S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

http://semr.math.nsc.ru

Том 10, стр. 566–582 (2013)

УДК 517.58
MSC 43A85

MEAN ASYMMETRY OF POLYNOMIALS ON COMPACT HOMOGENEOUS SPACES

V.M. GICHEV

Abstract. Let $M = G/H$ be a homogeneous space of a compact Lie group G, $\mathcal{E}$ be an G-invariant finite dimensional subspace of $L^2_{\mathbb{R}}(M)$, and $\mathcal{S}$ be the unit sphere in it. Set $\eta_a(u) = \int_M (u_+^a(x) - u_-^a(x))\,dx$, where $u_+(x) = \max\{u(x), 0\}$, $u_-(x) = -\min\{u(x), 0\}$. We consider the asymptotic behavior of the variance of the random variable η_a as $a \to \infty$ or $\dim \mathcal{E} \to \infty$ for the uniform distribution of u in $\mathcal{S}$. For instance, if $\mathcal{E}$ is the space of trigonometrical polynomials of degree less or equal to n, then $\mathrm{Var}(\eta_a) \sim \frac{A}{n}$ as $n \to \infty$.

Keywords: compact homogeneous space, sums of Laplace–Beltrami eigenfunctions, defect of symmetry.

1. Introduction

Let $M = G/H$ be a homogeneous space of a compact Lie group G. We shall say that an integrable function f on M is a polynomial if the linear span of its shifts is finite dimensional. Then f is real analytic. Let $\mathcal{E}$ be a finite dimensional linear G-invariant subspace of the space $L^2(M)$ of square integrable real functions, and $\mathcal{S}$ be the unit sphere in $\mathcal{E}$. We assume additionally that

$$1 \perp \mathcal{E}. \tag{1}$$

The homogeneous spaces $M, \mathcal{S}, G$ are equipped with the invariant probability measures. In this paper we consider the random variable

$$\eta_a(u) = \int_M (u_+(p)^a - u_-(p)^a)\,dp,$$

Gichev, V.M., Mean asymmetry of polynomials on compact homogeneous spaces.

The work is supported by the joint project 56 of Siberian and Far East Branches of RAS.
Received April, 16, 2013, published September, 27, 2013.

where $a \geq 0$, $u_+ = \max\{u, 0\}$, $u_- = -\min\{u, 0\}$, and u is uniformly distributed in $\mathcal{S}$. The expectation of η_a is always zero. If u_+ and u_- are equidistributed (we shall say that $\mathcal{E}$ is symmetric in this case), then $\eta_a = 0$ for all $a \geq 0$. (It follows from Theorem 1 that for the homogeneous spaces the converse is also true.) Thus the variance of η_a coincides with its second moment and measures the asymmetry of $\mathcal{E}$. In the paper [4] we consider the expectations of some geometric quantities such as the Hausdorff measures of level sets and integrals of the powers of functions in $\mathcal{S}$ for polynomials on isotropy irreducible homogeneous spaces. It turns out that the expectations are almost independent of the geometry of M and the spectrum of $\mathcal{E}$ (the most essential parameter is the coefficient of the local metric homothety for the natural immersion $M \to \mathcal{S}$, which is equal to $\sqrt{\frac{\operatorname{Tr}(-\Delta)}{\dim M \dim \mathcal{E}}}$, where Δ is the Laplace–Beltrami operator on M). In this paper, we in particular show that the variance of η_a depends on the spectrum of $\mathcal{E}$. We use a simple version of the Kac–Rice formula for the second moment, which can be applied to a wide class of the random metric quantities for uniform distribution of u in $\mathcal{S}$, and specify it for η_a. Using this, we find or estimate the asymptotic of η_a and $\xi_a = \int_M |u(p)|^a \, dp$ in the following settings. First, M is an arbitrary compact homogeneous space, $\mathcal{E}$ is fixed and $a \to \infty$. Then either $\mathcal{E}$ is symmetric or $\operatorname{Var} \eta_a \sim \operatorname{Var} \xi_a$ (i.e., the ratio tends to 1). Second, M is a circle, $a \geq 0$ is fixed, and $\dim \mathcal{E} \to \infty$. Let $\varSigma = \{s_1, s_2, \dots\}$ be an infinite subset of $\mathbb{N}$ enumerated as an increasing sequence, $\varSigma_n = \{s_1, \dots, s_n\}$, and $\mathcal{E}_n$ be the real linear span of the functions $\cos s_k t$, $\sin s_k t$, $k = 1, \dots, n$. Then

$$\operatorname{Var} \eta_a = \frac{1}{2\pi} \int_{-\pi}^{\pi} V_a \left(\frac{1}{n} \sum_{k=1}^{n} \cos s_k t \right) dt.$$

where V_a is a function on $[-1, 1]$ which depends only on t and on a as a parameter (if $a = 0$, then $V_a(t) = \arcsin t - t$). The left-had side depends on $\mathcal{E}_n$; we drop n to avoid awkward notation. For a generic $\varSigma$, either $\eta_a = 0$ or η_a admits a lower bound as $\frac{C}{n^\kappa}$ (Theorem 2). If $\varSigma = \mathbb{N}$, then

$$\operatorname{Var} \eta_a = \frac{A}{n} + O\left(n^{-\frac{5}{3}}\right),$$

where $A = \int_0^\infty V_a \left(\frac{\sin t}{t}\right) dt$ (Theorem 3). This implies estimates from above and from below for spaces $\mathcal{E}_n$ on tori $\mathbb{T}^m$ with the spectrum $nU \cap \mathbb{Z}^m$, where U is a bounded symmetric subset of $\mathbb{R}^m$ which contains zero in its interior (Theorem 5). The problem of finding asymptotic or bounds for $\operatorname{Var} \eta_a$ for sparse sets $\varSigma \subset \mathbb{N}$, as a rule, is difficult. In case $s_k = k^2$, $\operatorname{Var} \eta_a$ can be estimated from above and from below as $\frac{C}{n^2}$. We prove this in Section 4 using results of the paper [7] by Jurkat and van Horne on the distribution function of theta sums. If $s_k = p(k)$, where p is a polynomial, then it is close to the known problem of estimation of the means of Weyl sums of a special kind (the recent paper [11] contains a short survey (in Section 5) and useful references).

In the seminal paper [1], Donelly and Fefferman proved the following local version of quasisymmetry of eigenfunctions for real analytic manifolds:

$$\min\{\operatorname{Vol}\left(U^+ \cap B(r)\right), \operatorname{Vol}\left(U^- \cap B(r)\right)\} \geq c \operatorname{Vol}\left(B(r)\right),$$

where U^+, U^- are the sets of positivity and negativity of the eigenfunction, respectively, $B(r)$ is a ball of radius r, and c is a constant which depends only on the geometry of M. The inequality holds for sufficiently large eigenvalues.

Another version was proved in [2]: if U is a connected component of U^+ such that $U \cap B\left(\frac{r}{2}\right) \neq \varnothing$, where $B\left(\frac{r}{2}\right)$ is concentric to $B(r)$, then

$$\mathrm{Vol}\,(U \cap B(r)) \geq \frac{a}{\lambda^k}\,\mathrm{Vol}(B(r)),$$

where λ is the eigenvalue, a depends only on the metric in M, and k only on $\dim M$. Nazarov, Polterovich, and Sodin in [10] proved for surfaces a better estimate replacing U with U^+:

$$\frac{\mathrm{Vol}\,(U^+ \cap B(r))}{\mathrm{Vol}(B(r))} \geq \frac{a}{\log \lambda \sqrt{\log \log \lambda}}.$$

They also showed that for the standard 2-sphere the reverse inequality with $\frac{a}{\log \lambda}$ instead of $\frac{a}{\log \lambda \sqrt{\log \log \lambda}}$ is fulfilled for certain spherical harmonics and large λ. For manifolds of dimension $n > 2$ Mangoubi obtained analogous inequality with $a\lambda^{-\frac{n-1}{2}}$ on the right. In [9], Nadirashvili showed for compact two-dimensional real analytic Riemannian manifolds that the volumes of the sets of positivity of eigenfunctions of the Laplace–Beltrami operator are separated from zero by a positive number which is independent of the eigenvalue. Another theorem in this paper states for n-dimensional smooth compact Riemannian manifolds that the ratio of L^∞-norms of the positive and negative parts of the eigenfunctions are bounded from above and from below. Jakobson and Nadirashvili proved in the paper [5] that the ratio of L^p-norms of the positive and negative parts of an eigenfunction admits upper and lower bounds which depend only on the manifold and p. In the paper [8], Marinucci and Wigman considered the difference between measures of the sets of positivity or and negativity of u (in our notation, this is η_0). They proved for the two-dimensional sphere and Gaussian distribution in the space $\mathcal{E} = \mathcal{H}_n$ of spherical harmonics of degree n that

$$\mathrm{Var}(\eta_0) = C_1 \int_0^{\frac{\pi}{2}} \arcsin(P_n(\cos\theta)) \sin\theta \, d\theta > 0,$$

where P_n is the nth Legendre polynomial normalized by the condition $P_n(1) = 1$. Moreover, they proved that

$$\mathrm{Var}\,(\eta_0) \sim \frac{C_2}{n^2},$$

as $n \to \infty$, where n is even (for odd n the integral equals zero), where

$$C_2 = C_3 \int_0^{\infty} (\arcsin(J_0(x)) - J_0(x)) x \, dx,$$

J_0 is the Bessel function of the first kind, and C_1, C_3 are some explicitly calculated positive numbers.

Most of these results, which had been proven for the Laplace–Beltrami eigenfunctions, do not hold for the polynomials. The latter is the exception, this paper is motivated by [8] as well as [3] and [4], which deal only with the expectations. In contrast to them, the variances depend on the spectrum of $\mathcal{E}$ as the example of trigonometric polynomials shows.

2. Preliminaries

The notations introduced above and in this section will be used throughout the paper, in particular, the following ones:

$$\begin{aligned} m &= \dim M, \\ d &= \dim \mathcal{S} = \dim \mathcal{E} - 1, \\ c &= \sqrt{d+1}. \end{aligned}$$

The group G acts on $L^2(M)$ by $gu(x) = u(g^{-1}x)$. There is the standard equivariant mapping $\iota : M \to \mathcal{S}$. For $p \in M$, let $\phi_p \in \mathcal{E}$ be such that

$$u(p) = \langle \phi_p, u \rangle$$

for all $u \in \mathcal{E}$. Then $|\phi_p| = c$ independently of $p \in M$. Set

$$\begin{aligned} \bar{p} &= \iota(p) = \frac{\phi_p}{c}, \\ \bar{M} &= \iota(M) \subseteq \mathcal{S}. \end{aligned} \tag{2}$$

We denote as $\int_X f(x)\, dx$ the integral of a function f over the invariant probability measure on a compact homogeneous space X. If $\sigma : X \to Y$ is an equivariant surjective mapping and f is a function on Y, then $\int_Y f(y)\, dy = \int_X f(\sigma(x))\, dx$. We assume that

$$\dim M = \dim \bar{M}.$$

Then the mapping $\iota : M \to \mathcal{S}$ is a local diffeomorphism and an unbranched finite covering. Let ν denote the number of leaves. Let us equip $\bar{M}$ with the metric induced by $\mathcal{E}$. Then its pullback on M is an invariant metric and ι is a local isometry. Let

$$\begin{aligned} \varpi &= \mathrm{Vol}\,(M)\,, \\ \bar{\varpi} &= \mathrm{Vol}\,\big(\bar{M}\big) = \frac{\varpi}{\nu} \end{aligned}$$

be the corresponding volumes of $M, \bar{M}$, respectively. The measures defined by the Riemannian metric are G-invariant, hence they are proportional to the invariant probability measures with the coefficients ϖ for M and $\frac{\varpi}{\nu}$ for $\bar{M}$. The volume of the unit sphere in $\mathbb{R}^{k+1}$ is denoted as ϖ_k:

$$\varpi_k = \frac{2\pi^{\frac{k+1}{2}}}{\Gamma\left(\frac{k+1}{2}\right)}.$$

Since $\eta_1(u) = \int_M u(p)\, dp$ for all $u \in \mathcal{E}$, the condition (1) implies

$$\eta_1 = 0.$$

There are two special cases, $a = 0$ and $a = 2$. It is natural to compare η_a with

$$\xi_a(u) = \int_M |u(p)|^a\, dp = \int_M u_+(p)^a\, dp + \int_M u_-(p)^a\, dp.$$

For $a = 0, 2$ and every $u \in \mathcal{S}$ we have

$$\int_M |u(p)|^a\, dp = 1,$$

i.e., $\xi_a = 1$ on $\mathcal{S}$. Let M_k denote the kth moment; we drop the index if $k = 1$. It follows that $\mathsf{M}_2(\xi_a) = 1$, $\mathrm{Var}(\xi_a) = 0$, and there is a simple relation between the variances of ξ_a, η_a, and the random variable $\zeta_a(u) = \int_M u_+(p)^a\,dp$:

$$\mathrm{Var}\,\xi_a + \mathrm{Var}\,\eta_a = 4\,\mathrm{Var}\,\zeta_a.$$

In what follows, $f(t) \sim g(t)$ stands for $f(t) = g(t)(1 + o(1))$ as $t \to \infty$. The base point of M is denoted as o. Also, we assume the convention $0^0 = 0$.

3. Some formulas for the second moment and the asymptotic of the variance as $a \to \infty$

Let $f(u, p)$ be a function on $\mathcal{S} \times M$ which satisfies the condition

$$f(u, p) = f(gu, gp) \quad \text{for all } g \in G. \tag{3}$$

For example, this is true for the compositions $f(u, p) = F(u(p))$, where F is a fixed function on $\mathbb{R}$. Set

$$I_f(u) = \int_M f(u, p)\,dp.$$

It follows from (3) that

$$\mathsf{M}(I_f) = \int_M \Big(\int_{\mathcal{S}} f(u, p)\,du\Big)\,dp = \int_{\mathcal{S}} f(u, p)\,du, \tag{4}$$

where the integral in the right is independent of $p \in M$ due to the homogeneity of M. Similarly, for the second moment we have

$$\begin{aligned}\mathsf{M}_2(I_f) = \int_{\mathcal{S}} \Big(\int_M f(u, p)\,dp\Big)^2 du &= \int_{M\times M} \Big(\int_{\mathcal{S}} f(u, p)f(u, q)\,du\Big)\,dp\,dq \\ &= \int_M \Big(\int_{\mathcal{S}} f(u, p)f(u, o)\,du\Big)\,dp.\end{aligned} \tag{5}$$

An analogous formula holds for the higher moments. Suppose additionally that for all $p, q \in M$

$$f(\bar{p}, q) = \varphi(\langle \bar{p}, \bar{q}\rangle) = \varphi\left(\frac{\phi(p, q)}{c^2}\right), \tag{6}$$

where φ is a function on $[-1, 1]$. This is equivalent to the assumption that f is the composition of a function on $\mathcal{S} \times \mathcal{S}$ which depends only on the distance $|u - v|$, where $u, v \in \mathcal{S}$, and the immersion ι on the second variable. It follows from (6) that the integrand in the right of (5) depends only on $\langle u, \bar{p}\rangle$ and $\langle u, \bar{o}\rangle$. Hence for any fixed p it is constant on the preimages in $\mathcal{S}$ of the orthogonal projection in $\mathcal{E}$ onto the linear two-plane Π_p which passes through $\bar{o}$ and $\bar{p}$. We may choose an orthonormal basis in Π_p such that

$$\bar{o} = (1, 0), \quad \bar{p} = (\cos\theta, \sin\theta),$$

where

$$\theta = \theta(p) = \arccos\langle \bar{p}, \bar{o}\rangle \in [0, \pi]. \tag{7}$$

Thus, θ is the inner distance in $\mathcal{S}$ between o and p. The projection of the invariant probability measure on $\mathcal{S}$ onto Π_p has the density

$$\frac{d-1}{2\pi}(1 - x^2 - y^2)^{\frac{d-3}{2}}$$

with respect to the Lebesgue measure in the unit disc in Π_p. Using the polar coordinates, we get

$$\int_{\mathcal{S}} \varphi(\langle u, \bar{o}\rangle)\varphi(\langle u, \bar{p}\rangle)\,du$$

$$= \frac{d-1}{2\pi}\int_0^1 \Big(\int_{-\pi}^{\pi} \varphi(r\cos\tau)\varphi(r\cos(\theta-\tau))\,d\tau\Big)(1-r^2)^{\frac{d-3}{2}}\,rdr. \tag{8}$$

By (8) and (2),

$$\int_{\mathcal{S}} u_+(o)^a u_+(p)^a\,du = \frac{d-1}{2\pi}c^{2a}\int_{\mathcal{S}} \langle u, \bar{o}\rangle_+^a\,\langle u, \bar{p}\rangle_+^a\,du = K_a(d)Q_a(\theta), \tag{9}$$

where

$$K_a(d) = \frac{d-1}{2\pi}\,c^{2a}\int_0^1 r^{2a+1}(1-r^2)^{\frac{d-3}{2}}\,dr = \frac{\Gamma(a+1)\Gamma\left(\frac{d+1}{2}\right)(d+1)^a}{2\pi\,\Gamma\left(a+\frac{d+1}{2}\right)}, \tag{10}$$

$$Q_a(\theta) = \int_{-\pi}^{\pi} \cos_+^a\tau\cos_+^a(\theta-\tau)\,d\tau = \int_{\theta-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^a\tau\cos^a(\theta-\tau)\,d\tau. \tag{11}$$

The latter equality holds since $0\le\theta\le\pi$. Note that Q_a is the convolution of $\cos_+^a$ with itself.

Lemma 1. *For any $a\ge 0$ the function Q_a is real analytic and strictly decreasing on $(0,\pi)$.*

Proof. For $a=0$ we have $Q_a(\theta)=\pi-\theta$ and both assertions are obvious. Thus we may assume $a>0$. The integrand in the second integral in (11) extends analytically on θ onto a neighborhood of $(0,\pi)$. Integrating over the segment $[\theta-\frac{\pi}{2},\frac{\pi}{2}]$ in $\mathbb{C}$, we get an extension of Q_a in a neighborhood of this segment, which is holomorphic. To verify this, note that the integrand always equals zero at the lower limit and the standard arguments and estimates show that Q_a is complex differentiable and

$$Q_a'(\theta) = -a\int_{\theta-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^a\tau\cos^{a-1}(\theta-\tau)\sin(\theta-\tau)\,d\tau. \tag{12}$$

If $\frac{\pi}{2}\le\theta\le\pi$, then the integrand in (12) is nonnegative. Since it does not vanish identically, we have $Q_a'(\theta)<0$ in this case. If $0\le\theta<\frac{\pi}{2}$, then $Q_a'(\theta)<0$ since

$$\int_{\theta-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^a\tau\cos^{a-1}(\theta-\tau)\sin(\theta-\tau)\,d\tau = \int_{\theta-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^a(\theta-\tau)\cos^{a-1}\tau\sin\tau\,d\tau$$

$$> \int_{\theta-\frac{\pi}{2}}^{\frac{\pi}{2}-\theta} \cos^a(\theta-\tau)\cos^{a-1}\tau\sin\tau\,d\tau > 0,$$

where the first inequality is evident and the second holds because $\cos^{a-1}\tau\sin\tau$ is odd on $\left(\theta-\frac{\pi}{2},\frac{\pi}{2}-\theta\right)$ and positive on $\left(0,\frac{\pi}{2}-\theta\right)$ and $\cos(\theta-\tau)>\cos(\theta+\tau)$ if $0<\tau<\frac{\pi}{2}-\theta$. $\square$

The equality $(-u)_\pm = u_\mp$ implies

$$\int_{\mathcal{S}} u_-(o)^a u_-(p)^a\,du = \int_{\mathcal{S}} u_+(o)^a u_+(p)^a\,du \tag{13}$$

Similarly,

$$\int_{\mathcal{S}} u_+(o)^a u_-(p)^a\,du = K_a(d)R_a(\theta), \tag{14}$$

where

$$(15)\quad R_a(\theta)=\int_{-\pi}^{\pi}\cos_+^a\tau\cos_-^a(\theta-\tau)\,d\tau=\int_{-\frac{\pi}{2}}^{\theta-\frac{\pi}{2}}\cos^a\tau|\cos(\theta-\tau)|^a\,d\tau.$$

Clearly, $\cos_-(\pi-\theta)=\cos_+\theta$. Replacing θ with $\pi-\theta$ and substituting $\tau\to-\tau$ in the integral, we get

$$(16)\qquad R_a(\pi-\theta)=Q_a(\theta).$$

By (9), (13), and (14)

$$\int_{\mathcal{S}}|u(o)|^a|u(p)|^a\,du=\int_{\mathcal{S}}(u_+(o)^a+u_-(o)^a)(u_+(p)^a+u_-(p)^a)\,du$$
$$=2K_a(d)(Q_a(\theta(p))+R_a(\theta(p))),$$
$$\int_{\mathcal{S}}u(o)|u(o)|^{a-1}u(p)|u(p)|^{a-1}\,du=\int_{\mathcal{S}}(u_+(o)^a-u_-(o)^a)(u_+(p)^a-u_-(p)^a)\,du$$
$$=2K_a(d)(Q_a(\theta(p))-R_a(\theta(p))).$$

Applying formulas (5) and (6) with $\varphi(x)=|x|^a$ and $\varphi(x)=x|x|^{a-1}$, respectively, we get

$$(17)\qquad \mathsf{M}_2(\xi_a)=2K_a(d)\int_M(Q_a(\theta(p))+R_a(\theta(p)))\,dp,$$

$$(18)\qquad \mathsf{M}_2(\eta_a)=2K_a(d)\int_M(Q_a(\theta(p))-R_a(\theta(p)))\,dp.$$

A calculation shows that for any $t,\tau\in\mathbb{R}$

$$(19)\qquad \lim_{a\to\infty}\cos^a\frac{\tau}{\sqrt{a}}\cos^a\frac{t-\tau}{\sqrt{a}}=e^{-\frac{t^2}{4}}e^{-\left(\frac{t}{2}-\tau\right)^2}$$

and, moreover, the function $\cos^a\frac{\tau}{\sqrt{a}}$ decreases on a if $\left|\frac{\tau}{\sqrt{a}}\right|<\frac{\pi}{2}$. According to (11),

$$(20)\qquad \sqrt{a}Q_a\left(\frac{t}{\sqrt{a}}\right)=\int_{(t-\frac{\pi}{2})\sqrt{a}}^{\frac{\pi\sqrt{a}}{2}}\cos^a\frac{\tau}{\sqrt{a}}\cos^a\frac{t-\tau}{\sqrt{a}}\,d\tau.$$

For all $x\in[-\frac{\pi}{2},\frac{\pi}{2}]$ we have $\cos x\le1-\frac{4x^2}{\pi^2}$. Consequently,

$$\frac{\ln\cos x}{x^2}\le\frac{\ln(1-\varepsilon x^2)}{x^2}\le-\varepsilon$$

if $0<\varepsilon<\frac{4}{\pi^2}$ (note that the function $\frac{\ln(1-y)}{y}$ is negative and decreases on $(0,1)$). Hence $\cos^a\frac{\tau}{\sqrt{a}}\le e^{-\varepsilon\tau^2}$ for $\tau\in\left(-\frac{\pi\sqrt{a}}{2},\frac{\pi\sqrt{a}}{2}\right)$. Extending the integrand in (20) onto $\mathbb{R}$ by zero we get the upper bound $e^{-\varepsilon\tau^2}e^{-\varepsilon(t-\tau)^2}$ for it on $\mathbb{R}$. By Lebesgue's dominated convergence theorem, (19), and the equality $\int_{-\infty}^{\infty}e^{-\left(\frac{t}{2}-\tau\right)^2}\,d\tau=\sqrt{\pi}$, for any fixed $t\ge0$

$$(21)\qquad \lim_{a\to\infty}\sqrt{a}Q_a\left(\frac{t}{\sqrt{a}}\right)=\begin{cases}\sqrt{\pi}e^{-\frac{t^2}{4}}, & 0\le t<\frac{\pi}{2},\\ \frac{\sqrt{\pi}}{2}e^{-\frac{t^2}{4}}, & t=\frac{\pi}{2},\\ 0, & t>\frac{\pi}{2}.\end{cases}$$

It follows that for any $\varepsilon > 0$ and $b > -1$

$$\lim_{a\to\infty} a^{\frac{b}{2}+1}\int_0^\varepsilon t^b Q_a(t)\,dt = \lim_{a\to\infty}\sqrt{a}\int_0^{\varepsilon\sqrt{a}} t^b Q_a\left(\frac{t}{\sqrt{a}}\right)dt$$

$$= \sqrt{\pi}\int_0^\infty t^b e^{-\frac{t^2}{4}}\,dt = 2^b\sqrt{\pi}\,\Gamma\left(\frac{b+1}{2}\right). \tag{22}$$

Let $B(\varepsilon)$ be the ball in M of radius ε centered at o. If ε is saficiently small, then $\bar{B}(\varepsilon) = \iota(B(\varepsilon))$ is the ball in $\bar{M}$ centered at $\bar{o}$ of the same radius. We have $\operatorname{Vol} B(\varepsilon) \sim \frac{\varpi_{m-1}}{m}\varepsilon^m$ as $\varepsilon \to 0$. Since $\theta(\bar{p}) = \theta(p)$ is the inner distance in $\mathcal{S}$ from $\bar{o}$ to $\bar{p}$, it follows from the definition of Q_a and (21) that

$$\int_{\bar{M}\setminus B(\varepsilon)} Q_a(\theta(\bar{p}))\,d\bar{p} = o(a^{-n})$$

for any fixed small $\varepsilon > 0$ and all $n > 0$ as $a \to \infty$. Using the polar coordinates in $B(\varepsilon)$ and (22) with $b = m-1$, we get

$$\bar{\varpi}\int_{\bar{B}(\varepsilon)} Q_a(\theta(\bar{p}))\,d\bar{p} = \varpi\int_{B(\varepsilon)} Q_a(\theta(p))\,dp \sim \varpi_{m-1}\int_0^\varepsilon r^{m-1}Q_a(r)\,dr$$

$$\sim 2^{m-1}\varpi_{m-1}\sqrt{\pi}\,\Gamma\left(\frac{m}{2}\right)a^{-\frac{m+1}{2}} = 2^m\pi^{\frac{m+1}{2}}a^{-\frac{m+1}{2}}. \tag{23}$$

Since M is homogeneous,

$$\bar{M} = -\bar{M} \iff -\bar{o} \in \bar{M}.$$

According to the definition of θ (see (7)), $-\bar{o} \notin \bar{M}$ implies $\theta(p) \neq \pi$ for all $p \in M$. Hence $\cos_+\tau\cos_-(\theta(p)-\tau) \neq 1$ for all $p \in M$ and $\tau \in [-\pi,\pi]$. Set

$$\delta = \operatorname*{dist}_{\mathcal{S}}(\bar{M}, -\bar{M}).$$

Clearly,

$$\delta = \operatorname*{dist}_{\mathcal{S}}(\bar{M}, -\bar{o}) = \min\{\pi - \theta(p) : p \in M\}.$$

Since $\cos_- t = \cos_+(\pi - t)$, we have

$$\cos_+\tau\cos_-(\theta(p)-\tau) \le \max_{|\tau|\le\frac{\pi}{2}}\cos\tau\cos(\tau+\delta) = \frac{1+\cos\delta}{2}.$$

Due to (15) and (23),

$$\int_M R_a(\theta(p))\,dp < \pi\beta^a, \tag{24}$$

where $\beta = \frac{1}{2}(1+\cos\delta)$.

Theorem 1. *The space $\mathcal{E}$ is symmetric if and only if $\bar{M} = -\bar{M}$. Moreover, the asymptotic behavior of ξ_a and η_a as $a \to \infty$ is subject to the following formulas.*

(i) *If $\bar{M} \neq -\bar{M}$, then*

$$\mathsf{M}_2(\xi_a) \sim Aa^{-\frac{d+m}{2}}(d+1)^a, \tag{25}$$

$$\mathsf{M}_2(\eta_a) = \operatorname{Var}(\eta_a) = \mathsf{M}_2(\xi_a)(1-\rho(a)) \tag{26}$$

where $A = 2^{m+1}\pi^{\frac{m-1}{2}}\bar{\varpi}$, $0 \le \rho(a) \le 2\pi\beta^a$, $\beta = \frac{1}{2}(1+\cos\delta)$, and δ is the inner distance in $\mathcal{S}$ between M and $-M$.

(ii) *If $\bar{M} = -\bar{M}$, then $\eta_a = 0$ and $\mathsf{M}_2(\xi_a)$ satisfies (25) with $2A$ instead of A.*

(iii) *In both cases above,*

$$\frac{\mathsf{M}(\xi_a)^2}{\mathsf{M}_2(\xi_a)} \sim Ba^{\frac{m-d}{2}}, \tag{27}$$

where $B = \frac{2^{d-m-1}}{\bar{\varpi}\pi^{\frac{m+1}{2}}}$.

It follows from (27) that $\mathsf{M}(\xi_a)^2 = o(\mathsf{M}_2(\xi_a))$, except for the trivial situation $\bar{M} = \mathcal{S}$. Hence $\mathrm{Var}(\xi_a)$ has the same rate of growth as $\mathsf{M}_2(\xi_a)$ and ξ_a does not concentrate near its mean value. Moreover, by (i) and (ii) either $\mathsf{M}_2(\eta_a) = 0$ or $\mathsf{M}_2(\eta_a) \sim \mathsf{M}_2(\xi_a)$ as $a \to \infty$.

Proof of Theorem 1. If $\bar{M} = -\bar{M}$, then the mapping $\bar{p} \to -\bar{p}$ is well defined on $\bar{M}$, moreover, it is an isometry with respect for the Riemannian metric in $\bar{M}$. Hence it keeps the invariant probability measure. Since $(-u)_\pm = u_\mp$, this implies that u_+ and u_- are equidistributed. Thus $\mathcal{E}$ is symmetric. The converse is a consequence of (i).

(i) It follows from (23), (24), and (17) that

$$\mathsf{M}_2(\xi_a) \sim 2^{m+1}\bar{\varpi}\pi^{\frac{m+1}{2}} K_a(d) a^{-\frac{m+1}{2}}.$$

The equivalence $\Gamma(t)t^b \sim \Gamma(t+b)$ as $t \to \infty$ and (10) imply

$$K_a(d) \sim \frac{1}{\pi} a^{-\frac{d-1}{2}} (d+1)^a.$$

Therefore, $\mathsf{M}_2(\xi_a) \sim 2^{m+1}\pi^{\frac{m-1}{2}}\bar{\varpi}a^{-\frac{m+d}{2}}(d+1)^a$ as $a \to \infty$. This proves (25). According to (17) and (18),

$$\mathsf{M}_2(\eta_a) = \mathsf{M}_2(\xi_a) - 4K_a(d)\int_M R_a(\theta(p))\,dp.$$

Together with (11) and (24) this implies (26) with

$$\rho(a) = \frac{\mathsf{M}_2(\xi_a) - \mathsf{M}_2(\eta_a)}{2K_a(d)} = 2\int_M R_a(\theta(p))\,dp,$$

that satisfies the theorem by (24).

(ii) If $\bar{M} = -\bar{M}$, then the functions u_+ and u_- are equidistributed for every $u \in \mathcal{E}$. Hence $\eta_a(u) = 0$ for all $a \geq 0$ and $u \in \mathcal{S}$. Due to (16) and the evident equality $\theta(-\bar{p}) = \pi - \theta(\bar{p})$,

$$\int_M R_a(\theta(p))\,dp = \int_M Q_a(\theta(p))\,dp.$$

By (17), $\mathsf{M}_2(\xi_a) = 4K_a(d)\int_M Q_a(\theta(p))\,dp$.

(iii) The expectation of ξ_a can be easily calculated due to (4):

$$\begin{aligned}\mathsf{M}(\xi_a) &= \int_{\mathcal{S}} |u(o)|^a\,du = c^a \int_{\mathcal{S}} |\langle u, \bar{o}\rangle|^a\,du = \frac{\varpi_{d-1}}{\varpi_d} c^a \int_{-1}^{1} |x|^a (1-x^2)^{\frac{d}{2}-1}\,dx \\ &= \frac{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{d+1}{2}\right)(d+1)^{\frac{a}{2}}}{\sqrt{\pi}\,\Gamma\left(\frac{a+d+1}{2}\right)} \sim \frac{2^{\frac{d}{2}}\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi}} a^{-\frac{d}{2}}(d+1)^{\frac{a}{2}}\end{aligned}$$

as $a \to \infty$. Together with (25) this implies (27). □

4. The asymptotic of variance as $\dim \mathcal{E} \to \infty$ for trigonometric polynomials

According to (10),

$$\lim_{d\to\infty} K_a(d) = \frac{2^{a-1}}{\pi}\Gamma(a+1). \tag{28}$$

Moreover, if $-1 < a < 0$ or $a > 1$ then K_a increases with d, if $0 < a < 1$ then it decreases, and it is constant if $a = 0$ or $a = 1$. It is convenient to replace $\theta(p)$ with

$$\vartheta(p) = \frac{\pi}{2} - \theta(p) = \arcsin\langle \bar{o}, \bar{p}\rangle . \tag{29}$$

The geometrical meaning of ϑ is evident: $\vartheta(p)$ is the signed inner distance in $\mathcal{S}$ from $\bar{p}$ to the equator $\bar{o}^\perp \cap \mathcal{S}$. Set

$$F_a(\vartheta) = \frac{1}{2}\left(Q_a(\theta) - Q_a(\pi-\theta)\right) = \frac{1}{2}\left(Q_a\left(\frac{\pi}{2}-\vartheta\right) - Q_a\left(\frac{\pi}{2}+\vartheta\right)\right). \tag{30}$$

Since $\mathsf{M}(\eta_a) = 0$, $\mathrm{Var}(\eta_a) = \mathsf{M}_2(\eta_a)$. By (18) and (16), we have

$$\mathrm{Var}(\eta_a) = 4K_a(d)\int_M F_a(\vartheta(p))\,dp.$$

Due to (28), the behavior of $\mathrm{Var}(\eta_a)$ as $d \to \infty$ is determined by the integral in the right-hand side. It depends on the geometry of M and the spectrum of $\mathcal{E}$.

Clearly, F_a is odd on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (this is an immediate consequence of (30)). Thus we shall assume $0 \le \vartheta \le \frac{\pi}{2}$. It follows from Lemma 1 that F_a is real analytic and strictly increases on $\left(0, \frac{\pi}{2}\right)$. By (12),

$$F_a'(0) = -Q_a'\left(\frac{\pi}{2}\right) = a\int_0^{\frac{\pi}{2}} \cos^{a+1}\tau \sin^{a-1}\tau\, d\tau = \frac{a\Gamma(\frac{a}{2})^2}{4\Gamma(a)}.$$

The function

$$V_a(x) = F_a(\arcsin(x)) - F_a'(0)x \tag{31}$$

has a critical point at zero and

$$\mathrm{Var}(\eta_a) = 4K_a(d)\int_M V_a\left(\frac{\phi_o(p)}{c^2}\right)dp \tag{32}$$

since $\int_M \phi_o(p)\,dp = 0$ by (1). Here is the result of computation of K_a, F_a, and Q_a for the first three integer a.

(33)

	$K_a(d)$	$F_a(\vartheta)$	$Q_a(\theta)$
$a=0:$	$\frac{1}{2\pi}$	ϑ	$\pi-\theta$
$a=1:$	$\frac{1}{\pi}$	$\frac{\pi\sin\vartheta}{4}$	$\frac{(\pi-\theta)\cos\theta+\sin\theta}{2}$
$a=2:$	$\frac{4}{\pi}\frac{d+1}{d+3}$	$\frac{2\vartheta(2-\cos 2\vartheta)+3\sin 2\vartheta}{16}$	$\frac{2(\pi-\theta)(2+\cos 2\theta)+3\sin 2\theta}{16}$.

For $a = 3$ we have $K_a(d) = \frac{24(d+1)^2}{\pi(d+3)(d+5)}$ and $F_a(\vartheta) = \frac{\pi\sin\vartheta(4-\cos 2\vartheta)}{32}$. The expression for Q_3 is too long.

Let Σ be a finite subset of $\mathbb{N} = \{1, 2, \dots\}$ and $\mathcal{T}(\Sigma)$ be the linear space on $M = \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ generated by $\sin kt, \cos kt$ with $k \in \Sigma$. Then $\mathcal{T}(\Sigma)$ is $\mathbb{T}$-invariant and every $\mathbb{T}$-invariant finite dimensional subspace of $L^2(\mathbb{T})$ which satisfies (1) is equal to $\mathcal{T}(\Sigma)$ for some $\Sigma \subset \mathbb{N}$. We shall say that Σ is the spectrum of $\mathcal{E} = \mathcal{T}(\Sigma)$.

Let Σ be an infinite subset of $\mathbb{N}$ enumerated as an increasing sequence $\{s_n\}$, $\Sigma_n = \{s_1, \dots, s_n\}$, $\mathcal{E}_n = \mathcal{T}(\Sigma_n)$. Then

$$\mathcal{S}_n = \left\{ \sum_{k=1}^{n} a_k \cos s_k t + b_k \sin s_k t : \sum_{k=1}^{n} a_k^2 + b_k^2 = 1 \right\}$$

is the unit sphere in $\mathcal{E}_n$, $c^2 = \dim \mathcal{E}_n = d+1 = 2n$. We always assume that the base point is the zero and drop the index o in ϕ_o replacing it usually with n,

$$\phi_n(t) = 2 \sum_{k=1}^{n} \cos s_k t,$$

except for the following lemma, where ϕ_j corresponds to Σ_j, $j = 1, 2$.

Lemma 2. *If $\Sigma_1, \Sigma_2 \subset \mathbb{N}$ are finite and $\Sigma_1 \subseteq \Sigma_2$, then for all $k \in \mathbb{N}$*

$$0 \le \int_{-\pi}^{\pi} \phi_1(x)^k \, dx \le \int_{-\pi}^{\pi} \phi_2(x)^k \, dx.$$

Proof. Since $2\cos s_j t = e^{is_j t} + e^{-is_j t}$, the integrals are equal to the number of solutions to the equations $\pm s_1 \pm \dots \pm s_k = 0$, where $s_j \in \Sigma_l$, $l = 1, 2$. This makes evident both inequalities. □

If Σ contains only odd numbers, then $\phi_n(t + \pi) = -\phi_n(t)$. Hence, $\bar{\mathbb{T}} = -\bar{\mathbb{T}}$ and $\eta_a = 0$ by Theorem 1. By (31) and (33), $K_0(d) = \frac{1}{2\pi}$,

$$V_0(x) = \arcsin x - x.$$

The Taylor coefficients α_k of V_0 are nonnegative, $V_0(1) < 1$, and $\frac{|\phi_n(x)|}{2n} \le 1$ for all $x \in [-1, 1]$. Therefore, the series $\sum_{k=1}^{\infty} \alpha_k \left(\frac{\phi_n(x)}{2n}\right)^k$ converge absolutely in the norm of $L^1(\mathbb{T})$ and we have

$$\int_{-\pi}^{\pi} V_0\left(\frac{\phi_n(x)}{2n}\right) dx = \sum_{k=1}^{\infty} \alpha_k \int_{-\pi}^{\pi} \left(\frac{\phi_n(x)}{2n}\right)^k dx. \tag{34}$$

Set

$$\kappa = \kappa(\Sigma) = \min\left\{ k : k \text{ is odd and } \int_{-\pi}^{\pi} \phi_n(x)^k \, dx \ne 0 \text{ for some } n \in \mathbb{N} \right\}$$

if such k exists and $\kappa = \infty$ otherwise. By (1), $\kappa > 1$. Obviously, $\kappa(\Sigma)$ is equal to the least odd number $2l + 1$ such that there exist $\epsilon_1, \dots, \epsilon_{2l+1} \in \{-1, 1\}$ and $s_{j_1}, \dots, s_{j_{2l+1}} \in \Sigma$ which satisfy the equality

$$\epsilon_1 s_{j_1} + \dots + \epsilon_{2l+1} s_{j_{2l+1}} = 0. \tag{35}$$

Theorem 2. *If $\kappa < \infty$, then there exists $C > 0$ such that for all sufficiently large $n \in \mathbb{N}$ we have*

$$\mathrm{Var}(\eta_0) \ge \frac{C}{n^\kappa}.$$

The equality $\kappa = \infty$ holds if and only if $\Sigma = \{2^q l_n\}_{n \in \mathbb{N}}$, where $q \ge 1$ is fixed and l_n are odd. In this case, $\eta_a = 0$ for all $a \ge 0$ and $\mathcal{E}_n$ are symmetric for all $n \in \mathbb{N}$.

Proof. If $\int_{-\pi}^{\pi} \phi_n(x)^k\, dx \neq 0$, then $\int_{-\pi}^{\pi} \phi_l(x)^k\, dx \neq 0$ for all $l > n$ by Lemma 2, and the first assertion follows.

If all s_n have a common divisor d, then κ does not change if we replace s_k with $\frac{s_k}{d}$. Hence we may assume that Σ contains an odd number x. If it contains also an even number y, then $\kappa < \infty$ since there exists an obvious combination of x and y as in (35) (x times y minus y times x). The converse is true since the sum of odd number of odd numbers is odd, hence cannot be zero. Thus, if 2^q is the greatest common divisor for s_n, then all quotients $\frac{s_n}{2^q}$ are odd. Then $\bar{\mathbb{T}}$ is symmetric since $\phi_n(t + \frac{\pi}{2^q}) = -\phi(t)$, $t \in \mathbb{R}$, hence $\eta_a = 0$ for all $a \geq 0$ by Theorem 1. □

Thus, $\mathrm{Var}(\eta_0)$ either vanishes or decreases polynomially. For example, if $\Sigma = \{2^n : n \in \mathbb{N}\}$, then $\kappa = 3$ since $4 - 2 - 2 = 0$ and $\kappa > 1$. If $\Sigma = \{n^q : n \in \mathbb{N}\}$ and $q > 2$, then $\kappa > 3$ according to Fermat's Last Theorem proved by Wales.

In what follows, we assume $a \geq 0$ fixed.

4.1. **Polynomials of degree n.**

Theorem 3. *Let $\mathcal{E}_n = \mathcal{T}(\Sigma_n)$ with $\Sigma_n = \{1, 2, \ldots, n\}$, $a \geq 0$. Then*

$$\mathrm{Var}(\eta_a) = \frac{A}{n} + O\left(n^{-\frac{5}{3}}\right) \tag{36}$$

as $n \to \infty$, where $A = \frac{1}{2\pi^2} \int\limits_0^{\infty} V_a\left(\frac{\sin\tau}{\tau}\right) d\tau > 0$.

Proof. We have $\frac{\phi_n(t)}{c^2} = \frac{1}{n}\sum_{k=1}^n \cos kt = \frac{\sin(n+\frac{1}{2})t}{2n\sin\frac{t}{2}} - \frac{1}{2n}$. By (32),

$$\mathrm{Var}(\eta_a) = \frac{1}{2\pi^2}\int_0^{\pi} V_a\left(\frac{\sin(n+\frac{1}{2})t}{2n\sin\frac{t}{2}} - \frac{1}{2n}\right) dt.$$

Substituting $\tau = nt$, we get

$$\mathrm{Var}(\eta_a) = \frac{1}{2\pi^2 n}\int_0^{\pi n} V_a\left(\frac{\sin(1+\frac{1}{2n})\tau}{2n\sin\frac{\tau}{2n}} - \frac{1}{2n}\right) d\tau = \frac{1}{2\pi^2 n}\int_0^{\infty} V_a\left(f_n(\tau)\right) d\tau,$$

where

$$f_n(\tau) = \begin{cases} \frac{\sin(1+\frac{1}{2n})\tau}{2n\sin\frac{\tau}{2n}} - \frac{1}{2n}, & 0 < \tau \leq \pi n, \\ 0, & \tau > \pi n. \end{cases}$$

If $0 < \tau < \pi n$, then

$$f_n(\tau) = \frac{\sin\tau}{\tau}\cdot\frac{\tau}{2n}\operatorname{ctg}\frac{\tau}{2n} - \frac{1-\cos\tau}{2n}. \tag{37}$$

Hence for any fixed $\tau > 0$

$$\lim_{n\to\infty} f_n(\tau) = \frac{\sin\tau}{\tau}. \tag{38}$$

The following inequalities hold because the Taylor coefficients of $1 - x\operatorname{ctg} x$ are nonnegative:

$$0 < x\operatorname{ctg} x < 1, \quad 0 < x < \frac{\pi}{2}, \tag{39}$$

$$\frac{1}{3} < \frac{1 - x\operatorname{ctg} x}{x^2} < \frac{4}{\pi^2}, \quad 0 < x < \frac{\pi}{2}. \tag{40}$$

Furthermore, since V_a is odd and $V'(0) = 0$, for some C depending only on a and all $t \in (0, 1)$

$$|V_a(t)| \le Ct^3. \tag{41}$$

(In what follows, we assume that C may be different in distinct inequalities and depends only on a.) Using (39) and the inequality $\frac{\pi}{\tau} > \frac{1}{n}$ which holds for $\tau \in (0, \pi n)$ we get

$$|f_n(\tau)| \le \left|\frac{\sin\tau}{\tau} \cdot \frac{\tau}{2n} \operatorname{ctg} \frac{\tau}{2n}\right| + \left|\frac{1-\cos\tau}{2n}\right| < \frac{1+\pi}{\tau}. \tag{42}$$

By (41) and (42),

$$|V_a(f_n(\tau))| \le \frac{C}{(1+\tau)^3} \tag{43}$$

for some $C > 0$ and all $\tau > 0$ independently of n. Due to (38) and (43) we may apply Lebesgue's dominated convergence theorem to the sequence $V_a \circ f_n$:

$$\lim_{n\to\infty} \int_0^\infty V_a\left(f_n(t)\right) dt = \int_0^\infty V_a\left(\frac{\sin\tau}{\tau}\right) d\tau. \tag{44}$$

Let us estimate the rate of convergence. Set

$$B_n(\tau) = \left|V_a\left(\frac{\sin\tau}{\tau}\right) - V_a\left(f_n(\tau)\right)\right|.$$

If $0 < \tau \le \pi n$, then

$$\begin{aligned} \left|\frac{\sin\tau}{\tau} - f_n(\tau)\right| &= \left|\frac{\sin\tau}{\tau} \cdot \left(1 - \frac{\tau}{2n} \operatorname{ctg} \frac{\tau}{2n}\right) + \frac{1-\cos\tau}{2n}\right| \\ &\le \frac{4}{\pi^2\tau}\left(\frac{\tau}{2n}\right)^2 + \frac{1}{n} < \frac{2}{n} \end{aligned} \tag{45}$$

by (40). Since $f_n(\tau) = 0$ for $\tau > \pi n$, the inequality holds for all $\tau > 0$. Since the function F_a is continuously differentiable, for some $C > 0$ and any $t \in (0, 1)$ we have

$$|V_a'(t)| = |F_a'(\arcsin t)(1-t^2)^{-\frac{1}{2}} - F_a'(0)| \le C(1-t^2)^{-\frac{1}{2}}.$$

For $\tau \in (0, \pi)$, (37) and (39) imply the inequality $f_n(\tau) < \frac{\sin\tau}{\tau}$. By Legendre's theorem, for some t between $f_n(\tau)$ and $\frac{\sin\tau}{\tau}$

$$\begin{aligned} \left|V_a\left(\frac{\sin\tau}{\tau}\right) - V_a(f_n(\tau))\right| &= \left|F_a'(t)(1-t^2)^{-\frac{1}{2}} - F_a'(0)\right| \left(\frac{\sin\tau}{\tau} - f_n(\tau)\right) \\ &\le C\left(1 - \frac{\sin^2\tau}{\tau^2}\right)^{-\frac{1}{2}} \left(\frac{\sin\tau}{\tau} - f_n(\tau)\right). \end{aligned}$$

The equivalence $1 - \frac{\sin\tau}{\tau} \sim \frac{1}{6}\tau^2$ as $\tau \to 0$ implies $\left(1 - \frac{\sin^2\tau}{\tau^2}\right)^{-\frac{1}{2}} < \frac{\alpha}{\tau}$ for some $\alpha > 0$ and all $\tau \in (0, \pi)$. Thus, we have to estimate $\frac{1}{\tau}\left(\frac{\sin\tau}{\tau} - f_n(\tau)\right)$. We do it as in (45) with minor changes. The inequalities $0 < \tau < \pi$ and (40) imply

$$\frac{1}{\tau}\left(1 - \frac{\tau}{2n} \operatorname{ctg} \frac{\tau}{2n}\right) < \frac{4}{\pi^2\tau}\left(\frac{\tau}{2n}\right)^2 < \frac{1}{\pi n^2}.$$

Since $\frac{1-\cos\tau}{\tau}$ is bounded, we get

$$\left|V_a\left(\frac{\sin\tau}{\tau}\right) - V_a(f_n(\tau))\right| < \frac{C}{n}$$

for some $C > 0$ and all $\tau \in (0, \pi)$. Therefore,

$$\int_0^{\pi} B_n(\tau)\, d\tau = O\left(\frac{1}{n}\right). \tag{46}$$

Clearly, $f_n(t) = 1$ if and only if $t = 0$ and the convergence in (38) is locally uniform. Hence, there exist $\beta < 1$ and $N > 0$ such that $|f_n(t)| < \beta$ if $t > \pi$ and $n > N$. Due to Lemma 1, V_a is Lipschitz on $[-\beta, \beta]$. Thus (45) implies the inequality

$$\left|V_a\left(\frac{\sin\tau}{\tau}\right) - V_a\left(f_n(\tau)\right)\right| < \frac{C}{n},$$

for some $C > 0$ and all $\tau > \pi$, $n > N$. It follows that

$$\int_{\pi}^{\sqrt[3]{n}} B_n(\tau)\, d\tau = O\left(n^{-\frac{2}{3}}\right). \tag{47}$$

Let C be as in (43). Then

$$\int_{\sqrt[3]{n}}^{\infty} B_n(\tau)\, d\tau < C \int_{\sqrt[3]{n}}^{\infty} \frac{d\tau}{\tau^3} = \frac{C}{2}\, n^{-\frac{2}{3}}.$$

Together with (46) and (47) this implies

$$\int_0^{\infty} B_n(\tau)\, d\tau = O\left(n^{-\frac{2}{3}}\right).$$

This proves (36). Since V_a is odd and increasing, for any integer $k \geq 0$ and $2k\pi < \tau < (2k+1)\pi$ we have

$$V_a\left(\frac{\sin\tau}{\tau}\right) + V_a\left(\frac{\sin(\tau+\pi)}{\tau+\pi}\right) = V_a\left(\frac{\sin\tau}{\tau}\right) - V_a\left(\frac{\sin\tau}{\tau+\pi}\right) > 0.$$

Therefore, $A > 0$. $\square$

A computation shows that $A \approx 0.0208$.

Remark. Let $\Sigma_n = \{1, 3, \ldots, 2n-1\}$. Then $\phi_n(t) = \frac{\sin 2nt}{\sin t}$ and $\frac{1}{2n}\phi_n\left(\frac{\tau}{2n}\right) \to \frac{\sin\tau}{\tau}$ as $n \to \infty$. One might expect that the asymptotic of variance is subject to (36) but in fact it vanishes as it was noted in the beginning of this section. Hence, in this case there is no common integrable majorant for the functions $\frac{1}{2n}\left|\phi_n\left(\frac{\tau}{2n}\right)\right|$.

4.2. **Theta sums.** We shall prove that $\frac{A}{n^2} \leq \mathrm{Var}(\eta_0) \leq \frac{B}{n^2}$ if $\Sigma_n = \{k^2 : k = 1, 2, \ldots, n\}$ using results of the paper [7] by Jurkat and van Horne. For the sums

$$S_n(x) = \frac{1}{2} + \sum_{k=1}^{n-1} e^{\pi i k^2 x} + \frac{1}{2} e^{\pi i n^2 x}$$

they proved that the sequence Ψ_n of the distribution functions for the normalized sums $\frac{1}{\sqrt{n}} S_n$,

$$\Psi_n(t) = \mu\left(\left\{x \in [-1, 1] : |S_n(x)| \geq \sqrt{n}\, t\right\}\right),$$

where μ stands for the Lebesgue measure, converges pointwise outside some at most countable set to a function Ψ such that $\Psi(t) = O(t^{-4})$ (Theorem 3) and, moreover, Ψ_n are uniformly bounded on $t \geq 0$ and $n \geq 1$ (Proposition 1):

$$\Psi_n(t) < \frac{C}{(1+t)^4} \tag{48}$$

(the authors use the notation $\Psi_n(t) = O((1+t)^{-4})$ which is explained on the first page of the paper).

Theorem 4. *Let Σ_n be as above. Then there exist $A, B > 0$ such that for all $n \in \mathbb{N}$*

$$\frac{A}{n^2} < \operatorname{Var}(\eta_0) < \frac{B}{n^2}.$$

Proof. Let Φ_n be the distribution function for $\frac{|\phi_n|}{2n}$:

$$\Phi_n(t) = \frac{1}{2\pi}\,\mu\left(\{x \in [-\pi,\pi] : |\phi_n(x)| \geq 2nt\}\right).$$

Since $\phi_n(x) = 2\Re S_n(\frac{x}{\pi}) - 1 - \cos(n+1)^2 x$, we have $|\phi_n(x)| \leq 2(|S_n(\frac{x}{\pi})| + 1)$. Hence

$$\Phi_n(t) \leq \pi\Psi_n\left(\sqrt{n}\,t - \frac{1}{\sqrt{n}}\right).$$

Set $U_0(t) = V_0(t) - \frac{t^3}{6}$. Then $U_0'(t) = \frac{1}{\sqrt{1-t^2}} - 1 - \frac{t^2}{2} = O(t^4)$ as $t \to 0$ and $U_0(t) > 0$ on $(-1,1)$. We replace $V_0(t)$ with $U_0(t) + \frac{t^3}{6}$ in (32) and estimate the integrals of the summands separately.

Since ϕ_n is real analytic on $\mathbb{R}$ its level sets are finite whence the function $\Phi_n(t)$ is piecewise smooth and continuous on $[0,1]$ and $\Phi_n(1) = 0$. This verifies the integration by parts below. Furthermore, $|U_0(t)| = U_0(|t|)$ because U_0 is odd and increasing. Using this relation, (48), and assuming $n > 1$ we get

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} U_0\left(\frac{\phi_n(x)}{2n}\right)dx &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|U_0\left(\frac{\phi_n(x)}{2n}\right)\right|dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} U_0\left(\frac{|\phi_n(x)|}{2n}\right)dx \\ &= -\int_0^1 U_0(t)\,d\Phi_n(t) = \int_0^1 U_0'(t)\Phi_n(t)\,dt < \int_0^1 \frac{U_0'(t)}{\left(1+\sqrt{n}\,t - \frac{1}{\sqrt{n}}\right)^4}\,dt \\ &< \frac{1}{n^2}\int_0^1 \frac{U_0'(t)}{t^4}\,dt < \frac{K}{n^2}\end{aligned}$$

for some $K > 0$ and all $n \in \mathbb{N}$. The integral

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\phi_n(x)^3\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{k=1}^{n}\left(e^{ik^2x} + e^{-k^2x}\right)\right)^3 dx$$

is equal to the number of solutions to the equations $\pm k_1^2 \pm k_2^2 \pm k_3^2 = 0$ under the restrictions $1 \leq k_j \leq n$, $j = 1,2,3$, which is known to be estimated by the area of the disc of radius $\sqrt{n}$ up to a multiplicative constant. Indeed, if $k_1, k_2 \leq k_3$, then $k_1^2 + k_2^2 = k_3^2$, hence the condition $k_3 \leq n$ is equivalent to $|k_1 + ik_2| \leq n$. Since (k_1,k_2,k_3) is a Pythagorean triple we have $k_1 + ik_2 = z^2$, where z is a Gauss integer complex number such that $|z| \leq \sqrt{n}$. Thus,

$$\frac{A}{n^2} < \frac{1}{12\pi}\int_{-\pi}^{\pi}\left(\frac{\phi_n(x)}{2n}\right)^3 dx < \frac{L}{n^2}$$

for some $A, L > 0$. Furthermore,

$$\mathrm{Var}\,(\eta_0) = \frac{1}{12\pi}\int_{-\pi}^{\pi}\left(\frac{\phi_n(x)}{2n}\right)^3 dx + \frac{1}{2\pi}\int_{-\pi}^{\pi} U_0\left(\frac{\phi_n(x)}{2n}\right) dx < \frac{B}{n^2},$$

where $B = K + L$. By Lemma 2 (the first inequality) and (34), the second integral above is nonnegative. This proves the theorem. □

4.3. **A consequence for tori.** Let $M = G = \mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$, let the dual group $\mathbb{Z}^m$ be naturally embedded to $\mathbb{R}^m$, and U be a bounded symmetric neighborhood of zero in $\mathbb{R}^m$ which contains a nonzero point of $\mathbb{Z}^m$. Set $\Sigma_n = \mathbb{Z}^m \cap nU$ and let $\mathcal{E}_n = \mathcal{T}\,(\Sigma_n)$ be the linear span of the real and imaginary parts of the exponents $e^{i\langle s,t\rangle}$, where $s \in \Sigma_n$, $t \in \mathbb{R}^m$. Clearly, the analog of Lemma 2 holds in this setting: if $U_1 \subseteq U_2$, then $0 \le \int_{\mathbb{T}^m} \phi_1(p)^k\,dp \le \int_{\mathbb{T}^m} \phi_2(p)^k\,dp$, where ϕ_j correspond to $\Sigma_j = U_j \cap \mathbb{Z}^m$, $j = 1, 2$.

Theorem 5. *Let U be as above and $a \ge 0$. There are $A, B > 0$ such that for all $n \in \mathbb{N}$*

$$\frac{A}{n^m} < \mathrm{Var}\,(\eta_a) < \frac{B}{n^m}.$$

Proof. Suppose U is the cube $Q : \max_j\{|x_j|\} \le 1$. It follows from Theorem 3 that $\mathrm{Var}(\eta_a) \sim Cn^{-m}$ in this case. Since $rQ \subseteq U \subseteq RQ$ for some $r, R > 0$, the assertion follows from the analog of Lemma 2 mentioned above. □

References

[1] H. Donnelly, C. Fefferman, *Nodal sets of eigenfunctions on Riemannian manifolds*, Invent. Math., **93** (1988), 161–183. MR0943927

[2] H. Donnelly, C. Fefferman, *Growth and geometry of eigenfunctions of the Laplacian*, Analysis and partial differential equations, 635–655, Lecture Notes in Pure and Appl. Math., **122**, Dekker, New York, 1990. MR1044811

[3] В.М. Гичев, *Несколько замечаний о сферических гармониках*, Алгебра и анализ, **20**:4 (2008), 64–86 (English translation: V.M. Gichev, *Some remarks on spherical harmonics*, St. Petersburg Math. J., **20** (2009), 553–567). MR2473744

[4] V.M. Gichev, *Metric properties in the mean of polynomials on compact isotropy irreducible homogeneous spaces*, Analysis and Math. Physics, **3** (2013), No. 2, 119–144. MR3057234

[5] D. Jakobson, N. Nadirashvili, *Quasi-symmetry of L^p norms of eigenfunctions*, Communications in Analysis and Geometry, **10** (2002), 397-408. MR1900757

[6] Н.С. Надирашвили, Д. Тот, Д. Якобсон, *Геометрические свойства собственных функций*, УМН, **56**:6(342) (2001), 67–88 (English translation: D. Jakobson, N. Nadirashvili, J. Toth, *Geometric properties of eigenfunctions*, Russ. Math. Surv. **56** (2001), No. 6, 1085–1105). MR1886720

[7] W.B. Jurkat, J.W. van Horne, *The proof of the central limit theorem for theta sums*, Duke Math. J., **48** (1981), 873-885. MR0782582

[8] D. Marinucci, I. Wigman, *The defect variance of random spherical harmonics*, J. Phys. A, Math. Theor., **44** (2011), No. 35, Article ID 355206, 16 p. MR2867816

[9] N.S. Nadirashvili, *Metric properties of eigenfunctions of the Laplace operator on manifolds*, Ann. Inst. Fourier, **41** (1991), No. 1, 259-265. MR1112199

[10] F. Nazarov, L. Polterovich, and M. Sodin, *Sign and area in nodal geometry of Laplace eigenfunctions*, Amer. J. Math., **127** (2005), no. 4, 879–910. MR2154374

[11] L.B. Pierce, *On Discrete Fractional Integral Operators and Mean Values of Weyl Sums*, Bull. Lond. Math. Soc., **43** (2011), No. 3, 597-612. MR2820148

[12] R. Salem, A. Zygmund, *On Lacunary Trigonometric Series*, Proc. Nat. Acad. Sci. USA, **33** (1947), No. 11 (November), 333–338. MR0022263

[13] I. Wigman, *Fluctuations of the nodal length of random spherical harmonics*, Comm. Math. Phys., **298** (2010), Issue 3, 787–831. MR2670928

[14] I. Wigman, *Erratum to: Fluctuations of the Nodal Length of Random Spherical Harmonics*, Comm. Math. Phys., **309** (2012), Issue 1, 293-294. MR2864794

Victor Matveevich Gichev
Omsk Branch of Sobolev Institute of Mathematics,
ul. Pevtsova, 13,
644099, Omsk, Russia
E-mail address: gichev@ofim.oscsbras.ru