AMENABILITY OF CLOSED SUBGROUPS
AND ORLICZ SPACES

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ABSTRACT. We prove that a closed subgroup $H$ of a second countable locally compact group $G$ is amenable if and only if its left regular representation on an Orlicz space $L^\Phi(G)$ for some $\Delta_2$-regular $N$-function $\Phi$ almost has invariant vectors. We also show that a noncompact second countable locally compact group $G$ is amenable if and only if the first cohomology space $H^1(G, L^\Phi(G))$ is non-Hausdorff for some $\Delta_2$-regular $N$-function $\Phi$.

Keywords: locally compact group, amenable group, second countable group, closed subgroup, $N$-function, Orlicz space, 1-cohomology.

1. Introduction

Throughout, we assume all topological groups separated.

A locally compact topological group is called amenable [7] if there exists a $G$-invariant mean on $L^\infty(G)$ or, equivalently, $G$ possesses the fixed point property: for every action of $G$ by continuous affine transformations on a nonempty convex compact subset $Q$ of a locally convex space $W$, there is a fixed point in $Q$.

Let $V$ be a Banach $G$-module, i.e., a real or complex Banach space endowed with a continuous linear representation $\alpha : G \to B(V)$. We say that $V$ almost has invariant vectors if, for every compact subset $F \subset G$ and every $\varepsilon > 0$, there exists a unit vector $v \in V$ such that $\|\alpha(g)v - v\| \leq \varepsilon$ for all $g \in F$. Here $B(V)$ stands for the space of all bounded linear endomorphisms of a Banach space $V$.

Let $V$ be a normed space of functions $f : G \to \mathbb{R}$ ($f : G \to \mathbb{C}$) such that if $f \in V$ then, for every $g \in G$, the function

$$\lambda_G(g)f(x) = f(g^{-1}x), \quad x \in G,$$

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lies in $V$ and $\|\lambda_G(g)f\|_V = \|f\|_V$. Then $\lambda_G : G \to B(V)$ is called the left regular representation of $G$ in $V$.

Examples of such function spaces $V$ are given by the space $L^p(G)$ of all real-valued functions on $G$ integrable to the power $p$ over $G$ with respect to a left-invariant Haar measure $\mu_G$. Instead of the $L^p$ spaces, one can consider more general Orlicz spaces $L^\Phi(G)$ of real-valued functions on $G$ with the finite “gauge” norm

$$\|f\|_\Phi = \inf \left\{ k > 0 : \int_G \Phi \left( \frac{f(x)}{k} \right) \, d\mu_G(x) \leq 1 \right\}.$$ 

for an $N$-function $\Phi$ ($N$-functions are defined in Section 2). Orlicz spaces on locally compact groups were considered in [5, 10] and more recently in [1, 2, 14, 15].

In [20] (see also [19, Theorem 8.3.2]), Stegeman proved that, for a locally compact group, the following conditions (for every compact set $F \subset G$ and every $\varepsilon > 0$, there exists a function $f \in L^\Phi(G)$ with $f \geq 0$ and $\|f\|_{L^\Phi(G)} = 1$ such that $\|\lambda_G(z)f - f\|_{L^\Phi(G)} < \varepsilon$ for all $z \in F$).

In [7] Eymard extended this equivalence to quotients $G/H$ of locally compact groups by closed subgroups and proved that conditions $(P_p)$ are equivalent to the amenability of $G/H$.

In [15, Proposition 2, pp. 387–389], Rao proved that a locally compact group $G$ is amenable if and only if, given a $\Delta_2$-regular $N$-function $\Phi$, $G$ satisfies the property $(P_\Phi)$ for every compact set $F \subset G$ and every $\varepsilon > 0$, there exists a function $f \in L^\Phi(G)$ with $f \geq 0$ and $\|f\|_{L^{\Phi(G)}} = 1$ such that $\|\lambda_G(z)f - f\|_{L^{\Phi(G)}} < \varepsilon$ for all $z \in F$.

Here $\| \cdot \|_{L^{\Phi(G)}}$ stands for the gauge norm in the space $L^\Phi(G)$.

In 2005, Bourdon, Martin, and Valette established the following [4, Lemma 2]:

**Theorem A.** Suppose that $p \geq 1$. Let $X$ be a countable set on which a countable group $H$ acts freely. The following are equivalent:

(i) The natural “permutation” representation $\lambda_X$ of $H$ on $L^p(X)$ almost has invariant vectors;

(ii) $H$ is amenable.

In [11], in an attempt to generalize this assertion, we proved:

**Theorem B.** Assume that $p \geq 1$. Let $G$ be a second countable locally compact group and let $H$ be a closed subgroup in $G$. The following are equivalent:

(i) The left regular representation of $H$ on $L^p(G)$ almost has invariant vectors;

(ii) $H$ is amenable.

The paper is organized as follows: In Section 2, we recall some basic notions concerning $N$-functions and Orlicz spaces. Section 3 contains some necessary information on integration on locally compact groups and homogeneous spaces. In Section 4, we prove a generalization of Theorem B, where $L^p(G)$ is replaced by the Orlicz space $L^\Phi(G)$ for any $\Delta_2$-regular $N$-function $\Phi$ (Theorem 1). In Section 5, using the equivalence of amenability and the fulfillment of the above condition $(P_\Phi)$ and a general result by Guichardet, we deduce that the nonreduced and reduced first cohomology of a noncompact second countable locally compact group coincide if and only if it is not amenable.

2. **$N$-FUNCTIONS AND ORLICZ FUNCTION SPACES**

**Definition 1.** A function $\Phi : \mathbb{R} \to \mathbb{R}$ is called an $N$-function if

(i) $\Phi$ is even and convex;

(ii) $\Phi(x) = 0 \iff x = 0$.
(iii) $\lim_{x \to 0} \frac{\Phi(x)}{x} = 0$; $\lim_{x \to \infty} \frac{\Phi(x)}{x} = \infty$.

An N-function $\Phi$ has left and right derivatives (which can differ only on an at most countable set, see, for instance, [17, Theorem 1, p. 7]). The left derivative $\varphi$ of $\Phi$ (we write $\varphi = \Phi'$ below) is left continuous, nondecreasing on $(0, \infty)$, and such that $0 < \varphi(t) < \infty$ for $t > 0$, $\varphi(0) = 0$, $\lim_{t \to \infty} \varphi(t) = \infty$. The function
$$\psi(s) = \inf\{t > 0 : \varphi(t) > s\}, \quad s > 0,$$
is called the left inverse of $\varphi$.

The functions $\Phi, \psi$ given by
$$\Phi(x) = \int_0^{|x|} \varphi(t)dt, \quad \Psi(x) = \int_0^{|x|} \psi(t)dt$$
are called complementary N-functions.

The N-function $\Psi$ complementary to an N-function $\Phi$ can also be expressed as
$$\Psi(y) = \sup\{x|y| - \Phi(x) : x \geq 0\}, \quad y \in \mathbb{R}.$$

N-functions are classified in accordance with their growth rates as follows:

**Definition 2.** An N-function $\Phi$ is said to satisfy the $\Delta_2$-condition for large $x$ (for small $x$, for all $x$), which is written as $\Phi \in \Delta_2(\infty)$ ($\Phi \in \Delta_2(0)$, or $\Phi \in \Delta_2$), if there exist constants $x_0 > 0$, $K > 2$ such that $\Phi(2x) \leq K\varphi(x)$ for $x \geq x_0$ (for $0 \leq x \leq x_0$, or for all $x \geq 0$); and it satisfies the $\nabla_2$-condition for large $x$ (for small $x$, or for all $x$), denoted symbolically as $\Phi \in \nabla_2(\infty)$ ($\Phi \in \nabla_2(0)$, or $\Phi \in \nabla_2$) if there are constants $x_0 > 0$ and $c > 1$ such that $\Phi(x) \leq \frac{1}{2c}\Phi(cx)$ for $x \geq x_0$ (for $0 \leq x \leq x_0$, or for all $x \geq 0$).

Henceforth, let $\Phi$ be an N-function and let $(\Omega, \Sigma, \mu)$ be a measure space.

**Definition 3.** The set $L^\Phi = \tilde{L}^\Phi(\Omega) = L^\Phi(\Omega, \Sigma, \mu)$ is defined to be the set of measurable functions $f : \Omega \to \mathbb{R}$ such that
$$\rho_\Phi(f) := \int_\Omega \Phi(f)d\mu < \infty.$$

**Proposition 1.** [18] The set $\tilde{L}^\Phi$ is a vector space in the following cases:
(i) $\mu(\Omega) < \infty$, $\Phi \in \Delta_2(\infty)$;
(ii) $\mu(\Omega) = \infty$, $\Phi \in \Delta_2$;
(iii) $\Omega$ is countable, $\mu$ is the counting measure on $\Omega$, $\Phi \in \Delta_2(0)$.

**Definition 4.** The linear space
$L^\Phi = L^\Phi(\Omega) = L^\Phi(\Omega, \Sigma, \mu) = \{f : \Omega \to \mathbb{R} \text{ measurable} : \rho_\Phi(af) < \infty \text{ for some } a > 0\}$
is called an Orlicz space on $(\Omega, \Sigma, \mu)$.

For an Orlicz space $L^\Phi = L^\Phi(\Omega, \Sigma, \mu)$, the N-function $\Phi$ is called $\Delta_2$-regular if $\Phi \in \Delta_2(\infty)$ when $\mu(\Omega) < \infty$ or $\Phi \in \Delta_2$ when $\mu(\Omega) = \infty$ or $\Phi \in \Delta_2(0)$ for $\mu$ a counting measure.

Let $\Psi$ be the complementary N-function to $\Phi$.

Below we use a usual identify two functions equal on a set of measure zero.

If $f \in L^\Phi$ then the functional $\| \cdot \|_\Phi$ (called the Orlicz norm) defined by
$$\|f\|_\Phi = \|f\|_{L^\Phi(\Omega)} = \sup\left\{ \left( \int_\Omega |fg|d\mu : \rho_\Psi(g) \leq 1 \right) \right\}$$
is a seminorm. It becomes a norm if \( \mu \) satisfies the finite subset property (see [17, p. 59]); if \( A \in \Sigma \) and \( \mu(A) > 0 \) then there exists \( B \in \Sigma, B \subset A \), such that \( 0 < \mu(B) < \infty \).

The gauge (or Luxembury) norm of a function \( f \in L^\Phi \) is defined by the formula

\[
\|f\|_\Phi = \|f\|_{L^\Phi(\Omega)} = \inf \left\{ k > 0 : \rho_\Phi \left( \frac{|f|}{k} \right) \leq 1 \right\}.
\]

This is a norm without any constraint on the measure \( \mu \) (see [17, p. 54, Theorem 3]).

Suppose that the measure \( \mu \) satisfies the finite subset property. As is proved in [16, Chapter 10], a left-invariant Haar measure on a locally compact group has this property.

It is well known that the Orlicz and gauge norms are equivalent, namely (see, for example, [17, pp. 61–62]):

\[
\|f\|_\Phi \leq \|f\|_\Psi \leq 2\|f\|_\Phi.
\]

We will need the following version of Hölder’s inequality for Orlicz spaces [17, p. 62]:

**Hölder’s Inequality.** If \( \Phi \) and \( \Psi \) are two complementary \( N \)-functions then \( fg \in L^1 \) and

\[
\|fg\|_1 \leq \|f\|_{\Phi} \|g\|_{\Psi} \quad (\|fg\|_1 \leq \|f\|_{\Psi} \|g\|_{\Phi}).
\]

### 3. Integration on Locally Compact Groups and Borel Sections

Recall some basic facts and definitions from the theory of integration on locally compact groups.

Let \( G \) be a locally compact group and let \( H \) be a closed subgroup in \( G \). Denote by \( \mu_G \) and \( \mu_H \) left-invariant Haar measures on \( G \) and \( H \) respectively and denote by \( \pi \) the projection \( G \to G/H \).

Denote by \( \Delta_K \) the modulus of a locally compact group \( K \).

Given a function \( f \) and a class \( u \in G/H \), take an arbitrary representative \( x \) in \( u \) and consider the function \( \alpha : y \to f(xy) \) on \( H \). If \( \alpha \) is integrable over \( H \), the left invariance of \( \mu_H \) implies that \( \int_H f(xy) \, d\mu_H(y) \) is independent of the choice of \( x \) with \( \pi(x) = u \).

It is well known that the homogeneous space \( G/H \) admits a quasi-\( G \)-invariant measure \( \mu_{G/H} \) on \( H \) which is unique up to equivalence. Here the “quasi-\( G \)-invariance” means that all left translates of \( \mu_{G/H} \) by the elements of \( G \) are equivalent to \( \mu_{G/H} \). The measure \( \mu_{G/H} \) can be described as follows (see [3, Chapter VII, 2.5] or [7]).

(a) There exists a positive continuous function \( \rho \) on \( G \) such that \( \rho(xy) = \frac{\Delta_K(y)}{\Delta_G(y)} \rho(x) \) for all \( x \in G \) and \( y \in H \).

Put \( \mu_{G/H} = (\rho \mu_G)/\mu_H \) (see [3, Definition 1 in Chapter VII, 2.2]).

(b) If \( f \in L^1(G, \mu_G) \) then the set of \( \pi = \pi(x) \in G/H \) for which \( y \mapsto f(xy) \) is not \( \mu_H \)-integrable is \( \mu_{G/H} \)-negligible, the function \( \pi = \pi(x) \mapsto \int_H f(xy) \, d\mu_H(y) \) is \( \mu_{G/H} \)-integrable, and

\[
\int_G f(x) \rho(x) \, d\mu_G(x) = \int_{G/H} \mu_{G/H}([\pi]) \int_H f(xy) \, d\mu_H(y).
\]

(c) There exists a nonnegative continuous function \( h \) on \( G \) with \( \int_H h(xy) \, dy = 1 \) for all \( x \in G \) such that a function \( w \) on \( G/H \) is \( \mu_{G/H} \)-measurable (\( \mu_{G/H} \)-integrable).
if and only if \( h(w \circ \pi) \) is \( \rho \mu_G \)-measurable (\( \rho \mu_G \)-integrable). If \( w \in L^1(G/H, \mu_{G/H}) \) then
\[
\int_{G/H} w(x) \, d\mu_{G/H}(x) = \int_G h(w(\pi(x))) \rho(x) \, d\mu_G(x).
\]

Note that a second countable locally compact space is Polish (polonais) (see [3]). As follows from Dixmier’s lemma (see [6]), if \( G \) is a Polish group and \( H \) is a closed subgroup in \( G \) then there exists a Borel section \( \sigma : G/H \to G \) (in particular, \( \pi \circ \sigma = \text{id}_{G/H} \)). We will need the following technical assertion (see [11] for a proof):

**Lemma 1.** Suppose that \( G \) is a second countable locally compact group, \( H \) is a closed subgroup in \( G \), \( \sigma : G/H \to G \) is a Borel section, and \( f \in L^1(G, \rho \mu_G) \). Then, in the above notations,
\[
\int_G f(x) \rho(x) \, d\mu_G(x) = \int_H d\mu_H(y) \int_{G/H} f(\sigma(x)y) \, d\mu_{G/H}(x).
\]

### 4. Amenability of Closed Subgroups

The main result of this section is as follows:

**Theorem 1.** Assume that \( \Phi \) is a \( \Delta_2 \)-regular \( N \)-function. Let \( G \) be a second countable locally compact group and let \( H \) be a closed subgroup in \( G \). The following are equivalent:

(i) The left regular representation of \( H \) on \( L^\Phi(G) \) almost has invariant vectors;

(ii) \( H \) is amenable.

**Proof.** Put \( \Phi' = \varphi \) and let \( \Psi \) be the complementary \( N \)-function to \( \Phi \).

Observe first that (ii) implies (i) by the equivalence of amenability and the fulfillment of the Rao–Reiter condition (\( P_\Phi \)), established by Rao in [15, Proposition 2, pp. 387–389]:

(\( P_\Phi \)) For every compact set \( F \) and every \( \varepsilon > 0 \), there exists a function \( f \in L^\Phi(H) \) with \( f \geq 0 \) and \( \| f \|_{L^\Phi(H)} = 1 \) such that \( \| \lambda_H(z)f - f \|_{L^\Phi(H)} < \varepsilon \) for all \( z \in F \).

Now, prove (i) \( \Rightarrow \) (ii). Suppose that \( L^\Phi(G) \) almost has invariant vectors for \( H \) and deduce from this that \( H \) meets Reiter’s condition (\( P_1 \)).

By the \( \Delta_2 \)-regularity of \( \Phi \), we conclude from [17, Proposition 8, p. 79] that
\[
S := \sup \{ \rho\Phi(\varphi \circ |v|) : v \in L^\Phi(G), \| v \|_{L^{\Phi(G)}} \leq 1 \} < \infty.
\]

Take \( \varepsilon > 0 \) and a compact set \( F \subset H \); choose \( f \in L^\Phi(G), \| f \|_{L^{\Phi(G)}} = 1 \), such that
\[
\| \lambda_G(z)f - f \|_{L^{\Phi(G)}} \leq \frac{\varepsilon}{2(S + 1)} \tag{2}
\]
for all \( z \in F \). Assume without loss of generality that \( f \geq 0 \) (taking \( |f| \) instead of \( f \) if necessary).

Note that
\[
|\Phi(a) - \Phi(b)| \leq |a - b| (\varphi(a) + \varphi(b)), \quad a, b \geq 0, \tag{3}
\]
because \( \varphi \) is monotone and nonnegative (cf. [15, p. 388]).

Put \( u = \Phi \circ f \). Since \( \Phi \) is \( \Delta_2 \)-regular, we have (cf. [12, p. 78]):
\[
\int_G u(x) \, d\mu_G(x) = \int_G \Phi(f(x)) \, d\mu_G(x) = \| f \|_{L^{\Phi(G)}} = 1.
\]

For \( z \in F \), using (3), Hölder’s inequality (1), (2), and the inequality
\[
\| v \|_{L^{\Phi(G)}} \leq \rho\Phi(v) + 1,
\]
we infer
\[ \|\lambda_G(z)u - u\|_{L^1(G)} = \int_G |\Phi(f(z^{-1}x)) - \Phi(f(x))| \, d\mu_G(x) \]
\[ \leq \int_G |f(z^{-1}x) - f(x)| \|\varphi(f(z^{-1}x)) + \varphi(f(x))\| \, d\mu_G(x) \]
\[ \leq \|\lambda_G(z)f - f\|_{L^1(G)} \|\varphi \circ \lambda_G(z)f + \varphi \circ f\|_{L^1(G)} \]
\[ \leq \|\lambda_G(z)f - f\|_{L^1(G)} (\|\varphi \circ \lambda_G(z)f\|_{L^1(G)} + \|\varphi \circ f\|_{L^1(G)}) \]
\[ = 2\|\varphi \circ f\|_{L^1(G)} \|\lambda_G(z)f - f\|_{L^1(G)} \leq 2(\rho_\varphi(\varphi \circ f) + 1) \|\lambda_G(z)f - f\|_{L^1(G)} \]
\[ \leq 2(S + 1) \|\lambda_G(z)f - f\|_{L^1(G)} < \varepsilon. \]

Now, let \( \sigma : G/H \to G \) be a Borel section. Consider the function
\[ U(y) = \int_{G/H} \frac{u(y\sigma(\overline{x}))}{\rho(\sigma(\overline{x}))} \, d\mu_{G/H}(\overline{x}), \quad y \in H, \]
where \( \rho \) is the function described in Section 3. By Lemma 1, since \( u \) is nonnegative, we have
\[ \|U\|_{L^1(H)} = \int_G u(x) \, d\mu_G(x) = 1. \]

Involving Lemma 1 again, we obtain the following estimates:
\[ \|\lambda_H(z)U - U\|_{L^1(H)} = \int_H \left| \int_{G/H} \frac{u(z^{-1}y\sigma(\overline{x})) - u(y\sigma(\overline{x}))}{\rho(\sigma(\overline{x}))} \, d\mu_{G/H}(\overline{x}) \right| \, d\mu_H(y) \]
\[ \leq \int_H \left| \int_{G/H} \frac{u(z^{-1}y\sigma(\overline{x})) - u(y\sigma(\overline{x}))}{\rho(\sigma(\overline{x}))} \, d\mu_{G/H}(\overline{x}) \right| \, d\mu_H(y) \]
\[ \leq \int_G \frac{|u(z^{-1}x) - u(x)|}{\rho(x)} \, d\mu_G(x) = \int_G |u(z^{-1}x) - u(x)| \, d\mu_G(x) = \|\lambda_H(z)u - u\|_{L^1(G)}. \]

So, if \( x \in F \) then \( \|\lambda_H(z)U - U\|_{L^1(H)} < \varepsilon \). Thus, \( H \) has property \( (P_1) \) and hence is amenable. Theorem 1 is proved. \( \square \)

**Remark 1.** Theorem 1 is informative only if \( H \) is noncompact since (i) and (ii) are both fulfilled when \( H \) is compact.

## 5. First Cohomology and Amenability

Let \( G \) be a topological group and let \( V \) be a topological \( G \)-module, i.e., a real or complex topological vector space endowed with a linear representation \( \pi : G \times V \to V \). The space \( V \) is called a *Banach \( G \)-module* if \( V \) is a Banach space and \( \pi \) is a representation of \( G \) by isometries of \( V \). Introduce the notation:
\[ Z^1(G, V) := \{ b : B \to V \text{ continuous} \mid b(gh) = b(g) + \pi(g)b(h) \} \quad (1\text{-cocycles}); \]
\[ B^1(G, V) = \{ b \in Z^1(G, V) \mid \exists v \in V \forall g \in G \ b(g) = \pi(g)v \} \quad (1\text{-coboundaries}); \]
\[ H^1(G, V) = Z^1(G, V)/B^1(G, V) \quad (1\text{-cohomology with coefficients in} \ V). \]

Endow \( Z^1(G, V) \) with the topology of uniform convergence on compact subsets of \( G \) and denote by \( \overline{B}^1(G, V) \) the closure of \( B^1(G, V) \) in this topology. The quotient \( \overline{H}^1(G, V) = Z^1(G, V)/\overline{B}^1(G, V) \) is called the *reduced 1-cohomology* of \( G \) with coefficients in the \( G \)-module \( V \).

The following assertion was established by Guichardet (see [8, Théorème 1]):
Lemma 2. Let $G$ be a locally compact second countable group and let $V$ be a Banach module such that

$$V^G := \{ v \in V \mid \pi(g)v = v \text{ for all } g \in G \} = 0.$$  

Then the following are equivalent:

(i) $H^1(G, V) = \overline{H}^1(G, V)$;

(ii) $V$ does not almost have invariant vectors, that is, there exists a compact subset $F \subset G$ and $\varepsilon > 0$ such that $\sup_{g \in F} \| \pi(g)v - v \| \geq \varepsilon \| v \|$ for all $v \in V$.

As is well known, if a locally compact group $G$ is noncompact then the Haar measure of the whole group is infinite [9, Theorem 15.9]. Hence, constant functions are not integrable over $G$. Therefore, $L^\Phi(G)^G = 0$. Thus, combining Lemma 2 with the Rao–Reiter condition ($P_\Phi$), we obtain the following generalization of Corollary 2.4 in [13, p. 86] to coefficients in an Orlicz space:

Proposition 2. Suppose that $\Phi$ is a $\Delta_2$-regular $N$-function. If $G$ is a noncompact second countable locally compact group then the following are equivalent:

(i) $H^1(G, L^\Phi(G)) = \overline{H}^1(G, L^\Phi(G))$;

(ii) $G$ is not amenable.

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