S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 10, стр. 666–675 (2013) УДК 512.542
MSC 20D06, 20D60

ON GROUPS CRITICAL WITH RESPECT TO A SET OF NATURAL NUMBERS

YURI V. LYTKIN

ABSTRACT. The spectrum of a finite group is the set of its element orders. A finite group G is critical with respect to a subset ω of natural numbers, if ω is equal to the spectrum of G and not equal to the spectrum of any proper section of G. For any natural number n, we construct n finite critical groups with the same spectrum. We also give a complete description of finite groups critical with respect to the spectrum of the alternating group of degree 6 and the spectrum of the special linear group of dimension 3 over a field of order 3.

Keywords: finite group, spectrum, critical group.

1. Introduction

Let G be a finite group. The set of all element orders of G is called the *spectrum* and denoted by $\omega(G)$. The set $\omega(G)$ is closed under divisibility and thus, it is uniquely defined by its subset $\mu(G)$, which consists of elements maximal under divisibility. We call groups G and H *isospectral* if $\omega(G) = \omega(H)$. By a *section* of G we mean an arbitrary quotient group H/N, where $N, H \leq G$ and $N \trianglelefteq H$.

Let ω be a subset of natural numbers. Following [1], we call a group G *critical with respect to* ω (or just *critical*) if $\omega(G) = \omega$ and $\omega(H/N) \neq \omega$ for every proper section H/N of G.

In [1] it was proved that the number of finite groups critical with respect to any fixed set of natural numbers is always finite. In this connection V. D. Mazurov stated the following

LYTKIN, YU.V., ON GROUPS CRITICAL WITH RESPECT TO A SET OF NATURAL NUMBERS.

The work is partially supported by RFBR (grant 13-01-00505).
Received November, 7, 2013, published December, 14, 2013.

Conjecture. *For any natural number n there exist n pairwise non-isomorphic finite groups critical with respect to some fixed set of natural numbers.*

In this article we prove a theorem that confirms this conjecture.

Theorem 1. *Suppose that s is a natural number and*

$$\{p_{ij} \mid i = 1, \dots, s, j = 1, 2\}$$

is a set of pairwise distinct primes. Then there exist 2^s pairwise non-isomorphic finite groups critical with respect to

$$\Omega = \{n_1 \dots n_s \mid n_j \in \{1, p_{j1}, p_{j2}\}\}.$$

It follows from Theorem 1 that if we do not impose any restrictions on what set of natural numbers to choose, there is no boundary for the number of groups critical with respect to a fixed set. Thus the subsequent conjecture is that if we consider only sets that coincide with spectra of finite simple groups, then such boundary is going to exist.

It is known that generally, if L is a finite nonabelian simple group and G is a finite group isospectral to L, then $L \leq G \leq \operatorname{Aut} L$. Thus, if L is a finite nonabelian simple group with this property and ω is its spectrum, then L is the only group critical with respect to ω. Therefore, it is interesting to study finite simple groups that do not have this property. In some way the smallest of such groups are the alternating group A_6 and the special linear group $SL_3(3)$. Moreover, these groups have infinitely many groups isospectral to them [2, 3].

In this article we give the complete description of finite groups critical with respect to $\omega(A_6)$ and $\omega(SL_3(3))$, namely, we prove the following theorems.

Theorem 2. *Let G be a finite group critical with respect to $\{1, 2, 3, 4, 5\}$. Then G is either A_6, or $K \leftthreetimes A_5$, where K is the natural module of order 16 for the group $SL_2(4) \simeq A_5$.*

Theorem 3. *Let G be a finite group critical with respect to $\{1, 2, 3, 4, 6, 8, 13\}$. Then G is either $SL_3(3)$, or a Frobenius group AH, where A is an elementary abelian group of order 13^4 and H is an extension of the unique subgroup of order 2 in H by a group isomorphic to S_4.*

Moreover, H has a presentation $\langle x, y \mid x^4 = y^3 = (xy)^8 = 1;\ x^2 = (xy)^4\rangle$, and regarding A as a vector space over the field $\mathbb{F}_{13}$, a basis of A can be chosen in such a way that the action of H on A is defined by the following matrices:

$$x \sim \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad y \sim \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & 5 & 0 \end{pmatrix}.$$

2. Preliminaries

In this article by group we always mean finite group. We denote by $\mathbb{F}_q$ a field of order q and $\mathbb{Z}_n$ denotes a cyclic group of order n. The alternating group of degree n is denoted by A_n, the (generalized) quaternion group of order n is denoted by Q_n, $SL_n(q)$ denotes the special linear group of dimension n over a field of order q.

We denote by $G \rtimes H$ a semidirect product of groups G and H, where H acts on G via automorphisms.

Spectrum $\omega(G)$ of a finite group G defines so-called *prime graph* or *Gruenberg-Kegel graph* $GK(G)$. The vertices of this graph are the elements of $\pi(G)$, i. e., the set of all prime divisors of $|G|$. Distinct vertices p and q are adjacent, if $pq \in \omega(G)$. We denote by $s = s(G)$ the number of connected components of $GK(G)$, $\pi_i = \pi_i(G)$ denotes i-th connected component, $i = 1, \ldots, s$. If the order of G is even, we suppose that $2 \in \pi_1$. Let $\omega_i = \omega_i(G)$ be a set of $n \in \omega(G)$ such that every prime divisor of n is contained in π_i.

Lemma 1 (Gruenberg, Kegel, [4])**.** *If G is a finite group and $GK(G)$ is disconnected, then one of the following holds*:

(1) *$s(G) = 2$, G is a Frobenius group, i. e., G possesses a nontrivial normal nilpotent Hall subgroup A and $C_G(a) \leq A$ for every nonidentity $a \in A$*;
(2) *$s(G) = 2$, G is a 2-Frobenius group, i. e., $G = ABC$, where A, AB are normal in G, B is normal in BC and AB, BC are Frobenius groups*;
(3) *there exists a nonabelian simple group P, such that*

$$P \leq \overline{G} = G/K \leq \mathrm{Aut}(P)$$

for some nilpotent normal $\pi_1(G)$-subgroup K of G and $\overline{G}/P$ is a $\pi_1(G)$-group. Moreover, the graph $GK(P)$ is disconnected, $s(P) \geq s(G)$ and for every integer i, $2 \leq i \leq s(G)$ there exists j, $2 \leq j \leq s(P)$, such that $\omega_i(G) = \omega_j(P)$.

In particular, G possesses at most one unsolvable composition factor.

We need the following well-known results.

Lemma 2 ([5–7])**.** *Let G be a Frobenius group with kernel N and complement H. Then the following holds*:

(1) *N is nilpotent*; *in particular, $GK(N)$ is a complete graph and $|\mu(N)| = 1$*;
(2) *if U is a subgroup of H of order rs, where r and s are not necessarily distinct primes, then U is cyclic*; *in particular, Sylow r-subgroups of H are cyclic for odd primes r*;
(3) *if the order of H is even, then H contains a unique involution z*; *in particular, Sylow 2-subgroup of H is cyclic or (generalized) quaternion group, the group N is abelian and $n^z = n^{-1}$ for every element n of N*;
(4) *either H is solvable and $GK(H)$ is a complete graph, or H possesses a normal subgroup $L \simeq SL_2(5)$, such that $(|L|, |H : L|) \leq 2$ and $GK(H)$ can be constructed by removing the edge $\{3, 5\}$ from the complete graph on $\pi(H)$.*

Lemma 3 (Gaschütz, [8])**.** *Let G be a finite group, A is a normal abelian p-subgroup of G. If A is complemented in a Sylow p-subgroup of G, then A is complemented in G.*

Lemma 4 (Clifford, [9, Theorem 3.4.1])**.** *Let V be an irreducible G-module and $H \trianglelefteq G$. Then V is the direct sum of H-invariant subspaces V_i, $1 \leq i \leq r$, which satisfy the following conditions*:

(1) *$V_i = X_{i1} \oplus \cdots \oplus X_{it}$, where each X_{ij} is an irreducible H-module, t is independent of i, and X_{ij}, $X_{i'j'}$ are isomorphic H-modules if and only if $i = i'$;*
(2) *for any H-submodule U of V, we have $U = U_1 \oplus \cdots \oplus U_r$, where $U_i = U \cap V_i$, $1 \leq i \leq r$; in particular, any irreducible H-submodule of V lies in one of V_i;*
(3) *for $x \in G$ the mapping $x\pi : V_i \to V_i x$, $1 \leq i \leq r$, is a permutation of the set $S = \{V_1, \ldots, V_r\}$ and π induces a transitive permutation representation of G on S; furthermore, $HC_G(H)$ is contained in the kernel of π.*

3. On the number of groups critical with respect to a set of natural numbers

In this section we prove Theorem 1. We proceed by induction on s.

First, let $s = 1$, and put $p = p_{11}$ and $r = p_{12}$. Our goal is to construct two non-isomorphic groups critical with respect to $\omega = \{1, p, r\}$.

Consider a field $\mathbb{F}$ of order p^α, where α is a natural number such that r divides $p^\alpha - 1$. Note that such α always exists due to Fermat's little theorem. We denote by $A = \langle F, + \rangle$ and $B = \langle F^*, \cdot \rangle$ the additive and the multiplicative groups of $\mathbb{F}$ respectively. Define the action of B on A by the rule: $a^b = ab$, where $a \in A$, $b \in B$ and ab means a product of a and b in $\mathbb{F}$. Note that this action is *fixed-point-free*, i. e., if $a \neq 0$ and $b \neq 1$, then $a^b \neq a$. As r divides $p^\alpha - 1$, there is a subgroup $H \leq B$ of order r.

Let G be a semidirect product of groups A and H with respect to the action defined above. Then the following holds.

Lemma 5. *The spectrum of G is equal to ω.*

Proof. Let (h, a) be an arbitrary element of G. If $h = 1$, then the order of (h, a) is p for any nonidentity element $a \in A$.

Suppose that $h \neq 1$. For every $k \in \mathbb{N}$ we have $(h, a)^k = (h^k, a^{h^{k-1}} + \cdots + a^h + a)$. Since $h^{k-1} + \cdots + 1 = (h^k - 1)(h - 1)^{-1}$, the following holds:

$$a^{h^{k-1}} + \cdots + a^h + a = a(h^{k-1} + \cdots + 1) = a(h^k - 1)(h - 1)^{-1}.$$

Finally, $(h, a)^k = (h^k, a(h^k - 1)(h - 1)^{-1})$. Therefore, $|(h, a)| = r$. □

Lemma 6. *Let α have a property that r divides $p^\alpha - 1$ and does not divide $p^\beta - 1$ for any natural number $\beta < \alpha$. Then G is critical.*

Proof. Recall that if a group X acts fixed-point-freely on a group Y, then $|X|$ divides $|Y| - 1$. Let K be a nontrivial proper subgroup of G. The order of K is either $p^\beta r$, $0 \leq \beta < \alpha$, or p^β, $0 \leq \beta \leq \alpha$. Prove that the first case is impossible. Indeed, if $|K| = p^\beta r$, then K is isomorphic to a group $C \rtimes H$, where $C < A$. In this case r divides $p^\beta - 1$ and that contradicts the hypothesis of the lemma. Thus, the spectrum of every nontrivial proper subgroup of G is $\{1, p\}$. Now, if N is a nontrivial normal subgroup of G, then G acts on N via conjugation. There are no elements of order pr in G, so every element of order r acts on N fixed-point-freely, and thus r divides $p^\beta - 1$. It follows that $\beta = \alpha$ and the spectrum of G/N is $\{1, r\}$. This proves that G is critical. □

It follows from Lemmas 5 and 6 that there exists a group G of order $p^\alpha r$ critical with respect to ω. Now, consider a field $\mathbb{K}$ of order r^γ, where γ is such natural number that p divides $r^\gamma - 1$ and does not divide $r^\delta - 1$ for $\delta < \gamma$. Let G_1 be a semidirect product of groups C and D, where C is the additive group of $\mathbb{K}$ and D is a subgroup of order p in the multiplicative group of $\mathbb{K}$. Obviously, G_1 is also critical with respect to ω. Note that if $\alpha = 1$, then r divides $p-1$ and thus $r < p$. Therefore, γ cannot be equal to 1. Thus, the orders of G and G_1 are distinct, and in particular, $G_1 \not\simeq G$.

Now, let $s > 1$. First of all, note that if $A_1, \ldots, A_k$ are finite groups, then

$$\omega(A_1 \times \cdots \times A_k) = \{[n_1, \ldots, n_k] \mid n_i \in \omega(A_i), 1 \le i \le k\},$$

where we denote by $[n_1, \ldots, n_k]$ the least common multiple of integers $n_1, \ldots, n_k$.

For convenience we denote p_{i1} as p_i and p_{i2} as r_i for every $i \le s$. It follows from the proof of the case $s = 1$ that for every pair p_i, r_i there exist two non-isomorphic groups critical with respect to $\{1, p_i, r_i\}$. Denote them by G_{i1} and G_{i2}. Now the desired groups are $G_{1i_1} \times \cdots \times G_{si_s}$, where $i_k \in \{1, 2\}$. Spectra of these groups coincide with Ω. Moreover, these groups are pairwise non-isomorphic. Indeed, let $H = G_{1i_1} \times \cdots \times G_{si_s}$ and $K = G_{1j_1} \times \cdots \times G_{sj_s}$ be distinct groups. Since $H \ne K$, there exists a natural number k, such that $i_k \ne j_k$. Then $|G_{ki_k}| \ne |G_{kj_k}|$, wherefrom $|H| \ne |K|$. In particular, $H \not\simeq K$.

Finally, we prove that these groups are critical with respect to Ω. Consider a group $G = G_1 \times \cdots \times G_s$, where G_i is one of the groups G_{i1}, G_{i2} and $i \le s$. Let $|G_i| = p_i^{\alpha_i} r_i$.

Let us assume that G is not critical with respect to Ω. Suppose for instance that there is a subgroup K of G with the same spectrum. In this case, $|K| = \prod\limits_{i \le s} p_i^{\beta_i} r_i$, where $(\beta_1, \ldots, \beta_s) \ne (\alpha_1, \ldots, \alpha_s)$, and it is no loss assuming that $\beta_1 < \alpha_1$.

Since the spectrum of K is Ω, K contains elements of orders $p_1 p_2$ and $r_1 p_2$, i. e., elements of forms $k_1 = (g_1, g_2, 1, \ldots, 1)$ and $k_2 = (h_1, h_2, 1, \ldots, 1)$, where $|g_1| = p_1$, $|h_1| = r_1$, $|g_2| = |h_2| = p_2$. Then $k_1^{p_2} = (g_1^{p_2}, 1, \ldots, 1)$, $k_2^{p_2} = (h_1^{p_2}, 1, \ldots, 1)$, where $g_1^{p_2}$ and $h_1^{p_2}$ are not equal to 1 becase $p_2 \notin \{p_1, r_1\}$. Since $k_1^{p_2}$ and $k_2^{p_2}$ lie in K, a group $R = \langle k_1^{p_2}, k_2^{p_2} \rangle$ also lies in K. The group R is isomorphic to a subgroup $\langle g_1^{p_2}, h_1^{p_2} \rangle$ of G_1. However, due to Lemma 6 such subgroup in G_1 exists only if $\beta_1 = \alpha_1$ and this contradicts the initial assumption. Therefore, G does not contain subgroups with the spectrum equal to Ω.

Now assume that G contains a normal subgroup N such that the spectrum of the quotient group G/N coincides with Ω. It follows that $|N| = p_1^{\beta_1} \ldots p_s^{\beta_s}$, where $\beta_i < \alpha_i$ for every $i \le s$. Put $r = p_2 \ldots p_s$ and $M = \langle n^r \mid n \in N \rangle$. Since $N \trianglelefteq G$, we have $M \trianglelefteq G$. Furthermore, for every $n \in N$ we have $n^r = (g, 1 \ldots, 1)$, where $g \in G_1$, so M is isomorphic to some subgroup M_1 of G_1. It follows that $|M| \le p_1^{\beta_1}$ and $M_1 \trianglelefteq G_1$. However, Lemma 6 implies that such normal subgroup of G_1 exists only if $\beta_1 = \alpha_1$. Again, this contradicts the initial assumption. Therefore, G is critical. □

4. Groups critical with respect to $\omega(A_6)$

Here we prove Theorem 2. Let G be a group critical with respect to $\omega = \{1, 2, 3, 4, 5\}$. The number of connected components of $GK(G)$ is 3. Thus, Lemma 1 implies that there exists a nonabelian simple group P such that

$$P \le G/K \le \mathrm{Aut}(P)$$

for some 2-subgroup K of G. It follows from [10] that $P \in \{A_5, A_6, U_4(2)\}$. The case $P = U_4(2)$ cannot occur because $U_4(2)$ possesses an element of order $6 \notin \omega$.

Suppose that $P = A_6$. If $P < G/K$, then the inverse image of P in G is a proper subgroup, but $\omega(P) = \omega(G)$, so this case contradicts the assumption that G is critical. If $P = G/K$, then K must be trivial; in this case $G \simeq A_6$.

Thus we have only the case $P = A_5$ to deal with. Since $\mathrm{Aut}(A_5) = S_5$ and $|S_5 : A_5| = 2$, either $G/K = A_5$, or $G/K = S_5$. The latter case cannot occur because there is an element of order 6 in S_5. Hence $G/K = A_5$.

Now we show that K is an elementary abelian group. Let $\overline{G} = G/\Phi(K)$ and $\overline{K} = K/\Phi(K)$, where $\Phi(K)$ is the Frattini subgroup of K. Our goal is to show that $\omega(\overline{G}) = \omega(G)$. It suffices to check that $4 \in \overline{G}$. Indeed, if that is not true, then Sylow 2-subgroups of $\overline{G}$ are elementary abelian. Let S be a Sylow 2-subgroup of G and $\overline{K} < S$. Since $C_{\overline{G}}(\overline{K}) \ge S > \overline{K}$, it follows that $C_{\overline{G}}(\overline{K})/\overline{K} \trianglelefteq A_5$. The group A_5 is simple, therefore, $C_{\overline{G}}(\overline{K})/\overline{K} = A_5$, and so $\overline{G} = C_{\overline{G}}(\overline{K})$. But in this case $\overline{G}$ possesses elements of order 6 and 10; this contradicts the fact that $\omega(\overline{G}) \subseteq \omega(G)$. Thus $\omega(\overline{G}) = \omega(G)$. Since G is critical, we have $\Phi(K) = 1$.

The group K is of order 2^n for some natural number n. Also, it is normal in G, so there is a natural action of G on K via conjugation. The elements of K act on K trivially, therefore, this action is defined by the action of G/K on K, i. e., by the action the group A_5. The elements of A_5 of order 3 and 5 act on K fixed-point-freely (otherwise G would contain elements of order 6 or 10), so it follows from the proof of Lemma 6 that 3 divides $2^n - 1$ and 5 divides $2^n - 1$. Thus $n = 4k$, $k \in \mathbb{N}$.

Sometimes it is convenient for us to switch to the language of representation theory.

Lemma 7. *Let V be an irreducible A-module over a field $\mathbb{F}_2$, where $A = \langle \rho \rangle$ is a cyclic group of order* 3*. Suppose that V does not contain trivial submodules. Then* $\dim V = 2$.

Proof. First, suppose that $\dim V = 1$. In this case $v\rho = \lambda v$ for every $v \in V$. But $\lambda \neq 0$, so $\lambda = 1$, and V is a trivial module. Next, assume that $\dim V > 2$. Let v be a non-zero vector from V. Then $v + v\rho + v\rho^2$ is fixed by A. Thus, $v + v\rho + v\rho^2 = 0$ and so $v\rho^2 = v + v\rho$. This means that $\langle v, v\rho \rangle$ is an A-submodule of dimension 2; but this contradicts the irreducibility of V. □

Lemma 8. *Under assumptions of Theorem* 2 *a group K is elementary abelian of order* 16*, and the action of A_5 on K is defined uniquely.*

Proof. There is a general fact that we can use: if V is an irreducible P-module, where V and P are p-groups, then $\dim V = 1$ and P acts trivially on V. Indeed, define a group X as the semidirect product of V and P. Since X is a p-group, $V_1 = V \cap Z(X) > 1$. As $V_1 \trianglelefteq X$, the case $V_1 < V$ cannot occur, because V is irreducible. Thus, $V_1 = V$ and so $V \le Z(X)$. Therefore, P acts trivially on V. Since V is irreducible, $\dim V = 1$.

Now, G is critical, so K should be an irreducible A_5-module over a field $\mathbb{F}_2$. Indeed, if there is a proper submodule K_1 in K, then there is a proper subgroup G_1 in G, such that $K_1 \le G_1$, $G_1/K_1 = A_5$. We check that in this case $\omega(G_1) = \omega(G)$. Obviously, G_1 contains elements of orders 2, 3, and 5. Assume that there are no elements of order 4. In this case Sylow 2-subgroups in G_1 are elementary abelian. Let S be one of these subgroups, $K_1 < S$. Since $C_{G_1}(K_1) \geq S > K_1$, it follows that $C_{G_1}(K_1)/K_1 \trianglelefteq A_5$. The group A_5 is simple, so $C_{G_1}(K_1) = G_1$. But in this case G_1 contains elements of orders 6 and 10, which is impossible. Thus $\omega(G_1) = \omega(G)$, contrary to the assumption that G is critical. So K is an irreducible A_5-module. Let ρ and σ be elements of orders 3 and 5 from A_5, respectively. Since G contains no elements of orders 6 and 10, there are no trivial $\langle\rho\rangle$- and $\langle\sigma\rangle$-submodules in K.

Consider in A_5 a subgroup, isomorphic to A_4, that contains ρ. This group is $K_4\langle\rho\rangle$, where K_4 is a Klein four-group. Let us examine an irreducible module for K_4 in K. It follows from the proved fact that this is a one-dimensional module, on which K_4 acts trivially. Let v be a non-zero vector from this module. Vectors v and $v\rho$ are linearly independent, so $V = \langle v, v\rho\rangle$ is an irreducible module for $\langle\rho\rangle$. As $A_4 = K_4\langle\rho\rangle$, V is an irreducible module for A_4. Since $A_5 = A_4 \cup A_4\sigma \cup \cdots \cup A_4\sigma^4$, it follows that $V + V\sigma + \cdots + V\sigma^4$ is an A_5-invariant space, so it coincides with K. Therefore, $n = \dim K \leq 10$.

It is no loss assuming that $\rho = (123)$ and $\sigma = (12345)$. Let v_1, v_2 be a base of V, $v_1\rho = v_2$, $v_2\rho = v_1 + v_2$. It can be checked directly that $\sigma\rho = (13)(24)\rho\sigma^2$ and $\sigma^2\rho = \rho^{-1}\sigma^3$. As $(13)(24) \in K_4$ and K_4 acts trivially on V by Lemma 4, the vectors $v_1\sigma + v_2\sigma^2 + v_1\sigma^3$ and $v_2\sigma + v_1\sigma^2 + v_2\sigma^2 + v_2\sigma^3$ are fixed by ρ and thus these are zero vectors. Thus, by acting on this vectors with σ an appropriate number of times, we get three independent identities with v_1:

$$v_1 = v_2\sigma + v_1\sigma^2,\ v_1 = v_1\sigma^3 + v_2\sigma^4,\ v_1 = v_2 + v_2\sigma + v_2\sigma^4.$$

Therefore, in the set $\{v_i\sigma^j \mid i = 1, 2, j = 0, \dots, 4\}$ at least three vectors are linearly dependent. Thus, $n = \dim K \leq 7$. Also n is divisible by 4, so $n = 4$. Therefore, $|K| = 16$. Besides, this construction depends only on a choice of $v \in K$, and so it is defined uniquely up to isomorphism. □

Lemma 9. *The action of A_5 on K is defined as an action of the group $SL_2(4) \simeq A_5$ on a vector space $\mathbb{F}_4^2$.*

Proof. In Lemma 8 it was proved that the action of A_5 on K is defined uniquely. Thus it suffices to show that the action described in the hypothesis is suitable, i. e., that the elements of orders 3 and 5 in $SL_2(4)$ do not fix any points.

Let $v \in K$, $\rho \in SL_2(4)$. Suppose that $v\rho = v$. In this case the matrix ρ in the base containing v has the following form: $\rho = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$, and it can be checked directly that in this case ρ is of order at most 2. □

So we have proven that in G there is a normal elementary abelian subgroup K of order 16 and $G/K = A_5$. We also know how this quotient group acts on K. If we prove that this subgroup K is complemented in a Sylow 2-subgroup in G, then our theorem would follow from Lemma 3.

First, prove that the set $G \setminus K$ contains an element of order 2. Let D be the dihedral group or order 6, which is contained in A_5, and let D_1 be the preimage of D in G, i. e., $D_1 \geq K$ and $D_1/K = D$. The group D_1 contains a normal subgroup N_1 of index 2. Let S be a Sylow 3-subgroup of N_1. By Frattini argument, $D_1 = N_1 N_{D_1}(S)$. Since $|D_1 : N_1| = |N_{D_1}(S) : N_1 \cap N_{D_1}(S)| = 2$, in $N_{D_1}(S)$ there is a 2-element x. If $S = \langle y\rangle$, then $y^x = y^{-1}$, wherefrom $x^2 = 1$. Thus, x is the desired element of order 2. Indeed, if $x \in K$ then $[x, y] \in S \cap K$, so $[x, y] = 1$. This is impossible because there are no elements of order 6 in G. Thus, $x \in G \setminus K$.

Let T be a Sylow 2-subgroup in G that contains x. Since T/K is a Sylow 2-subgroup in $G/K = A_5$ and $N_{G/K}(T/K) \simeq A_4$, the normalizer $N_G(T)$ contains an element z of order 3. As $(xz)^3 \in T$, it must be equal to 1. The following holds:

$$1 = xzxzxz = x \cdot zxz^{-1} \cdot z^{-1}xz = xx^{z^{-1}}x^z.$$

Since x is an involution, it follows that $xx^z = x^zx$. Thus, $X = \langle x, x^z \rangle$ is an abelian subgroup of order 4 in T and $K \cap X = 1$. As $|T/K| = 4$, we have $KX = T$. So the subgroup K is complemented in the Sylow 2-subgroup T, and thus, it is complemented in G by the group A_5. Therefore, $G = K \leftthreetimes A_5$. □

5. Groups critical with respect to $\omega(SL_3(3))$

We proceed to the proof of Theorem 3. Let G be a group critical with respect to $\{1, 2, 3, 4, 6, 8, 13\}$. The number of connected components of $GK(G)$ is 2. Thus, it follows from Lemma 1, [10], and [11] that G is either $SL_3(3)$, or a Frobenius group.

Let G be a Frobenius group with kernel A and complement H. Due to Lemma 2, A is a nilpotent group with a complete graph, and $|\mu(A)| = 1$, H is solvable and also has a complete graph. Suppose that $13 \in \omega(H)$. In this case $2, 3 \notin \omega(H)$, because otherwise H would contain elements of order 26 or 39 (since $GK(H)$ is complete). Thus, $2, 3 \in \omega(A)$. It follows that $\mu(A) = \{6, 8\}$ and $|\mu(A)| = 2$. This contradicts Lemma 2. Therefore, $13 \in \omega(A)$, $\mu(A) = \{13\}$, $\mu(H) = \{6, 8\}$. Since $|H|$ is even, it follows from Lemma 2 that A is abelian and H contains the unique involution z.

Define the subgroup $\Omega(H)$ of H to be the group generated by all prime order elements of H. As H acts fixed-point-freely on A, [12, Theorem 2] implies that $\Omega(H) = Z \times K$, where Z is a cyclic Hall subgroup, p^2 does not divide $|Z|$ for any prime p, and K is either trivial, or isomorphic to one of the groups $SL_2(3)$, $SL_2(5)$. The group $SL_2(5)$ contains an element of order 5, so this is not the case.

Assume that $K = 1$. In this case $\Omega(H) \simeq \mathbb{Z}_6$, Sylow 3-subgroup S of H is of order 3 and normal in H. Thus, H acts on S via conjugation, and there is a homomorphism $\varphi : H \to \mathrm{Aut}(S)$. Note that $\ker \varphi$ is exactly $C_H(S)$. Since $\mathrm{Aut}(S)$ is of order 2, we have $|C_H(S)| = |\ker \varphi| = |H|/2$. Let T be a Sylow 2-subgroup of $C_H(S)$ and P be a Sylow 2-subgroup of H. It follows from Lemma 2 that P is either cyclic, or a generalized quaternion group. If P is cyclic, then it is of order 8. In this case T is of order 4 and there is an element of order 4 that centralizes an element of order 3. Hence $12 \in \omega(H)$, which is impossible. If P is a quaternion group, then it is of order 16 and T is of order 8. Every subgroup of order 8 in Q_{16} contains an element of order 4. Again, in this case $12 \in \omega(H)$. Thus, the case $K = 1$ cannot occur.

Let K be isomorphic to $SL_2(3)$. Since $\pi(H) = \pi(SL_2(3))$, the fact that Z is a Hall subgroup implies $Z = 1$.

We conclude that $\Omega(H) \simeq SL_2(3)$. As a Sylow 2-subgroup in $SL_2(3)$ is isomorphic to Q_8, it follows from Lemma 2 that a Sylow 2-subgroup in H must also be isomorphic to a quaternion group. This quaternion group must be Q_{16}, because all quaternion 2-groups of order greater than 16 contain an element of order 16. Thus, a Sylow 2-subgroup of H is isomorphic to Q_{16}, so $|H : \Omega(H)| = 2$.

Put $\overline{H} = H/\langle z \rangle$. We now prove that $\overline{H} \simeq S_4$. As $PSL_2(3) \simeq A_4$, the group $\overline{H}$ contains a subgroup of index 2, isomorphic to A_4. Also, $|\overline{H}| = 24$ and $\omega(\overline{H}) = \{1, 2, 3, 4\}$. Now the isomorphism follows from [13, Theorem 1].

The group S_4 has a presentation $\langle \overline{x}, \overline{y} \mid \overline{x}^2 = \overline{y}^3 = (\overline{xy})^4 = 1\rangle$. Thus, as z is the unique involution in H, we must have $H = \langle x, y \mid x^4 = y^3 = (xy)^8 = 1; x^2 = (xy)^4\rangle$. Using the coset enumeration algorithm (performed in GAP [14]), we check that such group exists and is uniquely defined, also that it satisfies the properties of H. Now we only have to define A in order to define the group G.

In H elements y and $r = y^x$ generate the normal subgroup L isomorphic to $SL_2(3)$, with presentation $L = \langle y, r \mid y^3 = r^3 = (yr)^4 = [(yr)^2, y] = [(yr)^2, r] = 1\rangle$. We can regard A as a vector space over the field $\mathbb{F}_{13}$. As $|\mathbb{F}^*| = 12$, for every element $x \in L$ in A there exists an eigenvector. Let $t = yr$ and v be an eigenvector for t with $vt = \lambda v$. Here $\lambda^2 = -1$ and hence $\lambda = \pm 5 \pmod{13}$.

Lemma 10. *The subspace $V = \langle v, vy\rangle$ of A is L-invariant.*

Proof. Let w be a non-zero vector from V. Then $w(y^2 + y + 1)$ is y-invariant. As $13 \cdot 3 \notin \omega(G)$, we must have $y^2 + y + 1 = 0$, and thus, $y^2 = -y - 1$. Similarly, $r^2 = -r - 1$.

To prove that V is L-invariant, it is sufficient to show that V is y-invariant and r-invariant. The following holds:

$$(vy)y = vy^2 = v(-y - 1) = -vy - v,$$

$$(vy)r = vt = \lambda v,$$

$$vr = v(yr)^4 r = v(yr)^3 yr^2 = v(yr)^3 y(-r - 1) = -v(yr)^4 - v(yr)^3 y = -v - \lambda^{-1} vy.$$

□

Since H acts on A irreducibly, Lemma 4 yields that $A = V_1 \oplus \cdots \oplus V_k$, where V_i is an irreducible L-module, $i \leq k$. Thus, $V_i \simeq V$. Furthermore, $k \leq 2$, since $|H : L| = 2$. Thus, $A = V_1$ or $A = V_1 \oplus V_2$.

Suppose that $A = V_1$. Then H is isomorphic to a subgroup of $GL_2(13)$. Let S be a Sylow 2-subgroup of H. Then $S \simeq Q_{16} = \langle c, d \mid c^8 = d^4 = 1;\ c^d = c^{-1};\ d^2 = c^4\rangle$ and $[S, S] = \langle a\rangle$, where $|a| = 4$. Since $\det a = 1$ and the eigenvalues of a are distinct, it follows that $a \sim \begin{pmatrix} i & 0 \\ 0 & i^{-1} \end{pmatrix}$, where $i^2 = -1$. As $[c, d] = c^{-2} \in \langle a\rangle$, we may assume that $a = c^2$. In this case $c \sim \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$, $\mu_1, \mu_2 \in \mathbb{F}_{13}$, $|\mu_1| = |\mu_2| = 8$. But 8 does not divide the order of $\mathbb{F}_{13}^*$. Thus, in $GL_2(13)$ there are no subgroups isomorphic to Q_{16}, and therefore, no subgroups isomorphic to H.

We conclude that $A = V_1 \oplus V_2$. In this case, A is the induced representation of H with respect to L. Since $H = L \cup Lx$ and $A = V_1 \oplus V_1 x$, it is easy to calculate the action of x and y on the basis v, vy, vx, vyx. Without loss of generality we put $\lambda = 5$, and it can be checked directly, that

$$x \sim \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad y \sim \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & 5 & 0 \end{pmatrix}.$$

This finishes the proof of Theorem 3. □

References

[1] V. D. Mazurov and W. J. Shi, *A criterion of unrecignizability by spectrum for finite groups*, Algebra and Logic, **51**:2 (2012), 239–243. MR2986582

[2] R. Brandl and W. J. Shi, *Finite groups whose element orders are consecutive integers, J. Algebra,* **143***:2 (1991), 388–400. MR1132578*

[3] V. D. Mazurov, Recognition of finite simple groups $S_4(q)$ *by their element orders, Algebra and Logic,* **41***:2 (2002), 166–198. MR1922988*

[4] J. S. Williams, Prime graph components of finite groups, J. Algebra, **69** *(1981), N2, 487–513. MR0617092*

[5] J. G. Thompson, *Normal p-complements for finite groups*, Math. Z., **72**:2 (1960), 332–354. MR0117289

[6] H. Zassenhaus, *Kennzeichnung endlicher linearean Gruppen als Permutationsgruppen*, Abh. Math. Sem. Univ. Hamburg, **11** (1936), 17–40.

[7] H. Zassenhaus, *Über endliche Fastkörper*, Abh. Math. Sem. Univ. Hamburg, **11** (1936), 187–220.

[8] B. Huppert, *Endliche Gruppen*, Springer Verlag, 1979.

[9] D. Gorenstein, *Finite groups*, Chelsea Publishing Company, New York, N. Y., 1980. MR0569209

[10] A. V. Zavarnitsine, *Finite simple groups with narrow prime spectrum*, Siberian electronic mathematical reports, **6** (2009), 1–12. MR2586673

[11] M. R. Aleeva, *On finite simple groups with the set of element orders as in a Frobenius group or a double Frobenius group*, Mathematical Notes, **73**:3 (2003), 299–313. MR1992593

[12] V. D. Mazurov, *A generalization of a theorem of Zassenhaus*, Vladikavkaz. Mat. Zh., **10**:1 (2008), 40–52. MR2434653

[13] D. V. Lytkina, *Structure of a group with elements of order at most 4*, Siberian Math. J., **48**:2 (2007), 283–287. MR2330064

[14] *GAP: Groups, algorithms, and programming*, http://www/gap-system.org.

Yuri Vsevolodovich Lytkin
Novosibirsk State University,
Pirogova st., 2,
630090, Novosibirsk, Russia
E-mail address: jurasicus@gmail.com