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SUBEXTENSIONS FOR A PERMUTATION $\mathrm{PSL}_2(q)$-MODULE

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Abstract. Using cohomological methods, we solve the problem of embedding $\mathrm{SL}_2(q)$ into the permutation wreath product for the permutation $\mathrm{PSL}_2(q)$-module in characteristic 2 that arises from the action on the projective line. We also prove some useful auxiliary results.

Keywords: finite simple groups, permutation module, group cohomology.

1. Introduction

We denote by $\mathbb{F}_q$ a finite field of order q and by $\mathbb{Z}_n$ a cyclic group of order n.

Let q be an odd prime power and let $G = \mathrm{PSL}_2(q)$. From the Universal Embedding Theorem [4, Theorem 2.6.A], it follows that the regular wreath product $\mathbb{Z}_2 \wr G$ contains a subgroup isomorphic to $\mathrm{SL}_2(q)$. It is of interest to know if the same is true for a permutation wreath product that is not necessarily regular. In particular, let ρ be the natural permutation representation of G of degree $q+1$ on the projective line over $\mathbb{F}_q$. The following problem arose in the research [8].

Problem 1. *Does the permutation wreath product $\mathbb{Z}_2 \wr_\rho G$ contain a subgroup isomorphic to $\mathrm{SL}_2(q)$?*

Although stated in purely group-theoretic terms, this problem is cohomological in nature. In the next section, we reformulate a generalized version of this question as an assertion about a homomorphism between second cohomology groups of group modules. We then apply some basic theory to obtain the following solution to Problem 1.

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Theorem 1. *In the above notation,* $\mathrm{SL}_2(q)$ *is embedded into* $\mathbb{Z}_2 \wr_\rho G$ *if* $q \equiv -1 \pmod 4$ *and is not embedded if* $q \equiv 1 \pmod 4$*.*

The case $q \equiv 1 \pmod 4$ of Problem 1 can also be treated without applying cohomological methods. In the last section, we present an alternative proof which was kindly provided by the anonymous referee.

2. Subextensions for group modules

Let G be a group and let L, M be right G-modules. Let

$$0 \to L \to M \tag{1}$$

and

$$1 \to M \to E \xrightarrow{\pi} G \to 1$$

be exact sequences of modules and groups, where the conjugation action of E on M agrees with the G-module structure, i. e. $m^e = m \cdot \pi(e)$ for all $m \in M$ and $e \in E$, and we identify M with its image in E. Then we call E an *extension of M by G*. It is natural to ask if there is a subgroup $S \leqslant E$ such that

$$S \cap M = L, \qquad SM = E, \tag{2}$$

where we implicitly identify L with its image in M. A subgroup S with these properties is itself an extension of L by G, and will thus be called a *subextension* of E that corresponds to the embedding (1). The classification of all such subextensions of E (whenever they exist) up to equivalence is also of interest.

Recall that extensions S_1, S_2 of L by G are *equivalent* if there is a homomorphism α that makes the diagram

$$\begin{tikzcd} & S_1 \arrow[dr] \arrow[dd, dotted, "\alpha"] & \\ L \arrow[ur] \arrow[dr] & & G \\ & S_2 \arrow[ur] & \end{tikzcd}$$

commutative. It is known [6] that the equivalence classes of such extensions are in a one-to-one correspondence with (thus are *defined by*) the elements of the second cohomology group $H^2(G, L)$. Furthermore, the sequence (1) gives rise to a homomorphism

$$H^2(G, L) \xrightarrow{\varphi} H^2(G, M). \tag{3}$$

The following assertion is nothing more than an interpretation of this homomorphism in group-theoretic terms.

Lemma 2. *Let L, M be G-modules and E an extension as specified above. Let $\bar{\gamma} \in H^2(G, M)$ be the element that defines E. Then the set of elements of $H^2(G, L)$ that define the subextensions S of E corresponding to the embedding* (1) *coincides with $\varphi^{-1}(\bar{\gamma})$, where φ is the induced homomorphism* (3)*. In particular, E has such a subextension S if and only if $\bar{\gamma} \in \operatorname{Im} \varphi$.*

Proof. Let S be a required subextension of E. Choose a transversal $\tau : G \to S$ of L in S. Then, for all $g_1, g_2 \in G$, we have $\tau(g_1)\tau(g_2) = \tau(g_1 g_2)\beta(g_1, g_2)$ for a 2-cocycle $\beta \in Z^2(G, L)$ and the element $\bar{\beta} = \beta + B^2(G, L)$ of $H^2(G, L)$ defines S. Let γ be the composition of β with the embedding (1). Then $\gamma \in Z^2(G, M)$ arises from the same transversal τ (composed with the embedding $S \to E$), hence $\gamma + B^2(G, M)$ is the element of $H^2(G, M)$ that defines E which is $\bar{\gamma}$. Therefore, $\varphi(\bar{\beta}) = \bar{\gamma}$.

Conversely, let $\varphi(\bar{\beta}) = \bar{\gamma}$ for some $\bar{\beta} \in H^2(G, L)$. Then there is a representative 2-cocycle $\gamma \in Z^2(G, M)$ whose values lie in L and which, when viewed as a map $G \times G \to L$, is a 2-cocycle $\beta \in Z^2(G, L)$ representative for $\bar{\beta}$. Now E can be identified with the set of pairs (g, m) with $g \in G$, $m \in M$ subject to the multiplication

$$(g_1, m_1)(g_2, m_2) = (g_1 g_2, m_1 \cdot g_2 + m_2 + \beta(g_1, g_2))$$

and, if we set $S = \{(g, m) \mid g \in G, m \in L\}$, then S is clearly a subextension of E defined by $\bar{\beta}$. □

It is known that the zero element of $H^2(G, M)$ defines the split extension (which fact is also a particular case of Lemma 2 with $L = 0$). Therefore, we have

Corollary 3. *Let L, M be G-modules as above and let E be the split extension of M by G. Then the following holds.*

(*i*) *The subextensions of E that correspond to the embedding (1) are defined by the elements of* $\operatorname{Ker}\varphi$, *where φ is the induced homomorphism* (3).

(*ii*) *If $H^2(G, M) = 0$ then every extension of L by G is a subextension of E.*

3. Notation and auxiliary results

Basic facts of homological algebra can be found in [6, 10]. For abelian groups A and B, we denote $\mathrm{Hom}(A, B) = \mathrm{Hom}_{\mathbb{Z}}(A, B)$ and $\mathrm{Ext}(A, B) = \mathrm{Ext}^1_{\mathbb{Z}}(A, B)$.

Lemma 4 (The Universal Coefficient Theorem for Cohomology, [6, Theorem 3])**.** *For all $i \geqslant 1$ and every trivial G-module A,*

$$H^i(G, A) \cong \mathrm{Hom}(H_i(G, \mathbb{Z}), A) \oplus \mathrm{Ext}(H_{i-1}(G, \mathbb{Z}), A).$$

The following corollary to Lemma 4 can also be proved independently.

Lemma 5. *For a trivial G-module A, we have $H^1(G, A) \cong \mathrm{Hom}(G, A)$.*

Lemma 6 (Shapiro's lemma, [10, §6.3])**.** *Let $H \leqslant G$ with $|G : H|$ finite. If V is an H-module and $i \geqslant 0$ then $H^i(G, V^G) \cong H^i(H, V)$, where V^G is the induced G-module.*

Given a group G, we denote by $M(G)$ the Schur multiplier of G. If A is a finite abelian group and p a prime then $A_{(p)}$ denotes the p-primary component of A.

Lemma 7. [7, Theorem 25.1] *Let G be a finite group, p a prime, and let $P \in \mathrm{Syl}_p(G)$. Then $M(G)_{(p)}$ is isomorphic to a subgroup of $M(P)$.*

Lemma 8. [1, Proposition III.10.1] *Let G be a finite group and let M be finite G-module such that $(|G|, |M|) = 1$. Then $H^i(G, M) = 0$ for all $i \geqslant 1$.*

4. Projective action of $\mathrm{PSL}_2(q)$

We denote $G = \mathrm{PSL}_2(q)$ for q odd. Let $\mathcal{P}$ be the projective line over $\mathbb{F}_q$ and let V be the permutation $\mathbb{F}_2 G$-module that corresponds to the natural action of G on $\mathcal{P}$. The sum of the basis vectors of V, which are permuted by G, spans a 1-dimensional submodule I, and we have the exact sequence

$$0 \to I \to V \to W \to 0, \qquad (4)$$

where $W \cong V/I$. The following result clarifies the composition structure of the module V. Let $k = \overline{\mathbb{F}}_2$ be the algebraic closure of $\mathbb{F}_2$.

Lemma 9. [2, Lemma 1.6] *In the notation above, I is the unique minimal submodule of V and W has a unique maximal submodule U such that*

$$0 \to U \to W \to I \to 0 \tag{5}$$

is a nonsplit short exact sequence. Moreover, $U \otimes k = U_+ \oplus U_-$, where U_+ and U_- are the two nontrivial absolutely irreducible kG-modules in the principal 2-block of G.

Using the knowledge of the Schur multiplier of G we can determine $H^2(G, I)$.

Lemma 10. *Let q be an odd prime power. For $\mathrm{PSL}_2(q)$ acting trivially on $\mathbb{Z}_2$, we have*

$$H^2(\mathrm{PSL}_2(q), \mathbb{Z}_2) \cong \mathbb{Z}_2.$$

Proof. Applying Lemma 4 for the trivial action of $G = \mathrm{PSL}_2(q)$ on $\mathbb{Z}_2$, we have

$$H^2(G, \mathbb{Z}_2) \cong \mathrm{Hom}(H_2(G, \mathbb{Z}), \mathbb{Z}_2) \oplus \mathrm{Ext}(H_1(G, \mathbb{Z}), \mathbb{Z}_2). \tag{6}$$

Since $H_2(G, \mathbb{Z}) = M(G)$, according to [7, Theorem 25.7], we have

$$H_2(G, \mathbb{Z}) = \begin{cases} \mathbb{Z}_2, & q \neq 9, \\ \mathbb{Z}_6, & q = 9. \end{cases}$$

It follows that $\mathrm{Hom}(H_2(G, \mathbb{Z}), \mathbb{Z}_2) \cong \mathbb{Z}_2$. Since the first integral homology group $H_1(G, \mathbb{Z})$ is isomorphic to the abelianization G/G', we have

$$H_1(G, \mathbb{Z}) = \begin{cases} 0, & q \neq 3, \\ \mathbb{Z}_3, & q = 3. \end{cases}$$

Therefore, we always have $\mathrm{Ext}(H_1(G, \mathbb{Z}), \mathbb{Z}_2) = 0$, and the claim follows. $\square$

We now determine the group $H^2(G, V)$.

Lemma 11. *Let V be the above-defined permutation module. Then we have*

$$H^2(G, V) \cong \begin{cases} 0, & q \equiv -1 \pmod 4, \\ \mathbb{Z}_2, & q \equiv 1 \pmod 4. \end{cases}$$

Proof. Since the action of G on $\mathcal{P}$ is transitive, we have $V \cong T^G$, where T is the principal $\mathbb{F}_2H$-module for a point stabilizer $H \leqslant G$. By Lemma 6, $H^2(G, V) \cong H^2(H, T)$. We have $H \cong \mathbb{F}_q \rtimes \mathbb{Z}_{(q-1)/2}$, a Frobenius group. If $q \equiv -1 \pmod 4$, the order $|H|$ is odd. By Lemma 8, $H^2(H, T) = 0$. Suppose that $q \equiv 1 \pmod 4$. Let $P \in \mathrm{Syl}_2(H)$. Lemma 7 implies that $H_2(H, \mathbb{Z})_{(2)}$ is a subgroup of $H_2(P, \mathbb{Z})$ which is 0, since cyclic groups have trivial Schur multiplier. Therefore,

$$\mathrm{Hom}(H_2(H, \mathbb{Z}), \mathbb{Z}_2) = \mathrm{Hom}(H_2(H, \mathbb{Z})_{(2)}, \mathbb{Z}_2) = 0.$$

Note also that $H_1(H, \mathbb{Z}) = \mathbb{Z}_{(q-1)/2}$. Now, H acts trivially on $T \cong \mathbb{Z}_2$, so we can use again the universal coefficient formula (6) to obtain

$$H^2(H, T) = \mathrm{Ext}(\mathbb{Z}_{(q-1)/2}, \mathbb{Z}_2) \cong \mathbb{Z}_2,$$

since $(q-1)/2$ is even by assumption. This completes the proof. $\square$

5. The case $q \equiv -1 \pmod 4$

We begin the proof of Theorem 1. Given a permutation representation ρ of $G = \mathrm{PSL}_2(q)$ as in the introduction, observe that the permutation wreath product $E = \mathbb{Z}_2 \wr_\rho G$ is isomorphic to the split extension of the permutation $\mathbb{F}_2G$-module V by G. Since $\mathrm{SL}_2(q)$ is an extension of the principal $\mathbb{F}_2G$ module I by G, and I is the trivial submodule of V by Lemma 6 with $i = 0$, it follows that Problem 1 is equivalent to the question of whether $\mathrm{SL}_2(q)$ is a subextension of E that corresponds to the embedding $I \to V$. Corollary 3(i) implies that such subextensions are defined by the elements of $\operatorname{Ker}\varphi$, where φ is the induced homomorphism

$$H^2(G, I) \xrightarrow{\varphi} H^2(G, V). \tag{7}$$

For $q \equiv -1 \pmod 4$, we have $H^2(G, V) = 0$ by Lemma 11. Consequently, $\mathrm{SL}_2(q)$, which is an extension of I by G, is a subextension of E by Corollary 3(ii).

6. The case $q \equiv 1 \pmod 4$

The above argument does not clarify what $\operatorname{Ker}\varphi$ is if $q \equiv 1 \pmod 4$, because in this case the homomorphism (7) becomes

$$\mathbb{Z}_2 \xrightarrow{\varphi} \mathbb{Z}_2$$

by Lemmas 10 and 11. (Of course, the fact that $H^2(G, I)$ is nonzero also follows from the existence of groups $\mathrm{SL}_2(q)$.)

We will consider the long sequence

$$H^1(G, I) \to H^1(G, V) \to H^1(G, W) \xrightarrow{\delta} H^2(G, I) \xrightarrow{\varphi} H^2(G, V) \tag{8}$$

induced by (4). Since this sequence is exact, it follows that $\operatorname{Im}\delta = \operatorname{Ker}\varphi$, and we might as well study the connecting homomorphism δ.

Lemma 12. *In the above notation, we have*

(i) $H^1(G, I) = 0$;

(ii) $H^1(G, V) \cong \begin{cases} 0, & q \equiv -1 \pmod 4, \\ \mathbb{Z}_2, & q \equiv 1 \pmod 4. \end{cases}$

Proof. (i) This holds, since $H^1(G, I) \cong \operatorname{Hom}(G, I) = \{0\}$ by Lemma 5.

(ii) We again use the fact that $V \cong T^G$ as in the proof of Lemma 11, where T is the principal $\mathbb{F}_2H$-module for the Borel subgroup $H \leqslant G$. We have

$$H^1(G, V) \cong H^1(H, T) \cong \operatorname{Hom}(H, T)$$

by Lemmas 5 and 6. Assume that $q \equiv -1 \pmod 4$. (Although we have covered this case in the previous sections, we still consider it for the sake of completeness.) Then $|H|$ is odd, and so $|\operatorname{Hom}(H, T)| = \{0\}$. Let $q \equiv 1 \pmod 4$. Then

$$\operatorname{Hom}(H, T) = \operatorname{Hom}(\mathbb{Z}_{(q-1)/2}, \mathbb{Z}_2) = \mathbb{Z}_2,$$

since $(q-1)/2$ is even by assumption. The claim follows. □

Lemma 12 implies that

$$\operatorname{Im}\delta \cong H^1(G, W)/H^1(G, V) \tag{9}$$

and so it remains to determine $H^1(G, W)$. To this end, we consider the long exact sequence

$$H^0(G, W) \to H^0(G, I) \to H^1(G, U) \to H^1(G, W) \to H^1(G, I) \tag{10}$$

induced by (5). We have $H^1(G,I) = 0$ by Lemma 12 and $H^0(G,W) = 0$ by Lemma 9. Therefore,

$$H^1(G,W) \cong H^1(G,U)/H^0(G,I), \tag{11}$$

and, since $H^0(G,I) = \mathbb{Z}_2$ is known, in view of Lemma 9 it remains to determine $H^1(G,U)$. Observe that $H^1(G,U) \cong \mathbb{Z}_2^m$ for some $m \geqslant 0$.

The first cohomology groups $H^1(G,U_\pm)$, where the modules $U_\pm$ are as defined in Lemma 9, can be calculated using the structure of the principal indecomposable modules in the principal 2-block of G described in [5]. This was done in [9]. Other calculations are announced in [3]. All these sources imply the following

Lemma 13. $\dim_k H^1(G,U_\pm) = 1$.

Consequently, we have

$$H^1(G,U)\otimes k \cong H^1(G,U\otimes k) \cong H^1(G,U_+ \oplus U_-) \cong k\oplus k,$$

which yields $H^1(G,U) \cong \mathbb{Z}_2^2$. Hence, $H^1(G,W) \cong \mathbb{Z}_2$ by (11) and $\operatorname{Im}\delta = 0$ by (9). We see that φ is an isomorphism in this case and the nonzero element of $H^2(G,I)$ which defines $\mathrm{SL}_2(q)$ does not lie in $\operatorname{Ker}\varphi$. Therefore, $\mathrm{SL}_2(q)$ is not a subextension of $V \leftthreetimes G$ by Corollary 3(i). The proof of Theorem 1 is complete.

7. Another proof

Here we present an alternative beautiful proof in the case $q \equiv 1 \pmod 4$ which was kindly proposed by the referee and is included here with his/her permission.

We preserve the above notation. Elements of the permutation $\mathbb{F}_2G$-module V will be written as $\sum_{x\in\mathcal{P}} a_x x$, where $a_x \in \mathbb{F}_2$. In particular, we have $I = \langle t\rangle$, where $t = \sum_{x\in\mathcal{P}} x$.

Suppose that $S = \mathrm{SL}_2(q)$ is a subextension of $V \leftthreetimes G$ corresponding to the embedding $I \to V$. Then $Z(S) = I$ and t is the (unique) involution of S. Let $s \in S$ be of order $q-1$. We have $s^{(q-1)/2} = t$, because $q \equiv 1 \pmod 4$. Since $s \in V \leftthreetimes G$, there are $v \in V$ and $g \in G$ such that $s = gv$. Note that $|g| = (q-1)/2$ and $t = s^{(q-1)/2} = vh$, where

$$h = 1 + g + \ldots + g^{|g|-1}.$$

Let $x \in P$ be a fixed point of $\rho(g)$. (It is readily checked that $\rho(g)$ has precisely two fixed points.) We can write $v = a_x x + w$, where $w = \sum_{y\in\mathcal{P}\setminus\{x\}} a_y y$. Clearly, wh is a linear combination of elements of $\mathcal{P}\setminus\{x\}$ and

$$(a_x x)h = a_x(xh) = a_x\frac{q-1}{2}x = 0.$$

Hence, the coefficient of x in $t = vh$ is zero, a contradiction.

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