APPLICATIONS OF (PROXIMAL) TAIMANOV THEOREM

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Abstract. Let $P^*(X)$ be the algebra of bounded, real-valued proximally continuous functions on an $EF$-proximity space $(X, \delta)$, where $X$ is a dense subspace of a Tychonoff topological space $S$. Mattson obtained several conditions which are equivalent to the following property: every member of $P^*(X)$ has a continuous extension to $S$. In this paper, we generalize the above problem to $L$-proximity via proximal Taimanov theorem when $S$ is a $T_1$ space.

Keywords: Taimanov Theorem, $EF$-proximity, $L$-proximity, extension of continuous functions, bunch, Wallman topology.

1. Introduction

A continuous extension of continuous functions from dense subspaces is an important topic in topology/analysis and has a vast literature. Taimanov [11] proved the following valuable result: “Let $S$ be a $T_1$-space, $X$ a dense subspace of $S$, and $Y$ a compact Hausdorff space. A continuous function $f$ on $X$ to $Y$ admits a continuous extension over $S$ if and only if for all disjoint closed subsets $A, B$ of $Y$, the relation $(f^{-1}(A))^- \cap (f^{-1}(B))^- = \emptyset$. From this result, a theorem of Smirnov [10] is easily proved, as well as a theorem of Vrát [12]. A final corollary is a special case of a theorem of Katětov [3][2].

Proximal and nearness extensions of Taimanov theorem [1], [8] generalize many special results showing thereby the beauty and importance of Taimanov theorem.

Let us see how Taimanov theorem is connected to proximity. Define fine Leader–Lodato or $L$-proximity $\delta_0$ on $S$ and its subspace proximity $\delta$ on subsets $A, B$ of $X$ by:

$A \delta B$ in $X$ if and only if closures of $A, B$ in $S$ intersect.

Since $Y$ is compact Hausdorff the fine proximity $\eta_0$ on $Y$ is $EF$ or Efremovič.
[11] can now be expressed as:

(TT) Taimanov Theorem.
Let $S$ be a $T_1$-space, $X$ a dense subspace of $S$, and $Y$ a compact Hausdorff space.
Let $X$ have $L$-proximity $\delta$, which is the subspace proximity induced by $\delta_0$ on $S$. A continuous function $f$ on $X$ to $Y$ admits a continuous extension over $S$ if and only if $f : (X, \delta) \to (Y, \eta_0)$ is proximally continuous.

By replacing the condition of compactness on $Y$ by Tychonoff, we get the

(PTT) Proximal Taimanov Theorem. [1]
Let $S$ be a $T_1$-space, $X$ a dense subspace of $S$, and $Y$ a Tychonoff space with EF-proximity $\eta$. Let $X$ have an $L$-proximity $\delta$ induced by fine $L$-proximity $\delta_0$ on $S$. Then a continuous function $f$ on $X$ to $Y$ admits a continuous extension over $S$ to the Smirnov compactification $Y^*$ of $Y$ if and only if $f : (X, \delta) \to (Y, \eta_0)$ is proximally continuous.

Above result includes, as special cases, almost all results in extension of continuous functions from dense subspaces [8].

2. Preliminaries

An $L$-proximity $\delta$ on a nonempty set $X$ is defined as follows. For subsets $A, B, C$ of $X$ and $x, y \in X$ we have:

(a) $A \delta B \Rightarrow B \delta A$, (symmetry)
(b) $A \delta B \Rightarrow A \neq \emptyset$ and $B \neq \emptyset$,
(c) $A \cap B \neq \emptyset \Rightarrow A \delta B$,
(d) $A \delta (B \cup C) \Leftrightarrow A \delta B$ or $A \delta C$, (union axiom)
(e) $A \delta B$ and $\{b\} \delta C$ for each $b \in B \Rightarrow A \delta C$, (L-axiom)
(f) $\{x\} \delta \{y\} \Rightarrow x = y$.

Every $T_1$-space $X$ has a compatible fine $L$-proximity $\delta_0$, defined by

$$A \delta_0 B \Leftrightarrow \overline{A \cap B} \neq \emptyset.$$  

That is $A \delta_0 B \Rightarrow A \delta B$ for any compatible $L$-proximity $\delta$. Further in EF-proximity, (e) is replaced by a stronger condition [9]:

(g) $A \delta B \Rightarrow$ there is a $C \subset X$ such that $A \delta C$ and $(X - C) \delta B$.

3. Extension of functions

Let $P^*(X)$ be the algebra of bounded, real-valued proximally continuous functions on an $L$-proximity space $(X, \delta)$, where $X$ is a dense subspace of a $T_1$ topological space $S$. Let $\delta$ be induced by fine $L$-proximity $\delta_0$ on $S$. If $f \in P^*(X)$, then the closure of $f(X)$, being bounded, is compact in $\mathbb{R}$. Hence by Taimanov theorem (TT), $f$ has an extension $F \in P^*(S)$. It is easy to see that the result follows even if $S$ has a proximity $\alpha$ which induces proximity on $X$ finer than its proximity $\delta$. Hence we have the following result:

(3.1) Theorem.
Let $P^*(X)$ be the algebra of bounded, real-valued proximally continuous functions on an $L$-proximity space $(X, \delta)$, where $X$ is a dense subspace of a $T_1$ topological space $S$ which has a compatible $L$-proximity $\alpha$. Then the following are equivalent:

(i) every $f \in P^*(X)$ has an extension $F \in P^*(S)$;
(ii) $\alpha$ induces a finer proximity than $\delta$ on $X$;
(iii) \( A \triangle B \) in \( X \) implies closures of \( A, B \) in \( S \) are disjoint.

Now we generalize Mattson’s result. Let \( P(X) \) be the algebra of real-valued proximally continuous functions on an \( L \)-proximity space \( (X, \delta) \), where \( X \) is a dense subspace of a \( T_1 \) topological space \( S \). Let \( \delta \) be induced by fine \( L \)-proximity \( \delta_0 \) on \( S \). Then by proximal Taimanov theorem (PTT), each \( f \in P(X) \), has an extension \( F : P(S) \to R^* \), the Stone-\v{C}ech compactification of \( R \). As in (3.1) the result follows even if \( S \) has a proximity \( \alpha \) which induces proximity on \( X \) finer than its proximity \( \delta \).

(3.2) Theorem.
Let \( P(X) \) be the algebra of real-valued proximally continuous functions on an \( L \)-proximity space \( (X, \delta) \), where \( X \) is a dense subspace of a \( T_1 \) topological space \( S \) which has a compatible \( L \)-proximity \( \alpha \). Then every \( f \in P(X) \) has an extension \( F : P(S) \to R^* \), the Stone-\v{C}ech compactification of \( R \) if and only if \( \alpha \) induces on \( X \) a finer \( L \)-proximity than \( \delta \).