S(e)MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 10, стр. 591–614 (2013) УДК 517.9
MSC 35C05

REPRESENTATION OF SOLUTIONS TO FUNCTIONAL AND EVOLUTION EQUATIONS AND IDENTIFICATION PROBLEMS

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Abstract. In this paper, some problems of representing solutions to operator-functional and evolution equations are studied. Formulas of such representations for use in some identification problems are presented.

Keywords: functional equations, evolution equations, iterations, identification problems.

1. Introduction

Let X be a set of elements x, $\lambda(x)$, a mapping of X into X, and E, a Banach space. Here $u(x,t)$ and $w(x,p)$ denote elements of a vector space E depending on $x \in X$ and numerical parameters t and p.

Let $A(x)$ be a linear operator from E into E depending on $x \in X$, and let $L = \sum_{k=0}^{N} c_k \frac{\partial^k}{\partial t^k}$, $N \geq 1$ be a differential operator with constant coefficients c_k. Assume that not only $Lu(x,t) \in E$, $A(x)u(\lambda(x),t) \in E$, $A(x)w(\lambda(x),p) \in E$, but also the powers $A^k(x)w(\lambda(x),p)$, $A^k(\lambda(x))w(\lambda(x),p)$, $A^k(x)u(\lambda(x),t)$, $A^k(\lambda(x))u(\lambda(x),t)$, $k = 1,2,\ldots$, are elements of the Banach space E, and $AL = LA$.

This paper considers direct and inverse problems for operator-functional equations with parameter and evolution equations, namely, for the operator-functional equations

$$pw(x,p) = A(x)w(\lambda(x),p) + a(x,p) \tag{1}$$

Anikonov, Yu.E., Representation of solutions to functional and evolution equations and identification problems.

This work was supported by the Mathematical Sciences Department of the Russian Academy of Sciences (project no. 1.3.1-2012).

Received September, 4, 2013, published October, 10, 2013.

and the evolution equations

$$Lu(x,t) \equiv \sum_{k=0}^{N} c_k \frac{\partial^k u}{\partial t^k} = A(x)u(\lambda(x),t) + R(x,t). \tag{2}$$

First, consider more specific examples of relations (1) and evolution equations (2).

We begin with functional equations. At a fixed p, for instance, $p = 1$, given $\lambda(x)$, $\widetilde{a}(x) = a(x,1)$ and a real function $A(x)$, $x \in X$, equality (1) can be considered a linear functional equation for $\widetilde{w}(x) = w(x,1)$, namely, it is necessary to find $\widetilde{w}(x) \in E$, $x \in X$, if

$$\widetilde{w}(x) = A(x)\widetilde{w}(\lambda(x)) + \widetilde{a}(x).$$

Numerous examples and results of the theory and applications of functional equations can be found in [1] – [5] and [6] – [7].

We shall restrict our consideration to one result from book [5]. Here X is a topological space, $A(x)$ is a real function, the vector space E is a Banach space, and the estimates $0 < |A(x)| \le q < 1$, $\|\widetilde{a}(x)\| \le \alpha$ are satisfied.

In this case, to solve $\widetilde{w}(x)$ of the functional equation

$$\widetilde{w}(x) = A(x)\widetilde{w}(\lambda(x)) + \widetilde{a}(x)$$

paper [5] presents the formula

$$\widetilde{w}(x) = \sum_{k=0}^{\infty} \prod_{j=1}^{k} A(\lambda^{[j-1]}(x))\widetilde{a}(\lambda^{[k]}(x)), \tag{3}$$

where $\prod_{j=1}^{k} A(\lambda^{[j-1]}(x)) = 1$ at $k = 0$, and $\lambda^{[k]}(x) = \underbrace{\lambda(\lambda(\dots(\lambda(x))))}_{k}$, $\lambda^{[0]}(x) = x$ is a superposition or iteration of order k. Notice that formally one can consider (which is reasonable) formula (3) for $\widetilde{w}(x)$ in the case where $A(x)$ and its powers $A^k(x)$ are linear operators from E into E, say, differential or integral as appropriate. It is self-evident that in this case additional conditions will be needed for the operator-functional equation and formula (3) to be well-posed.

Nevertheless, this line of research is of current interest and holds much promise.

As an example of iteration, we present a formula found by the author.

If X is a normed vector space with a norm $\|x\|$, $x \in X$, at $\lambda(x) = \dfrac{x}{(1+\|x\|^\alpha)^{1/\alpha}}$, $\alpha > 0$ is a fixed number, and the following equality holds:

$$\lambda^{[k]}(x) = \frac{x}{(1+k\|x\|^\alpha)^{1/\alpha}}.$$

In addition, if some mapping $\mu(x)$ of space X into itself satisfies the estimate

$$\|\mu(x)\| \le \frac{\|x\|}{(1+\|x\|^\alpha)^{1/\alpha}},$$

we have

$$\|\mu^{[k]}(x)\| \le \frac{\|x\|}{(1+k\|x\|^\alpha)^{1/\alpha}}, \quad k = 0, 1, \dots.$$

As for differential equations, let $D \subset \mathbb{R}^n$ be a domain of a real Euclidean space $\mathbb{R}^n$, $n \ge 1$, variables $y = (y_1, y_2, \dots, y_n)$; the vector space E consists of all infinitely differentiable with respect to y functions or vector-functions $w(x,y)$, $x \in X$, $y \in$

D, and $A(x)$ is a linear differential operator $A(x) = \sum_{|\alpha|\le m} A_\alpha(x,y)D_y^\alpha$, where the coefficients $A_\alpha(x,y)$ are functions or matrices infinitely differentiable with respect to y. In this case, relation (1) can be considered a differential equation for $w(x,y,p)$ or a system of equations for the variable y with parameters p and x:

$$pw(x,y,p) = \sum_{|\alpha|\le m} A_\alpha(x,y)D_y^\alpha w(\lambda(x),y,p) + a(x,y,p). \tag{4}$$

Differential equations can be obtained from relation (1) or in another way.

Let the set X be a ball $\omega = \{x : |x| < r\}$ of the Euclidean space $\mathbb{R}^n$ of variables $x = (x_1, x_2, \dots, x_n)$, $n \ge 1$, and let $\lambda(x)$ be a differentiable mapping of the ball ω into itself.

Set $\lambda(x) = x + h\mu(x)$, where h is a constant and $\mu(x)$ is a vector-function.

Assume that $w(x,p)$ is an infinitely differentiable function of the variable x, $x \in \omega$. Then with the Taylor formula

$$w(\lambda(x),p) = w(x + h\mu(x),p) = \sum_{|\alpha|=0}^{m} \frac{D^\alpha w(x,p)}{\alpha!}(h\mu(x))^\alpha + R_m(x,p). \tag{5}$$

Substituting (5) into (1), we have

$$pw(x,p) = \sum_{|\alpha|=0}^{m} A(x)\frac{D^\alpha w(x,p)}{\alpha!}(h\mu(x))^\alpha + AR_m(x,p) + a(x,p),$$

where $AR_m(x,p) = h^{m+1}\widetilde{R}_m(x,p)$. If h is small, the term AR_m can be ignored.

Thus, equality (1) turns into an approximate differential equation with the parameter p:

$$pw(x,p) = \sum_{|\alpha|=0}^{m} A(x)\frac{D^\alpha w(x,p)}{\alpha!}(h\mu(x))^\alpha + a(x,p),$$

and at $\lim_{m\to\infty} AR_m(x,p) = 0$, into a linear equation of infinite order:

$$pw(x,p) = \sum_{|\alpha|=0}^{\infty} A(x)\frac{D^\alpha w(x,p)}{\alpha!}(h\mu(x))^\alpha + a(x,p)$$

Here the operator $A(x)$ can also be differential, for instance, a Laplace operator, $A = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$.

Notice also that at $\lambda(x) = x$ equality (1) can be considered a linear operator equation with parameter p, since $A(x)$ is a differential or other linear operator, for instance, $A = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + q(x)$, $q(x)$ is some function. In spectral theory such equations are called characteristic [8]. Formally the equation $pw(x,p) = A(x)w(x,p) + a(x,p)$, where p is analytic in the neighborhood of zero, can be obtained from the characteristic equation $p_1\widetilde{w}(x,p_1) = \widetilde{A}(x)\widetilde{w}(x,p_1) + \widetilde{a}(x,p_1)$ with singularities of the resolvent in the neighborhood of zero by the transformation

$$A(x) = \widetilde{A}^{-1}(x), \quad p = \frac{1}{p_1}, \quad a(x,p) = -\frac{1}{p_1}A^{-1}\widetilde{a}(x,p_1).$$

Notice a relation between (1) and the evolution equations. Consider an evolution equation in a Banach space E:

$$\frac{\partial u(x,t)}{\partial t} = A(x)u(\lambda(x),t) + R(x,t), \quad t \geq 0, \quad x \in X \tag{6}$$

with an additional condition $u|_{t=0} = 0$. This equation is an evolution equation of type (2). By a formal Laplace transform in the variable t,

$$w(x,p) = \int_0^\infty u(x,t)\mathrm{e}^{-pt}dt, \quad a(x,p) = \int_0^\infty R(x,t)\mathrm{e}^{-pt}dt,$$

equation (2) transforms into the operator-functional equation

$$pw(x,p) = A(x)w(\lambda(x),p) + a(x,p),$$

that is, into (1).

In a more specific case of $X = D \subset \mathbb{R}^n$, where D is the domain of the Euclidean space $\mathbb{R}^n$ and $A(x)$ and $\lambda(x)$ are complex valued domains D that are continuous in the closure, the evolution equation

$$\frac{\partial v(y,t)}{\partial t} = \int_{\mathbb{R}^n} K(y,z)v(z,t)dz + R(y,t), \quad y \in \mathbb{R}^n, \quad t \geq 0$$

with a zero initial condition $v|_{t=0} = 0$, where

$$K(y,z) = \frac{1}{(2\pi)^n}\int_D A(\xi)\mathrm{e}^{\mathrm{i}[\xi y - \lambda(\xi)z]}d\xi,$$

by the formal transform

$$w(x,p) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_0^\infty v(y,t)\mathrm{e}^{-[\mathrm{i}xy+pt]}dydt,$$

$$a(x,p) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_0^\infty R(y,t)\mathrm{e}^{-[\mathrm{i}xy+pt]}dydt$$

turns into the functional equation (1):

$$pw(x,p) = A(x)w(\lambda(x),p) + a(x,p).$$

Notice that if $\lambda(\xi) = \xi$, the kernel $K(y,z)$ depends on the difference $y - z$:

$$K(y-z) = \frac{1}{(2\pi)^n}\int_D A(\xi)\mathrm{e}^{\mathrm{i}\xi(y-z)}d\xi.$$

It is known that the required analyticity of the Laplace transform $w(x,p) = \int_0^\infty u(x,t)\mathrm{e}^{-pt}dt$ with respect to the variable p for $p > -1$ corresponds to the existence of the integral $\int_0^\infty u(x,t)\mathrm{e}^{t}dt$, that is, to a fast decrease in the solution $u(x,t)$ of equation (6) in the variable t. This condition, together with the condition $u|_{t=0} = 0$, can be quite suitable information for identification problems for (6). In addition, sometimes finiteness of the source function $R(x,t)$ in the variable t implies finiteness of the solution to equation (6) in the variable t. Here the functions $w(x,p) =$

$\int\limits_0^\infty u(x,t)\mathrm{e}^{-pt}dt$, $a(x,p) = \int\limits_0^\infty R(x,t)\mathrm{e}^{-pt}dt$ are integer with respect to the variable p and, hence, they are expanded into a series $w(x,p) = \sum\limits_{k=0}^\infty w_k(x)p^k$, $a(x,p) = \sum\limits_{k=0}^\infty a_k(x)p^k$, $p \in \mathbb{C}$.

Examples of evolution equations (2) are many linear differential equations and systems of equations of mathematical physics, in particular, equations with a divergent argument.

2. Representation of solutions.

We begin our study of problems for the equation

$$pw(x,p) = A(x)w(\lambda(x),p) + a(x,p)$$

by using the method of generating functions [9], equating the coefficients at the same powers of the formal series.

Lemma 1. *Let* $w(x,p) = \sum\limits_{k=0}^\infty w_k(x)p^k$, $A(x)w(\lambda(x),p) = \sum\limits_{k=0}^\infty A(x)w_k(\lambda(x))p^k$, $a(x,p) = \sum\limits_{k=0}^\infty a_k(x)p^k$, $w_k(x) \in E$, *and* $a_k(x) \in E$ *be such that*

$$pw(x,p) = A(x)w(\lambda(x),p) + a(x,p), \quad x \in X.$$

Then we have the following equalities:

$$\begin{aligned} &A(x)w_0(\lambda(x)) + a_0(x) = 0,\\ &w_{k-1}(x) = A(x)w_k(\lambda(x)) + a_k(x), \quad k = 1,2,\ldots, \end{aligned} \tag{7}$$

and for any $m \geq 1$

$$a_0(x) + \sum_{k=1}^m \prod_{j=1}^k A(\lambda^{[j-1]}(x))a_k(\lambda^{[k]}(x)) + \prod_{j=1}^{m+1} A(\lambda^{[j-1]}(x))w_m(\lambda^{[m+1]}(x)) = 0,$$

the converse is also true.

Proof. Substituting the series $w(x,p) = \sum\limits_{k=0}^\infty w_k(x)p^k$, $a(x,p) = \sum\limits_{k=0}^\infty a_k(x)p^k$, $A(x)w(\lambda(x),p) = \sum\limits_{k=0}^\infty A(x)w_k(\lambda(x))p^k$ into the equation

$$pw(x,p) = A(x)w(\lambda(x),p) + a(x,p),$$

and equating the terms at the same powers p, we have

$$\begin{aligned} &A(x)w_0(\lambda(x)) + a_0(x) = 0,\\ &w_{k-1}(x) = A(x)w_k(\lambda(x)) + a_k(x), \quad k = 1,2,\ldots \end{aligned}$$

Hence, successively eliminating $w_k(x)$, $k = 0,1,2,\ldots,m$, we obtain the equalities

$$a_0(x) + \sum_{k=1}^m \prod_{j=1}^k A(\lambda^{[j-1]}(x))a_k(\lambda^{[k]}(x)) + \prod_{j=1}^{m+1} A(\lambda^{[j-1]}(x))w_m(\lambda^{[m+1]}(x)) = 0$$

for any $m \geq 1$.

The converse, multiplication of (7) to p^k and formal summation over k lead to the equation (1). □

Corollary.

If $\lim_{m\to\infty}\prod_{j=1}^{m+1} A(\lambda^{[j-1]}(x))w_m\lambda^{[m+1]}(x))=0$, *then*

$$a_0(x)+\sum_{k=1}^{\infty}\prod_{j=1}^{k} A(\lambda^{[j-1]}(x))a_k(\lambda^{[k]}(x))=0. \tag{8}$$

Remark. In the theory and applications of the generating functions $\sum_{k=0}^{\infty} a_k p^k$ (see, for instance, [9]), specification of p and convergence of the series are not assumed, but relations corresponding to the Cauchy power series algebra are used.

In this case the main recurrence equalities of type (7) determine specific numbers, functions, etc. Here it should be noted that if in (7) $x\in\mathbb{R}^1$, $A(x)=2x$, $\lambda(x)=x$, $a_k(x)=-w_{k+1}(x)$, relations (7) take the form

$$w_{k-1}(x)=2xw_k(x)-w_{k+1}(x),\quad k\geq 1.$$

A solution to this recurrence relation are the Chebyshev polynomials $w_k=\cos(k\arccos x)$ ([10]) at $w_0(x)=1$ and $w_1(x)=x$.

However, if $X=(1,2,\dots,n,\dots)$, $\lambda(n)=n+1$, $A(n)=1$, $a_k(n)=nw_k(n)$, (7) transforms to the recurrence equality

$$w_{k-1}(n)=w_k(n+1)+nw_k(n).$$

A solution to this recurrence relation are Stirling numbers of the first kind [9].

If $\lambda(n)=n+1$, $A(n)=1$, $a_k(n)=-kw_k(n)$, (7) has the form

$$w_{k-1}(n)=w_k(n+1)-kw_k(n),$$

which corresponds to Stirling numbers of the second kind [9].

Theorems 1, 2, and 3 presented below form henceforth a basis for constructing special solutions and identification problems.

Theorem 1. *So that*

$$w(x,p)=\sum_{k,s=0}^{\infty}\prod_{j=1}^{s} A(\lambda^{[j-1]}(x))a_{k+s+1}(\lambda^{[s]}(x))p^k$$

can formally, and at $a_{m+i}=0$, $i=1,2,\dots$, *exactly, satisfy the equation*

$$pw(x,p)=A(x)w(\lambda(x),p)+\sum_{k=0}^{\infty}a_k(x)p^k,\quad x\in X,$$

it is necessary and sufficient that the following equality hold:

$$a_0(x)+\sum_{k=1}^{\infty}\prod_{j=1}^{k} A(\lambda^{[j-1]}(x))a_k(\lambda^{[k]}(x))=0.$$

Proof. Consider the difference

$$A(x)w(\lambda(x),p)+\sum_{k=0}^{\infty}a_k(x)p^k-pw(x,p).$$

We have

$$pw(x,p) = \sum_{k,s=0}^{\infty} \prod_{j=1}^{s} A(\lambda^{[j-1]}(x)) a_{k+s+1}(\lambda^{[s]}(x)) p^{k+1} =$$
$$= \sum_{\substack{k=0\\s=1}}^{\infty} \prod_{j=1}^{s} A(\lambda^{[j-1]}(x)) a_{k+s+1}(\lambda^{[s]}(x)) p^{k+1} + \sum_{k=0}^{\infty} a_{k+1}(x) p^{k+1} =$$
$$= \sum_{\substack{k=1\\s=1}}^{\infty} \prod_{j=1}^{s} A(\lambda^{[j-1]}(x)) a_{k+s+1}(\lambda^{[s]}(x)) p^{k} + \sum_{k=1}^{\infty} a_k(x) p^k.$$

Here we use the fact that

$$\lambda^{[0]}(x) = x, \quad \prod_{j=1}^{0} A(\lambda^{[j-1]}(x)) = 1.$$

Then

$$A(x)w(\lambda(x),p) = A(x) \sum_{k,s=0}^{\infty} \prod_{j=1}^{s} A(\lambda^{[j]}(x)) a_{k+s+1}(\lambda^{[s+1]}(x)) p^k$$
$$= A(x) \sum_{s=0}^{\infty} \prod_{j=1}^{s} A(\lambda^{[j]}(x)) a_{s+1}(\lambda^{[s+1]}(x))$$
$$+ A(x) \sum_{\substack{k=1\\s=0}}^{\infty} \prod_{j=1}^{s} A(\lambda^{[j]}(x)) a_{k+s+1}(\lambda^{[s+1]}(x)) p^k$$
$$= \sum_{s=1}^{\infty} \prod_{j=1}^{s} A(\lambda^{[j-1]}(x)) a_s(\lambda^{[s]}(x))$$
$$+ A(x) \sum_{\substack{k=1\\s=1}}^{\infty} \prod_{j=1}^{s} A(\lambda^{[j-1]}(x)) a_{k+s}(\lambda^{[s]}(x)) p^k.$$

Thus,

$$A(x)w(\lambda(x),p) + a_0(x) + \sum_{k=1}^{\infty} a_k(x) p^k - pw(x,p) =$$
$$= a_0(x) + \sum_{k=1}^{\infty} \prod_{j=1}^{k} A(\lambda^{[j-1]}(x)) a_k(\lambda^{[k]}(x)).$$

The theorem has been proved. □

Now, consider, in a linear Banach space E, the evolution equation (2):

$$Lu(x,t) = A(x)u(\lambda(x),t) + R(x,t).$$

Let $b(t)$ be an infinitely differentiable real function of variable t, $t_0 \le t \le t_1$.

Theorem 2. *So that*

$$u(x,t) = \sum_{k,s=0}^{\infty} \prod_{j=1}^{s} A(\lambda^{[j-1]}(x)) a_{k+s+1}(\lambda^{[s]}(x)) L^k b(t)$$

can formally, and at $a_{m+i}(x) = 0$, $i = 1, 2, \ldots$, exactly, satisfy the equation

$$Lu(x,t) = \sum_{k=0}^{N} c_k \frac{\partial^k u}{\partial t^k} = A(x)u(\lambda(x),t) + \sum_{k=0}^{\infty} a_k(x)L^k b(t), \quad t_0 \le t \le t_1, \quad x \in X,$$

it is necessary and sufficient that the following equality hold:

$$a_0(x) + \sum_{k=1}^{\infty}\prod_{j=1}^{k} A(\lambda^{[j-1]}(x))a_k(\lambda^{[k]}(x)) = 0.$$

The proof is the same as that of Theorem 1.

Remark. In accordance with the theory of differential equations, the solution to evolution equation (2) determined by the formula of Theorem 2 is a particular solution, and a general solution to this equation is

$$u(x,t) = \widetilde{u}(x,t) + \sum_{k,s=0}^{\infty}\prod_{j=1}^{s} A(\lambda^{[j-1]}(x))a_{k+s+1}(\lambda^{[s]}(x))L^k b(t),$$

where $\widetilde{u}(x,t)$ is a general solution to the homogeneous equation

$$Lu(x,t) = A(x)u(\lambda(x),t).$$

Conditions sufficient for the series convergence in Theorems 1 and 2 are presented in Lemmas 2 and 3 below.

Lemma 2. *If the space E is a Banach one and $\|A(x)\| \le q < 1$, $\|a_k(x)\| \le \alpha$, $x \in X$, $k = 0, 1, \ldots$, the series*

$$a_0(x) + \sum_{k=1}^{\infty}\prod_{j=1}^{k} A(\lambda^{[j-1]}(x))a_k(\lambda^{[k]}(x)), \quad \sum_{k,s=0}^{\infty}\prod_{j=1}^{s} A(\lambda^{[j-1]}(x))a_{k+s+1}(\lambda^{[s]}(x))p^k,$$

$|p| < 1$, converge.

Proof. For $m \ge 1$ and

$$S_m(x) = a_0(x) + \sum_{k=1}^{m}\prod_{j=1}^{k} A(\lambda^{[j-1]}(x))a_k(\lambda^{[k]}(x)),$$

we have

$$\|S_m\| \le \alpha\left(1 + \sum_{k=1}^{m} q^k\right) < \alpha\frac{1}{1-q}.$$

Therefore, there exists $\lim_{m\to\infty} S_m(x) = a_0(x) + \sum_{k=1}^{\infty}\prod_{j=1}^{k} A(\lambda^{[j-1]}(x))a_k(\lambda^{[k]}(x))$. As for the series for $w(x,p)$, this series is majorized by the series

$$\alpha\sum_{k,s=0}^{\infty} q^s p^k = \alpha\frac{1}{(1-q)(1-p)}, \quad 0 < q < 1, \quad |p| < 1.$$

□

Lemma 3. *Let the space E be a Banach space, and let $\|A(x)\| \le q < 1$, $\|a_k(x)\| \le \alpha$, $|L^k b(t)| \le p_0$, $0 < p_0 < 1$, $x \in X$, $t_0 \le t \le t_1$, $k = 0, 1, \ldots$. Then the series*

$$a_0(x) + \sum_{k=1}^{\infty} \prod_{j=1}^{k} A(\lambda^{[j-1]}(x)) a_k(\lambda^{[k]}(x)),$$

$$\sum_{k,s=0}^{\infty} b^{(k)}(t) \prod_{j=1}^{s} A(\lambda^{[j-1]}(x)) a_{k+s+1}(\lambda^{[s]}(x)) p^k,$$

$|p| < p_0$ converge for $x \in X$ and $t \in [t_0, t_1]$.

The proof of Lemma 3 is the same as that of Lemma 2.

Consider the case of finite-dimensional expansion of elements $a(x,p) = \sum_{k=0}^{m} a_k(x) p^k$ and $R(x,t) = \sum_{k=0}^{m} L^k b(t) a_k(x)$, that is, when $a_{m+j} = 0$, $j = 1, 2, \ldots$. As already noted in Theorems 1 and 2, the series convergence need not be proved, and, hence, the corresponding constraints on $A(x)$ and $a_k(x)$ need not be placed.

Let us formulate this in Theorem 3, somewhat changing the order of summation.

Theorem 3. *Elements $w(x,p)$ and $u(x,t)$ of the Banach space E*

$$w(x,p) = \sum_{k=0}^{m-1} \sum_{l=k+1}^{m} \prod_{j=1}^{l-(k+1)} A(\lambda^{[j-1]}(x)) a_l(\lambda^{[l-(k+1)]}(x)) p^k,$$

$$u(x,t) = \sum_{k=0}^{m-1} \sum_{l=k+1}^{m} \prod_{j=1}^{l-(k+1)} A(\lambda^{[j-1]}(x)) a_l(\lambda^{[l-(k+1)]}(x)) L^k b(t)$$

satisfy the equations

$$pw(x,p) = A(x) w(\lambda(x), p) + \sum_{k=0}^{m} a_k(x) p^k,$$

$$Lu(x,t) = A(x) u(\lambda(x), t) + \sum_{k=0}^{m} a_k(x) L^k b(t),$$

respectively, if and only if the following equality holds:

$$a_0(x) + \sum_{k=1}^{m} \prod_{j=1}^{k} A(\lambda^{[j-1]}(x)) a_k(\lambda^{[k]}(x)) = 0.$$

Notice a particular case where $a_k(x) = 0$, $k = 2, 3, \ldots$, that is, $a(x,p) = a_0(x) + p a_1(x)$, $R(x,t) = b(t) a_0(x) + L b(t) a_1(x)$, $L = \sum_{k=0}^{N} c_k \dfrac{\partial^k}{\partial t^k}$. In this case the equation has the form $a_0(x) + A(x) a_1(\lambda(x)) = 0$, and the solutions to the equations

$$pw(x,p) = A(x) w(\lambda(x), p) + a(x,p),$$
$$Lu(x,t) = A(x) u(\lambda(x), t) + R(x,t)$$

are, respectively,

$$w(x,p) = a_1(x), \quad u(x,t) = b(t) a_1(x).$$

Notice that the relation $a_0(x) + A(x) a_1(\lambda(x)) = 0$ does not have iterations of the mapping $\lambda(x)$, which is more convenient when studying inverse problems.

Thus, construction of special solutions to equations (1) and (2) using formulas of Theorems 1, 2, and 3 is directly associated with the main equality founded here

$$a_0(x) + \sum_{k=1}^{\infty} \prod_{j=1}^{k} A(\lambda^{[j-1]}(x)) a_k(\lambda^{[k]}(x)) = 0, \tag{9}$$

which can (and must) be considered an identification equation for some elements $\lambda(x)$, $a_k(x)$, and $A(x)$.

In more detail, the formulas for $w(x,p)$ and $u(x,t)$,

$$w(x,p) = \sum_{k,s=0}^{\infty} \prod_{j=1}^{s} A(\lambda^{[j-1]}(x)) a_{k+s+1}(\lambda^{[s]}(x)) p^k,$$

$$u(x,t) = \sum_{k,s=0}^{\infty} L^k b(t) \prod_{j=1}^{s} A(\lambda^{[j-1]}(x)) a_{k+s+1}(\lambda^{[s]}(x)),$$

contain elements $a_k(x)$ of the Banach space E, $k = 1, 2, \dots$ ($a_0(x)$ is not included), the mapping $\lambda(x)$ of the set X into itself, and the linear operator $A(x)$. A situation is possible where a part of these elements is a priori unknown and has also to be found, which is typical for identification problems.

Equation (9), as a new constructive method for investigating inverse problems with applications in prognosis theory, control theory etc., is designed for this. Evidently, there are many problems associated with equation (9). Let us consider some of them.

3. Problems associated with equality (9).

First, consider problems of search for a mapping $\lambda(x)$ at fixed $A(x)$ and $a_k(x)$. If $A(x)$ is a real function, equation (9) for $\lambda(x)$ is called an equation with iterations (see [11] – [14] and the references therein).

Notice that if we assume that $X \subset E$, $A(x) = 1$, the element $y = a_1(x)$ has an inverse one in E, $x = g(y)$, the equation with iterations (9) can be written as an equation of the 2nd kind for $\lambda(x)$:

$$\lambda(x) = g(-a_0(x) - \sum_{k=2}^{\infty} a_k(\lambda^{[k]}(x))).$$

This representation is the major method for proving the theorems of existence and uniqueness of a solution to the iterative equations using the results of fixed point theory, mainly of the Schauder and Banach theorems ([15], [13], [14]).

Other problems of finding $\lambda(x)$ from equation (9) are initial–boundary value problems for iterative functional-differential equations ([16], [17]). Actually, when X is an Euclidean space domain, $a_k(x)$, $k = 1, 2, \dots$ are fixed vector-functions, and $A(x)$ is a function, for instance, $A(x) = 1$, given in this domain, we can set in (9) $a_0(x) = -\widetilde{L}\lambda$, where $\widetilde{L} = \sum_{|\alpha| \le m} \widetilde{A}_\alpha D^\alpha$ is a differential operator, since the solutions $w(x,p)$, $u(x,t)$ of equations (1), (2) of the theorems do not depend explicitly on $a_0(x)$. In this case the equation for $\lambda(x)$ becomes, according to terminology [16], [17], an iteratively functional-differential equation,

$$\widetilde{L}\lambda(x) = \sum_{k=1}^{\infty} \prod_{j=1}^{k} A(\lambda^{[j-1]}(x)) a_k(\lambda^{[k]}(x)). \tag{10}$$

In this case as well, the major method for investigating the initial–boundary value problems for equations (10) are methods of fixed point theory (see [16], [17]) based on the following representation of $\lambda(x)$:

$$\lambda(x) = \widetilde{L}^{-1}\left(a_0(x) + \sum_{k=1}^{\infty}\prod_{j=1}^{k} A(\lambda^{[j-1]}(x))a_k(\lambda^{[k]}(x))\right).$$

Consider an algebraic problem of search for the function $A(x)$ using equality (9). Let in (9) X be some compact of the Euclidean space, $\lambda(x) = x$, $a_{m+j}(x) = 0$, $j = 1, 2, \ldots$, and let $a_k(x)$ be fixed continuous complex-valued functions $k = 1, 2, \ldots, m$, $x \in X$. In this case equation (9) for the function $A(x)$ is algebraic:

$$a_0(x) + \sum_{k=1}^{m} A^k a_k(x) = 0, \quad x \in X.$$

If A is a differential operator with a symbol $\widetilde{A}(\xi)$, $\xi \in \mathbb{R}^n$, $Ae^{i\xi x} = e^{i\xi x}\widetilde{A}(\xi)$, and the functions $a_k(x)$, $k = 0, 1, 2, \ldots, m$, are integer functions of exponential type, using the Fourier transform for x and the Paley-Wiener theorem we also have, in the variables ξ, an algebraic equation for the symbol $\widetilde{A}(\xi)$,

$$\widehat{a}_0(\xi) + \sum_{k=1}^{m} \widetilde{A}^k \widehat{a}_k(\xi) = 0, 1 \quad a_k(x) = \int_K \widehat{a}_k(\xi) e^{i\xi x} d\xi,$$

and $K \subset \mathbb{R}^n$ is a compact.

The corresponding information and a description of the investigations can be found in [18] (see also [19]).

Here, the use of algebraic methods to study equations (9) at additional algebraic constraints on E, X, $A(x)$, $\lambda(x)$, and $a_k(x)$ holds much promise and is of interest.

The space E and the set X can be fields, groups, etc., and the operators $A(x)$, mappings $\lambda(x)$, and elements $a_k(x)$ can also be characterized by algebraic terms, for instance, $a_k(x)$ are p-adic numbers, [20], etc.

The question is in the following: what problems and results of algebra are directly connected with the search for elements $\lambda(x)$, $A(x)$, and $a_k(x)$ of the major relation, which is the essence of the method for investigating inverse problems,

$$a_0(x) + \sum_{k=1}^{\infty}\prod_{j=1}^{k} A(\lambda^{[j-1]}(x))a_k(\lambda^{[k]}(x)) = 0?$$

It would also be interesting to find a connection with holomorphic dynamics. Holomorphic dynamics studies the iterative properties of holomorphic functions, and the solution to functional and differential-functional equations with such iterations [21].

The use of methods of the theory of differential equations for finding, from (9), coefficients of the differential operator $A(x)$, the functions $a_k(x)$, etc, also seems important and promising. Let us present some examples.

Let in the evolution equation (2)

$$Lu = A(x)u(\lambda(x), t) + R(x, t),$$

$x \in \Omega$ is a domain in $\mathbb{R}^n$. The source function has the form

$$R(x,t) = b(t)a_0(x) + a(x)\sum_{k=1}^{m} L^k b(t),$$

where the function $a_0(x)$ is fixed, $a(x)$ is a sought-for function, and $A(x) = \sum_{|\alpha|\le m} A_\alpha(x)D^\alpha$ is a differential operator with infinitely differentiable known coefficients $A_\alpha(x)$, $\lambda(x) = x$. In this case equation (9) for $a(x)$ has the form of a linear differential equation:

$$a_0(x) + \sum_{k=1}^{m} A^k a(x) = 0,$$

which is important in problems of identification of the sources $R(x,t)$.

Consider an equation with a variable parameter,

(11)
$$c_1\frac{\partial w}{\partial t} + c_2\frac{\partial^2 w}{\partial t^2} = p\Delta w + R(x,t), \quad x \in D \subset \mathbb{R}^n, \quad t \ge 0, \quad 0 < p < 1, \quad ,n \ge 3.$$

Here c_1 and c_2 are fixed constants which are not simultaneously zero, $R(x,t)$ is a continuous function in the closure of the domain D with a sufficiently smooth boundary ∂D. Investigations of inverse problems for equations of this type can be found in [22] and the references therein. Sometimes problems of search for $w(x,t,p)$ and $R(x,t)$ at given and sufficiently smooth $w(x,t,p)$, $\dfrac{\partial w(x,t,p)}{\partial n}$ at the boundary ∂D of the domain D cause questions associated with relation (9).

Let

$$Aw = \frac{1}{(n-2)\omega_n}\int_D \frac{c_1\dfrac{\partial w(y,t,p)}{\partial t} + c_2\dfrac{\partial^2 w(y,t,p)}{\partial t^2}}{|x-y|^{n-2}}\,dy,$$

$$a(x,t,p) = \frac{p}{(n-2)\omega_n}\int_{\partial D}\left(w(y,t,p)\frac{\partial}{\partial n_y}\frac{1}{|x-y|^{n-2}} - \frac{1}{|x-y|^{n-2}}\frac{\partial w(y,t,p)}{\partial n_y}\right)dS_y,$$

$$\widetilde{R}(x,t) = -\frac{1}{(n-2)\omega_n}\int_D \frac{R(y,t)}{|x-y|^{n-2}}\,dy.$$

Then equation (11), by inverting the Laplace operator, [23], turns into an operator-functional equation (1) with a parameter p, namely,

$$pw(x,t,p) = Aw(x,t,p) + \widetilde{R}(x,t) + a(x,t,p)$$

with a given operator A and function $a(x,t,p)$.

Assuming analyticity of the solution $w(x,t,p)$ and function $a(x,t,p)$ with respect to the parameter p, that is,

$$w(x,t,p) = \sum_{k=0}^{\infty} w_k(x,t)p^k, \quad |p| < 1,$$

$$pa(x,t,p) = \sum_{k=1}^{\infty} a_k(x,t)p^k, \quad |p| < 1,$$

using Theorem 1 we obtain formal formulas for $w(x,t,p)$ and $\widetilde{R}(x,t)$:

$$w(x,t,p) = \sum_{k,s=0}^{\infty} A^s a_{k+s+1}(x,t)p^k,$$
$$\widetilde{R}(x,t) = \sum_{k=1}^{\infty} A^k a_k(x,t).$$

The problems of convergence of such series are partially discussed below. For finite expansions with respect to the variable p of the initial data $w|_{\partial D}$, $\left.\frac{\partial w}{\partial n}\right|_{\partial D}$, we obtain exact formulas for $w(x,t,p)$ and $\widetilde{R}(x,t)$.

Another example is the recurrence system of equations of the ray method [24].

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a sufficiently smooth boundary S, and let Ω_n be the Lebesgue measure of the domain Ω, $n > 2$. Here we restrict ourselves to the scalar case.

According to the ray method [24], for components $w_k(x)$ of the expansion

$$w(x,\omega) = \sum_{k=0}^{\infty} \frac{w_k(x)\mathrm{e}^{\mathrm{i}\omega\tau(x)}}{(-\mathrm{i}\omega)^{k+\gamma}}, \quad x \in \Omega$$

of the solution to the equation $\Delta w + \frac{\omega^2}{c^2(x)} w = 0$, we have the following recurrence equalities:

$$\begin{gathered} 2(\operatorname{grad}\tau, \operatorname{grad} w_k) + w_k\Delta\tau = \Delta w_{k-1}, \\ k = 0,1,2,\ldots, \quad w_{-1} = 0, \quad |\operatorname{grad}\tau|^2 = \frac{1}{c^2(x)}. \end{gathered} \tag{12}$$

Hence, using representation of the twice differentiable function in the closure $\overline{\Omega}$ [23], from (12) one can obtain the recurrence relations being considered in the present paper (see Lemma 1, (7)):

$$w_{k-1}(x) = A(x)w_k(x) + a_k(x) \tag{13}$$

and equation (1)

$$\frac{1}{(-\mathrm{i}\omega)} w(x,\omega) = A(x)w(x,\omega) + a(x,\omega)$$

at $p = \frac{1}{-\mathrm{i}\omega}$. Here the operator $A(x)$ is determined on $w_k(x)$ by inverting the Laplace operator using the formula

$$Aw_k = -\frac{1}{(n-2)\omega_n} \int\limits_{\Omega} \frac{2(\operatorname{grad}\tau(y), \operatorname{grad} w_k(y)) + w_k\Delta\tau(y)}{|x-y|^{n-2}} dy,$$

and the functions $a_k(x)$ can be found by using the boundary data on S of the functions $w_k(x)$,

$$a_k(x) = -\frac{1}{(n-2)\omega_n} \int\limits_{\Omega} \left[\frac{1}{|x-y|^{n-2}} \frac{\partial w_{k-1}(y)}{\partial n} - w_{k-1}(y) \frac{\partial}{\partial n_y} \frac{1}{|x-y|^{n-2}} \right] dy,$$

$k = 1,2,\ldots$, $a_0(x) = 0$, [23].

The operator-functional equation (1) for $w(x,\omega)$ is obtained by multiplying the both parts of (13) by $\frac{1}{(-\mathrm{i}\omega)^{k+\gamma}}\mathrm{e}^{\mathrm{i}\omega\tau(x)}$ and summing over k,

$$\frac{1}{(-\mathrm{i}\omega)}w(x,\omega) = A(x)w(x,\omega) + a(x,\omega).$$

With the parameter $p = \frac{1}{-\mathrm{i}\omega}$,

$$a(x,\omega) = \sum_{k=1}^{\infty} a_k(x)\frac{1}{(-\mathrm{i}\omega)^{k+\gamma}}\mathrm{e}^{\mathrm{i}\omega\tau(x)}.$$

It should be noted that the operator A, as well as the system of equations (12), greatly depends on the choice of the function $\tau(x)$ with the condition $|\operatorname{grad}\tau|^2 = \frac{1}{c^2(x)}$. If the velocity $c(x)$ is a priori unknown, the function $\tau(x)$ is also unknown and has to be sought for together with $w_k(x)$. It is important that at given boundary conditions $u_k|_S$, $\left.\frac{\partial u_k}{\partial n}\right|_S$, equation (9) can be considered an equation for $\tau(x)$. Let us explain it for a finite expansion at $a_{m+j}(x) = 0$, $j = 1,2,\ldots$. In this case the equation for $\tau(x)$,

$$\sum_{k=1}^{m} A^k a_k(x) = 0,$$

is obtained by eliminating the function $w_k(x)$ by the above-mentioned method from the system of equations

$$\left\{\begin{array}{l} 2(\operatorname{grad} w_0, \operatorname{grad}\tau) + w_0\Delta\tau = 0, \\ 2(\operatorname{grad} w_k, \operatorname{grad}\tau) + w_k\Delta\tau = \Delta w_{k-1}, \quad k = 1,2,\ldots,m, \\ \Delta w_m = 0. \end{array}\right. \tag{14}$$

The system of equations (14) is closed: the number of equations is equal to the number of sought-for functions $\tau(x)$, $w_k(x)$, $k = 0,1,2,\ldots,m$, that is, $m+2$.

As for system (14), notice that if $w_0 \neq 0$, from the first equation of (14) one can find $\Delta\tau$ and substitute it into the next ones. The thus obtained closed system of nonlinear differential equations for $\tau(x)$, $w_k(x)$ is a Cauchy-Kovalevskaya system with all consequences and applications. Notice also that for a general scheme of the ray method $A_1 w_k = A_2 w_{k-1}$, $w_k \in E$, where A_1 and A_2 are linear operators, this system of equations is reduced, by inverting the differential operator A_2 with allowance for the boundary conditions, to the system of equations (7):

$$w_{k-1} = Aw_k + a_k, \quad A = A_2^{-1}A_1.$$

4. Construction of special solutions.

Let us formulate some results pertaining to this subject. First we obtain, from Theorem 1 and Lemma 2, a formula for solving the functional equation $w(x) = A(x)w(\lambda(x)) + a(x)$.

Theorem 4. *Let in equation* (1) $\lambda(x)$, $A(x)$, $a(x,p) = a_0(x) + \sum_{k=1}^{\infty} a_k p^k$, $x \in X$, $|p| < 1$ *be fixed, and let the following conditions be satisfied:*

(1) $A(x)$ *is a real function*, $0 < |A(x)| \le q < 1$,
(2) $a_k(x) = a(x)$, $k = 1,2,\ldots$, *do not depend on* k *and* $\|a(x)\| \le \alpha$

(3) $a_0(x) = -v(x) + a(x)$, $v(x) = \sum\limits_{k=0}^{\infty} \prod\limits_{j=1}^{k} A(\lambda^{[j-1]}(x))a(\lambda^{[k]}(x))$.

Then the following equality holds: $a_0(x) + \sum\limits_{k=1}^{\infty} \prod\limits_{j=1}^{k} A(\lambda^{[j-1]}(x))a_k(\lambda^{[k]}(x)) = 0$ *and* $w(x,p) = \sum\limits_{k,s=0}^{\infty} \prod\limits_{j=1}^{s} A(\lambda^{[j-1]}(x))a_{k+s+1}(\lambda^{[s]}(x))p^k$ *is a solution to the functional equation with the parameter* p,

$$pw(x,p) = A(x)w(\lambda(x),p) + a(x,p),$$

if and only if

$$v(x) = \sum_{k=0}^{\infty} \prod_{j=1}^{k} A(\lambda^{[j-1]}(x))a(\lambda^{[k]}(x))$$

is a solution to the functional equation

$$v(x) = A(x)v(\lambda(x)) + a(x).$$

Proof. First, notice that the series being considered in the theorem are, according to Lemma 2, convergent, $x \in X$, $|p| < 1$.

Since $a_0(x) = -v(x) + a(x)$, $a_k(x) = a(x)$, $k = 1, 2, \ldots$, $v(x) = \sum\limits_{k=0}^{\infty} \prod\limits_{j=1}^{k} A(\lambda^{[j-1]}(x))a(\lambda^{[k]}(x))$, we have

$$a(x,p) = a_0(x) + \sum_{k=1}^{\infty} a_k(x)p^k = -v(x) + a(x) + a(x)\frac{p}{1-p} = -v(x) + \frac{a(x)}{1-p},$$

$$w(x,p) = \sum_{k,s=0}^{\infty} \prod_{j=1}^{s} A(\lambda^{[j-1]}(x))a_{k+s+1}(\lambda^{[s]}(x))p^k =$$
$$= \sum_{k,s=0}^{\infty} \prod_{j=1}^{s} A(\lambda^{[j-1]}(x))a(\lambda^{[s]}(x))p^k = \frac{v(x)}{1-p},$$

$$a_0(x) + \sum_{k=1}^{\infty} \prod_{j=1}^{k} A(\lambda^{[j-1]}(x))a_k(\lambda^{[k]}(x)) =$$
$$= -v(x) + a(x) + \sum_{k=1}^{\infty} \prod_{j=1}^{k} A(\lambda^{[j-1]}(x))a(\lambda^{[k]}(x)) = -v(x) + v(x) = 0.$$

Therefore, according to Theorem 1, $w(x,p) = \dfrac{v(x)}{1-p}$ satisfies the functional equation

$$pw(x,p) = A(x)w(\lambda(x),p) + \frac{a(x)}{1-p} - v(x).$$

Substituting $w(x,p) = \dfrac{v(x)}{1-p}$, we have

$$\frac{pv(x)}{1-p} = \frac{A(x)v(\lambda(x))}{1-p} + \frac{a(x)}{1-p} - v(x)$$

or

$$\frac{v(x)}{1-p} = \frac{A(x)v(\lambda(x))}{1-p} + \frac{a(x)}{1-p}.$$

The theorem has been proved. □

For the operator-functional equations with a parameter p, we have formulas for $a_0(x)$ and $w(x,p)$.

Theorem 5. *Let in equation* (1) $\|A(x)\| \le q < 1$, $\|a_k\| \le \alpha$, $x \in X$, $k = 1, 2, \ldots$, *and let the mapping* $\lambda(x)$ *of the set* X *into itself be fixed and*

$$a_0(x) = -\sum_{k=1}^{\infty} \prod_{j=1}^{k} A(\lambda^{[j-1]}(x))a_k(\lambda^{[k]}(x)).$$

Then

$$w(x,p) = \sum_{k,s=0}^{\infty} \prod_{j=1}^{s} A(\lambda^{[j-1]}(x))a_{k+s+1}(\lambda^{[s]}(x))p^k$$

is a solution to the operator-functional equation with the parameter p,

$$pw(x,p) = A(x)w(\lambda(x),p) + a_0(x) + \sum_{k=1}^{\infty} a_k(x)p^k, \quad x \in X, \quad |p| < 1.$$

The proof of Theorem 5 follows from Lemma 2 and Theorem 1.

The construction of particular solutions $u(x,t)$ and $a_0(x)$ is given by Theorem 6.

Theorem 6. *Let in equation* (2) $\|A(x)\| \le q < 1$, $\|a_k\| \le \alpha$, $|L^k b(t)| \le p_0 < 1$, $x \in X$, $t_0 \le t \le t_1$, $k = 1, 2, \ldots$, *and let the mapping* $\lambda(x)$ *of the set* X *into itself be fixed and*

$$a_0(x) = -\sum_{k=0}^{\infty} \prod_{j=1}^{k} A(\lambda^{[j-1]}(x))a_k(\lambda^{[k]}(x)).$$

Then

$$u(x,t) = \sum_{k,s=0}^{\infty} \prod_{j=1}^{s} A(\lambda^{[j-1]}(x))a_{k+s+1}(\lambda^{[s]}(x))L^k b(t)$$

is a particular solution to the evolution equation

$$Lu(x,t) = A(x)u(\lambda(x),t) + a_0(x)b(t) + \sum_{k=1}^{\infty} L^k b(t)a_k(x).$$

The proof of Theorem 6 follows from Lemma 3 and Theorem2.

5. Nonlinear identification problems.

First consider the questions of existence and uniqueness of a solution to nonlinear inverse problems for functional equation (1). We find the function $\lambda(x)$ by given $a_k(x)$ and $A(x) = 1$ from the equation (9).

First note that if the function $z = a_1(x)$ has an inverse function $x = b(z)$, equation (9) at $A(x) = 1$ can be written in the following form:

$$\lambda(x) = B\lambda, \quad B\lambda = b\left(-\left(a_0(x) + \sum_{k=2}^{\infty} a_k\left(\lambda^{[k]}(x)\right)\right)\right).$$

It is clear that in this case the questions of existence, uniqueness, and stability of the solution to (9) can be associated with fixed point theory of the operator B.

Let us present sufficient conditions of correctness of formal operations when the space E is a real Euclidean space $\mathbb{R}^n$, $n \geq 1$, X is a ball in $\mathbb{R}^n$, $X = \{x : |x| \leq A\}$, using the fixed point theorem. The conditions for the known vector-functions $a_k(x)$, $k = 0, 1, 2, \dots$ presented below are assumed to be satisfied:

I. The mapping $z = a_1(x)$ from $\mathbb{R}^n$ into $\mathbb{R}^n$, $a_1(0) = 0$ has a unique continuous inverse mapping $x = b(z)$, $b(0) = 0$, and the inequality $|b(z) - b(y)| \leq L\,|z - y|$, $|y| \leq A$, $|z| \leq A$ holds; $L \in (0, 1]$ is a fixed number.

II. The vector-functions $a_k(x)$, $a_k(0) = 0$, $k = 0, 1, 2, \dots$ satisfy, in the domain $|x| \leq A$, the relations

$$|a_k(x) - a_k(y)| \leq L_k|x - y|, \quad |x| \leq A, \quad |y| \leq A,$$

where $L_0 > 0$, $L_1 > 0$, $L_k \geq 0$, $k = 2, 3, \dots$ are fixed numbers, such that the series given below converge with the estimates:

$$L_0 + \sum_{k=2}^{\infty} L_k < 1, \quad \frac{1}{1-M} L \sum_{k=2}^{\infty} L_k = q, \quad 0 \leq q < 1,$$

M is the root of the equation $\psi(t) = 0$, where the convex function

$$\psi(t) = L_0 + \sum_{k=2}^{\infty} L_k t^k - t, \quad 0 \leq t \leq 1,$$

has a real single zero $t = M$ in the interval $(0, 1)$, since $\psi(0) = L_0 > 0$ and, according to condition II,

$$\psi(1) = L_0 + \sum_{k=2}^{\infty} L_k - 1 < 0.$$

Let Q be a full metric space of all continuous vector-functions $f(x) \in \mathbb{R}^n$, $f(0) = 0$, $|x| \leq A$ with a distance $\|f_1 - f_2\| = \sup\limits_{|x| \leq A} |f_1(x) - f_2(x)|$ such that $|f(x)| \leq A$, $|f(x) - f(y)| \leq M|x - y|$. Here M is the root of the equation $L_0 + \sum_{k=2}^{\infty} L_k M^k = M$, and the constants A, L_0, L, and L_k correspond to conditions I and II.

Theorem 7. *When conditions* I *and* II *are satisfied, the equation* $\sum\limits_{k=0}^{\infty} a_k(\lambda^{[k]}(x)) = 0$ *has the unique solution* $\lambda(x) \in Q$ *such that the series*

$$w(x, p) = \sum_{k,s=0}^{\infty} a_{k+1+s}(\lambda^{[s]}(x)) p^k \tag{15}$$

uniformly converges for $|p| < p_0$, $|x| \leq A$, *where* $p_0 < 1$ *is any number, and the thus determined* $\lambda(x)$ *and* $w(x, p)$ *satisfy the functional equation*

$$p w(x, p) = w(\lambda(x), p) + a(x, p), \quad |x| \leq A, \quad |p| < p_0.$$

Proof. At $\lambda(x) \in Q$ by virtue of the inequalities $|\lambda^{[k]}(x)| \leq M^k|x| \leq M^k A$, $|a_k(\lambda^{[k]}(x))| \leq M^k L_k A$ the series $a_0(x) + \sum\limits_{k=2}^{\infty} a_k\left(\lambda^{[k]}(x)\right)$ uniformly converges and is a continuous vector-function of x, since all terms of the series are continuous. Owing to continuity of $b(z)$, the vector-function $b\left(-\left(a_0(x) + \sum\limits_{k=2}^{\infty} a_k\left(\lambda^{[k]}(x)\right)\right)\right)$ is also continuous.

Consider the equation for $\lambda(x)$,

$$a_0(x) + \sum_{k=1}^{\infty} a_k\left(\lambda^{[k]}(x)\right) = 0,$$

and rewrite it in the form

$$\lambda(x) = b\left(-\left(a_0(x) + \sum_{k=2}^{\infty} a_k\left(\lambda^{[k]}(x)\right)\right)\right) \equiv B\lambda.$$

Let us show that the operator B maps the set Q into itself and is a contracting operator with a constant q. It should be noted that if all $L_k = 0$, $k = 2, 3, \ldots$, we have $\lambda(x) = b(-a_0(x))$. Otherwise, by virtue of the inequality

$$|a_k(\lambda^{[k]}(x)) - a_k(\lambda^{[k]}(y))| \le L_k M^k |x - y|,$$

we have

$$\begin{aligned}
&|B\lambda(x) - B\lambda(y)| = \\
&= \left|b\left(-\left(a_0(x) + \sum_{k=2}^{\infty} a_k\left(\lambda^{[k]}(x)\right)\right)\right) - b\left(-\left(a_0(y) + \sum_{k=2}^{\infty} a_k\left(\lambda^{[k]}(y)\right)\right)\right)\right| \\
&\le L\left[|a_0(x) - a_0(y)| + \sum_{k=2}^{\infty}\left|a_k\left(\lambda^{[k]}(x)\right) - a_k\left(\lambda^{[k]}(y)\right)\right|\right] \\
&\le L\left[L_0|x - y| + \sum_{k=2}^{\infty} L_k|\lambda^{[k]}(x) - \lambda^{[k]}(y)|\right] \\
&\le L[L_0 + \sum_{k=2}^{\infty} L_k M^k]|x - y| \\
&= L[L_0 + \sum_{k=2}^{\infty} L_k M^k - M]|x - y| + LM|x - y| \\
&= LM|x - y| \le M|x - y|,
\end{aligned}$$

that is,

$$|B\lambda(x) - B\lambda(y)| \le M|x - y|, \quad B(0) = 0.$$

Hence, $B\lambda(x) \in Q$.

It remains to prove that the operator B is contracting with a constant $q = \dfrac{1}{1-M} L \sum_{k=2}^{\infty} L_k$. Actually,

$$|B\lambda_1(x) - B\lambda_2(x)| \le \sum_{k=2}^{\infty} LL_k\left|\lambda_1^{[k]}(x) - \lambda_2^{[k]}(x)\right|.$$

Since

$$\begin{gathered}
|\lambda_1(\lambda_1(x)) - \lambda_2(\lambda_2(x)| \le \\
\le |\lambda_1(\lambda_1(x)) - \lambda_1(\lambda_2(x))| + |\lambda_1(\lambda_2(x)) - \lambda_2(\lambda_2(x))| \le \\
\le M\left|\lambda_1(x) - \lambda_2(x)\right| + \|\lambda_1 - \lambda_2\| \le (M+1)\left\|\lambda_1 - \lambda_2\right\|,
\end{gathered}$$

we inductively obtain the inequalities

$$\left|\lambda_1^{[k]}(x) - \lambda_2^{[k]}(x)\right| \le \left|\lambda_1\left(\lambda_1^{[k-1]}(x)\right) - \lambda_1\left(\lambda_2^{[k-1]}(x)\right)\right| +$$

$$+\left|\lambda_1\left(\lambda_2^{[k-1]}(x)\right)-\lambda_2\left(\lambda_2^{[k-1]}(x)\right)\right|\leq M\left|\lambda_1^{[k-1]}(x)-\lambda_2^{[k-1]}(x)\right|+$$

$$+\|\lambda_1-\lambda_2\|\leq(MM_{k-1}+1)\|\lambda_1-\lambda_2\|=M_k\|\lambda_1-\lambda_2\|,$$

where the numbers $M_2 = M + 1$, $M_k = MM_{k-1} + 1$, $k = 3, 4, \dots$ are uniquely determined by these recurrence relations, and $M_k = \dfrac{1-M^k}{1-M} = 1 + M + \dots + M^{k-1} < \dfrac{1}{1-M}$.

Therefore,

$$|B\lambda_1(x)-B\lambda_2(x)|\leq\sum_{k=2}^{\infty}LL_kM_k\|\lambda_1-\lambda_2\|\leq\frac{1}{1-M}\sum_{k=2}^{\infty}LL_k\|\lambda_1-\lambda_2\|=q\|\lambda_1-\lambda_2\|.$$

Since x is arbitrary, we have the following contraction estimate:

$$\|B\lambda_1-B\lambda_2\|\leq q\|\lambda_1-\lambda_2\|.$$

Thus, it has been proved that equation (9) has a unique solution, $\lambda(x) \in Q$.

Let us prove that series (15) uniformly converges for $|x| \leq A$, $|p| < p_0 < 1$, where p_0 is any number. We have $w(x, p_0) = \sum_{k=0}^{\infty} w_k(x)p_0^k$, $w_k(x) = \sum_{s=0}^{\infty} a_{k+1+s}(\lambda^{[s]}(x))$. Hence, we obtain

$$\begin{aligned}|w_k(x)| &\leq \sum_{s=0}^{\infty}|a_{k+1+s}(\lambda^{[s]}(x))|\leq\\ &= \sum_{s=0}^{\infty}L_{k+1+s}M^s|x|\leq\frac{A}{1-M}\sum_{s=0}^{\infty}L_{k+1+s}=\tilde{q}_k\xrightarrow[k\to\infty]{}0\end{aligned}$$

The estimate $|w_k(x)| \leq \tilde{q}_k$ guarantees uniform convergence of series (15) for $|x| \leq A$, $|p| < p_0 < 1$. By virtue of Theorem 1, the found $w(x, p)$ and $\lambda(x)$ satisfy the functional equation $pw(x, p) = w(\lambda(x), p) + a(x, p)$. □

In what follows, the property of quasi-monotonicity of the operators is used to investigate the uniqueness of the solution to mapping $\lambda(x)$ of equation (9), which corresponds also to the uniqueness of the solution to the inverse problem for functional equation (1).

Let the set X be a half-space $x_n > 0$ of the Euclidean space $\mathbb{R}^n$ of variables $x = (x_1, x_2, \dots, x_n) = (x', x_n)$, $x' = (x_1, x_2, \dots, x_{n-1})$.

Consider the mapping $\lambda(x)$ of the half-space X into itself, determined by the choice of the function $\widetilde{\lambda}(x)$, according to the formula $\lambda(x) = (x', \widetilde{\lambda}(x', x_n))$, where $\widetilde{\lambda}(x', x_n)$ is a real continuous strictly monotonically increasing function with respect to the variable x_n, such that $\lambda(x', 0) = 0$, $\lambda(x', x_n) < x_n$.

The set of such functions $\widetilde{\lambda}(x)$ is denoted by $\{\widetilde{\lambda}(x)\}$.

Assume that the real function $A(x) > 0$, $a_k(x) > 0$, $x \in X$, $k = 1, 2, \dots$ are continuous, monotonically increasing with respect to the variable x_n, and such that $|A(x)| \leq q < 1$, $|a_k(x)| \leq \alpha$. Recall that in this case according to Lemma 2 the series $\sum\limits_{k=0}^{\infty}\prod\limits_{j=1}^{k} A(\lambda^{[j-1]}(x))a_k(\lambda^{[k]}(x))$ is convergent.

For such fixed functions $A(x)$, $a_k(x)$, $k = 1, 2, \dots$, and $a_0(x) < 0$, $a_0(x)$ is a continuous function. Consider an equation with iterations for the function $\widetilde{\lambda}(x)$,

$x \in X$:

$$a_0(x) + \sum_{k=0}^{\infty} \prod_{j=1}^{k} A(\lambda^{[j-1]}(x)) a_k(\lambda^{[k]}(x)) = 0, \tag{16}$$

where $\lambda(x) = (x', \widetilde{\lambda}(x', x_n))$ is a mapping of the half-space X into itself.

Rewrite equation (16) in the form of an operator equation of the 1st kind for the sought-for function $\widetilde{\lambda}(x) \in \{\widetilde{\lambda}\}$:

$$B\widetilde{\lambda} \equiv \sum_{k=1}^{\infty} \prod_{j=1}^{k} A(\lambda^{[j-1]}(x)) a_k(\lambda^{[k]}(x)) = -a_0(x). \tag{17}$$

Let $\omega(\xi, r) \subset X$, $\xi = (\xi_1, \xi_2, \ldots, \xi_{n-1})$, $r > 0$ be an open half-ball with the center at a point $\xi \in \mathbb{R}^{n-1}$ of radius $r > 0$:

$$\omega(\xi, r) = \left\{ x, x_n > 0, \sum_{j=1}^{n-1} (x_j - \xi_j)^2 + x_n^2 < r^2 \right\}$$

The author's papers (see [25]) give a definition of a quasimonotone operator, formulation and proof of a theorem of uniqueness of the solution to operator equations of the first kind with quasimonotone operators in the class of infinitely differentiable functions that are analytic with respect to a selected variable. According to the definition of quasimonotonicity of the operator $B\widetilde{\lambda} = \sum_{k=1}^{\infty} \prod_{j=1}^{k} A(\lambda^{[j-1]}(x)) a_k(\lambda^{[k]}(x))$, $\lambda(x) = (x', \widetilde{\lambda}(x', x_n))$, it is sufficient to prove that if $\widetilde{\lambda}_i(x) \in \{\widetilde{\lambda}\}$, $i = 1, 2$ and $\widetilde{\lambda}_1(x) > \widetilde{\lambda}_2(x)$, $\forall x \in \omega_0 = \omega(\xi_0, r_0)$, $B\widetilde{\lambda}_1 \neq B\widetilde{\lambda}_2$.

Theorem 8. *If at least one of the functions $a_k(x)$, $k = 1, 2, \ldots$, strictly monotonically increases over the variable x_n, the operator*

$$B\widetilde{\lambda} = \sum_{k=1}^{\infty} \prod_{j=1}^{k} A(\lambda^{[j-1]}(x)) a_k(\lambda^{[k]}(x))$$

is quasimonotone.

Proof. Let $\widetilde{\lambda}_1(x) > \widetilde{\lambda}_2(x)$, $\forall x \in \omega_0$, $\widetilde{\lambda}_i(x) \in \{\widetilde{\lambda}\}$, $i = 1, 2$. It follows from the definition of the class of functions $\{\widetilde{\lambda}\}$ that if $x \in \omega_0 = \omega(\xi_0, r_0)$, we have $\lambda_i(x) = (x', \widetilde{\lambda}_i(x', x_n)) \in \omega_0$. By induction we have $\lambda_i^{[k]}(x) \in \omega_0$ for any $k = 1, 2, \ldots$ In this case, if $\lambda_1^{[k]}(x) = (x', \widetilde{x}_{nk})$, $\lambda_2^{[k]}(x) = (x', \widetilde{\widetilde{x}}_{nk})$, we have $\widetilde{x}_{nk} > \widetilde{\widetilde{x}}_{nk}$.

Let a_{k_0} be strictly monotonically increasing with respect to x_n. Owing to monotonicity of $A(x)$ and $a_k(x)$ with respect to x_n, $k = 1, 2, \ldots$, and strict monotonicity of a_{k_0}, we have $A(\lambda_1^{[j-1]}(x)) \geq A(\lambda_2^{[j-1]}(x))$, $a_k(\lambda_1^{[j-1]}(x)) \geq a_k(\lambda_2^{[j-1]}(x))$, $k \neq k_0$, $a_{k_0}(\lambda_1^{[j-1]}(x)) > a_{k_0}(\lambda_2^{[j-1]}(x))$. Hence, $B\widetilde{\lambda}_1 > B\widetilde{\lambda}_2$, $\forall x \in \omega_0$, in particular, $B\widetilde{\lambda}_1 \neq B\widetilde{\lambda}_2$ and the operator $B\widetilde{\lambda}$ is quasimonotone. □

Corollary. *The equation with iterations*

$$a_0(x) + \sum_{k=0}^{\infty} \prod_{j=1}^{k} A(\lambda^{[j-1]}(x)) a_k(\lambda^{[k]}(x)) = 0,$$

$\lambda(x) = (x', \widetilde{\lambda}(x))$, *has no more than one infinitely differentiable solution that is analytic with respect to x_n in the domain $x_n \geq 0$ $\widetilde{\lambda}(x) \in \{\widetilde{\lambda}\}$.*

Next consider the construction of exponential solutions with a finite sum of the basic equality (9)

$$a_0(x) + \sum_{k=1}^{m} \prod_{j=1}^{k} A(\lambda^{[j-1]}(x)) a_k(\lambda^{[k]}(x)) = 0. \tag{18}$$

Assume that $A(x) = 1$ and $a_k(x)$ are complex-valued functions. Set $b_0(x) = \mathrm{e}^{-a_0(x)}$, $b_k(x) = \mathrm{e}^{a_k(x)}$, $k = 1, 2, \dots, m$. Then equality (18) takes the following form:

$$b_0(x) = \prod_{k=1}^{m} b_k(\lambda^{[k]}(x)). \tag{19}$$

Let $v(y, z)$ be a function that is analytic with respect to $z = (z_1 + \mathrm{i}z_2)$, $z_1, z_2 \in \mathbb{R}$, in a circle $|z| < z_0$, $y \in D \subset \mathbb{R}^n$ such that $|v(y, z)| < z_0$. In equality (19), we set $x = (y, t)$, $y \in D \subset \mathbb{R}^n$, $|t| < z_0$, $\lambda^{[k]}(x) = v^k(y, t)$, where $v^{[1]}(y, t) = v(y, t)$, $v^{[k+1]}(y, t) = v^{[k]}(y, v(y, t))$, that is, iterations are made with respect to the variable t.

It is naturally assumed that $b_k(z)$ are also analytic functions in a circle $|z| < z_0$.

Define $b_0(x) = b_0(y, t)$ in (19) as a coefficient of the equation

$$\frac{\partial^s v(y, t)}{\partial t^s} = b_0(y, t) \widetilde{L} v(y, t),$$

where $\widetilde{L}$ is a linear differential operator with respect to the variable y and s is a natural number, $s \geq 1$.

At these assumptions, equality (19) becomes a nonlinear evolution equation with iterations for the function $v(y, t)$,

$$\frac{\partial^s v(y, t)}{\partial t^s} = \widetilde{L} v(y, t) \prod_{k=1}^{m} b_k(v^{[k]}(y, t)). \tag{20}$$

If in (20) $b_k(z) = z^{\alpha_k}$, $k = 1, 2, \dots, m$, where α_k are some given numbers, equation (20) takes the form

$$\frac{\partial^s v(y, t)}{\partial t^s} = \widetilde{L} v(y, t) \prod_{k=1}^{m} \left[v^{[k]}(y, t) \right]^{\alpha_k}. \tag{21}$$

A solution $v(y, t)$ of the equation will be sought for in the form $v(y, t) = \varphi(y) t^\beta$, where $\varphi(y)$ is a complex-valued function and β is a complex number. Substituting $v(y, t) = \varphi(y) t^\beta$ into (21), we obtain the equality

$$\beta(\beta - 1)(\beta - s + 1)(\varphi(y))^{1 - \sum_{k=1}^{m} \alpha_k (1 + \beta + \ldots + \beta^{k-1})} = \widetilde{L} \varphi(y) t^{\sum_{k=1}^{m} \alpha_k \beta^k + s}.$$

Hence, we have

Theorem 9. *Equation* (21) *has a solution $v(y, t) = \varphi(y) t^\beta$ if the function $\varphi(y)$ is a solution to the nonlinear equation*

$$\widetilde{L} \varphi(y) = \beta(\beta - 1)(\beta - s + 1)(\varphi(y))^{1 - \sum_{k=1}^{m} \alpha_k (1 + \beta + \ldots + \beta^{k-1})},$$

and the number β belongs to the set of roots of the equation $\sum_{k=1}^{m} \alpha_k \beta^k + s = 0$ combined with the set $(0, 1, 2, \dots, s - 1)$.

Remark. If t is time and the iterations $v^{[k]}$ are made with respect to the variable t, it should be considered that $v(y,t)$ is also the time of some past or future event at the time t and location y.

Notice also that if $\beta = \beta_1 + \mathrm{i}\beta_2$ is a complex root of the equation $\sum_{k=1}^{m} \alpha_k \beta^k + s = 0$, the solution $u(y,t) = \varphi(y)t^\beta$, $\varphi = \varphi_1 + \mathrm{i}\varphi_2$, is written in the form

$$u(y,t) = [\varphi_1(y)\cos(\beta_2 \ln t) - \varphi_2(y)\sin(\beta_2 \ln t)]t^{\beta_1} + \\ + \mathrm{i}[\varphi_2(y)\cos(\beta_2 \ln t) + \varphi_1(y)\sin(\beta_2 \ln t)]t^{\beta_1}.$$

If $\beta_1 < 0$, the regime with this solution is called blow-up [26].

According to Theorem 3, using the operator-functional equation as an example, we touch to the problem of decomposing the information into the sum of two terms.

Consider the operator-functional equation

$$w(x) = A(x)w(\lambda(x)) + a(x).$$

Here $X = E$, E is a Banach space, and $a(x)$, $w(x)$, and $\lambda(x)$ are mappings of E into E, $A(x)$ for each $x \in E$ is a linear operator from E to E having an inverse operator $A^{-1}(x)$, for instance, if $A(x) > 0$ is a real function, $A^{-1}(x) = \frac{1}{A(x)}$.

It turns out that if the mapping $a(x)$ of the space E into itself is decomposed in some way into the sum of two mappings $a(x) = a_0(x)+a_1(x)$ so that the mapping $y = a_1(x)$, $x \in E$, $y \in E$ has an inverse $x = a_1^{-1}(y)$, the mappings $w(x) = a_1(x)$ and $\lambda(x) = a_1^{-1}\left(-A^{-1}(x)a_0(x)\right)$ satisfy the equation $w(x) = A(x)w(\lambda(x)) + a(x)$.

A classical example of decomposition $a(x) = a_0(x) + a_1(x)$ is that of a real function $a(x)$, $x_0 \le x \le x_1$ of bounded variation into the sum of two monotone functions one of which increases, and the other decreases.

Summing up, it can be said that, generally speaking, the decomposition $a(x) = a_0(x) + a_1(x)$ into the sum of two terms is ambiguous; it makes it possible to solve the inverse problem of search for $w(x)$ and $\lambda(x)$. Further development and practical application of this approach is very important for the theory and applications of inverse problems. Return to the ray method/

As noted above, a closed system of nonlinear differential equations (14) for the functions $\tau(x)$, $w_k(x)$, $k = 0, 1, 2, \ldots, m$ at $w_0(x) \neq 0$ can be written as a system of equations of the Cauchy-Kovalevskaya type:

$$\begin{aligned} \Delta\tau &= -\frac{2}{w_0}(\operatorname{grad} w_0, \operatorname{grad}\tau), \\ \Delta w_{k-1} &= 2(\operatorname{grad} w_k, \operatorname{grad}\tau) - \frac{2w_k}{w_0}(\operatorname{grad} w_0, \operatorname{grad}\tau), \quad k = 1, 2, \ldots, m \\ \Delta w_m &= 0. \end{aligned} \tag{22}$$

Assume that the following analytic functions are given for this system of equations on a simply connected fragment $\widetilde{S}$ of a sphere of radius R:

$$\tau|_{\widetilde{S}}, \quad \left.\frac{\partial\tau}{\partial n}\right|_{\widetilde{S}}, \quad w_k|_{\widetilde{S}}, \quad \left.\frac{\partial w_k}{\partial n}\right|_{\widetilde{S}}, \quad k = 0, 1, 2, \ldots, m. \tag{23}$$

Using the general Cauchy-Kovalevskaya theorem, we obtain the theorem below.

Theorem 10. *For every point $s \in \widetilde{S}$, there exists a neighborhood in $\mathbb{R}^n$ of this point at which there exists a unique analytic solution to the system of equations* (22) *satisfying data* (23).

Remark. If $\tau(x)$ in some neighborhood is known, one can find, for instance,

(1) the velocity $c(x)$ by the formula $|\mathrm{grad}\,\tau|^2 = \dfrac{1}{c^2(x)}$,

(2) the set of zeroes $\tau(x)$,

which can correspond to the geometrical characteristics of a wave source, for instance, an earthquake source.

It should be noted that the solution $w(x,p)$ of equation (1) for some operators A can have singularities with respect to the variable p at the point $p = 0$.

An example is the solution

$$w(x,p) = \frac{1}{p}\left(a(x,p) + \frac{1}{p^2}\int_{x_0}^{x} a(t,p)\mathrm{e}^{\frac{x-t}{p}}\,dt\right)$$

of the equation $pw(x,p) = \int_{x_0}^{x} w(x,p)dt + a(x,p)$. With allowance for this, consider an operator-functional equation of another type,

$$w(x,p) = pA(x)w(\lambda(x),p) + a(x,p). \tag{24}$$

Here, as before, $x \in X$, $\lambda(x)$ is the mapping of X into X, $w(x,p) \in E$, $A(x)$ is a linear operator from E into E, $a(x,p) \in E$, and the parameter p, in contrast to (1), is multiplied not by $w(x,p)$, but by $A(x)w(\lambda(x),p)$.

In the case of finite decomposition $a(x,p) = \sum_{k=0}^{m} a_k p^k$, the result that is being directly verified is formulated in the theorem presented below.

Theorem 11. *If $a(x,p) = \sum_{k=0}^{m} a_k p^k$, $a_k(x) \in E$, the system of recurrence relations*

$$w_k(x) = A(x)w_{k-1}(\lambda(x)) + a_k(x), \quad k = 0,1,2,\dots,m-1, \quad w_{-1}(x) = 0$$

determines, by the equality $w(x,p) = \sum_{k=0}^{m-1} w_k p^k$, the solution to the operator-functional equation $w(x,p) = pA(x)w(\lambda(x),p) + a(x,p)$ if and only if

$$\sum_{k=0}^{m}\prod_{j=1}^{m-k} A(\lambda^{[j-1]}(x))a_k(\lambda^{[m-k]}(x)) = 0.$$

Also in this case, we obtain, in different symbols, the basic relation (9) in a finite-dimensional variant. Therefore, as before, the results of search for the elements $a_k(x)$, $A(x)$, and $\lambda(x)$ from (9) in the finite-dimensional case can be a result of identification problems for equation (24).

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