its proximity δ.

(3.2) Theorem.

Let $P(X)$ *be the algebra of real-valued proximally continuous functions on an* L*-proximity space* (X, δ)*, where* X *is a dense subspace of a* T_1 *topological space* S *which has a compatible* L*-proximity* α*. Then every* $f \in P(X)$ *has an extension* $F : P(S) \to \mathbb{R}^*$*, the Stone-Čech compactification of* $\mathbb{R}$ *if and only if* α *induces on* X *a finer* L*-proximity than* δ*.*

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