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ON THE GODSIL – HIGMAN NECESSARY CONDITION FOR EQUITABLE PARTITIONS OF ASSOCIATION SCHEMES

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Abstract. In his monograph 'Association schemes', C. Godsil derived a necessary condition for equitable partitions of association schemes and noticed that it could be used to show that certain equitable partitions do not exist. In this short note, we show that, in fact, this condition is not stronger than the well-known Lloyd theorem.

Keywords: association scheme, equitable partition.

1. Introduction

Equitable partitions of association schemes are often related to important and interesting combinatorial and geometric objects such as combinatorial designs, orthogonal arrays and Cameron – Liebler line classes in projective geometries. Thereby the question of existence of these structures in association schemes is very difficult in general.

In this short note, we show that a necessary condition for equitable partitions recently proposed by C. Godsil in his monograph [6] is not stronger than the well-known Lloyd theorem (see Theorem 1 in Section 2). In the next section, we recall some basic definitions and notions. Section 3 contains the proof of our result (Theorem 2).

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2. Association schemes and their equitable partitions

Let V be a finite set of size v and $\mathbf{C}^{V\times V}$ be the set of matrices over $\mathbf{C}$ with rows and columns indexed by V. Let $\mathcal{R} = \{R_0, R_1, \dots, R_d\}$ be a set of non-empty subsets of $V \times V$. For $i = 0, \dots, d$, let $A_i \in \mathbf{C}^{V\times V}$ be the adjacency matrix of the graph (V, R_i). The pair $\mathcal{A} = (V, \mathcal{R})$ is said to be an *association scheme* with d classes and vertex set V if the following properties hold:

(1) $A_0 = I$, the identity matrix,
(2) $\sum_{i=0}^{d} A_i = J$, where J is the all ones matrix,
(3) $A_i^{\mathrm{T}} \in \{A_0, \dots, A_d\}$, for every $i = 0, \dots, d$,
(4) A_iA_j is a linear combination of $A_0, \dots, A_d$, for all $i, j = 0, \dots, d$.

The matrix algebra $\mathbf{C}[\mathcal{A}]$ over $\mathbf{C}$ generated by $A_0, \dots, A_d$ is called the *Bose – Mesner algebra* of $\mathcal{A}$. It now follows from properties (1)-(4) that $\mathbf{C}[\mathcal{A}]$ has a basis consisting of the matrices $A_0, \dots, A_d$ and its dimension is $d + 1$. We say that $\mathcal{A}$ is commutative if $\mathbf{C}[\mathcal{A}]$ is commutative, and that $\mathcal{A}$ is symmetric if the matrices A_i are symmetric. Clearly, a symmetric association scheme is commutative.

In the paper, we assume that $\mathcal{A}$ is a symmetric association scheme with d classes.

Since $\mathcal{A}$ is commutative, it follows that the matrices A_0, A_1, …, A_d are simultaneously diagonalized by an appropriate unitary matrix. This means that $\mathbf{C}^V$ is decomposed as an orthogonal direct sum of $d + 1$ maximal common eigenspaces of A_0, A_1, …, A_d:

$$\mathbf{C}^V = W_0 \oplus W_1 \oplus \dots W_d. \tag{1}$$

For every $0 \le j \le d$, define $E_j \in \mathbf{C}^{V\times V}$ to be the orthogonal projection onto W_j.

Note [2] that the matrices E_j form another basis for $\mathcal{A}$ consisting of the primitive idempotents of $\mathcal{A}$, i.e., $E_iE_j = \delta_{ij}E_i$ and $\sum_{j=0}^{d} E_j = I$.

For the two basises $A_0, \dots, A_d$ and $E_0, \dots, E_d$ of $\mathbf{C}[\mathcal{A}]$, the change-of-basis matrices P and Q are defined by

$$A_i = \sum_{j=0}^{d} P_{ji}E_j, \quad E_i = \frac{1}{v}\sum_{j=0}^{d} Q_{ji}A_j,$$

where, in fact, P_{ji} is an eigenvalue of A_i on the eigenspace W_j. It now follows that

$$PQ = vI.$$

The numbers $v_i = P_{0i}$, $0 \le i \le d$, are called the *valencies* of the scheme. The numbers $f_j = \mathrm{tr}(E_j) = \mathrm{rank}(E_j) = \dim(W_j)$, $0 \le j \le d$, are called the *multiplicities* of the scheme. By [2, Lemma 2.2.1(iv)] the following relation holds:

$$\frac{Q_{ij}}{f_j} = \frac{P_{ji}}{v_i}. \tag{2}$$

The distance-regular graphs provide important but not only examples of symmetric association schemes. For more results and background on distance-regular graphs and association schemes, we refer the reader to [1], [2], [5].

Let π be a partition of V with t cells $C_1, C_2, \dots, C_t$. The *characteristic matrix* H of π is the $v \times t$-matrix whose columns are the characteristic vectors of $\mathbf{C}^V$ of the cells of π.

Define the *quotient matrix* N of $A \in \mathbf{C}[\mathcal{A}]$ with respect to π to be equal to

$$N_{k,j} = \sum_{x \in C_k, y \in C_j} A_{x,y}/|C_k|.$$

We say that a partition π of $\mathcal{A}$ is *equitable* [6] if the column space of H is $\mathbf{C}[\mathcal{A}]$-invariant, i.e., for every $A \in \mathbf{C}[\mathcal{A}]$, there exists a $t \times t$-matrix N such that

$$AH = HN.$$

One can easily see that N is the quotient matrix of A with respect to π.

One can give the following equivalent combinatorial definition of equitable partition: π is equitable if, for all $0 \leq i \leq d$, $1 \leq j,k \leq t$ and every vertex $x \in C_k$, there exist exactly n_{ij}^k vertices $y \in C_j$ such that $(x,y) \in R_i$. The $t \times t$-matrices N_i, $0 \leq i \leq d$, such that

$$(N_i)_{k,j} = n_{ij}^k,$$

are, again, the quotient matrices of A_i with respect to π, i.e., $A_iH = HN_i$.

Let C be a subset of the vertex set of a graph Γ. The covering radius ρ_C of C is defined to be $\rho_C := \max\{\mathrm{d}(x,C) \mid x \in \Gamma\}$, where $\mathrm{d}(x,C) := \min\{\mathrm{d}(x,y) \mid y \in C\}$, and $\mathrm{d}(x,y)$ is the usual graph distance. For $i = 0,\ldots,\rho_C$, define $\Gamma_i(C)$ to be the set of vertices that are at distance i from C. The partition $\{C = \Gamma_0(C), \Gamma_1(C), \ldots, \Gamma_{\rho_C}(C)\}$ is referred to as the *distance partition* of the vertex set of Γ with respect to C.

For an association scheme $\mathcal{A} = (V, \mathcal{R})$, assume that a graph $\Gamma = (V, R_i)$ is distance-regular for some i. An important type of equitable partitions is provided by completely regular codes. A vertex subset C of Γ is called a *completely regular code* if the distance partition with respect to C is equitable. Clearly, a single vertex of the graph Γ is a completely regular code.

For instance, the completely regular codes of the Hamming graphs give rise to orthogonal arrays, of the Johnson graphs — to combinatorial designs [4], [9], and of the Grassmann graphs of diameter 2 — to Cameron – Liebler line classes in the projective geometry of dimension 3 [10].

Another interesting type of equitable partitions is provided by the partition of the vertex set of antipodal distance-regular graphs into antipodal classes [9].

The following result is well known in Algebraic Combinatorics and is sometimes referred to as Lloyd's theorem [5].

Theorem 1. *If π is an equitable partition of symmetric association scheme $\mathcal{A}$, then, for any matrix $A \in \mathcal{A}$, the characteristic polynomial of its quotient matrix B divides the characteristic polynomial of A.*

In other words, Theorem 1 states that every eigenvalue of the quotient matrix B is an eigenvalue of A, and its multiplicity as an eigenvalue of B is not greater than its multiplicity as an eigenvalue of A.

3. Main result

Following Godsil [6], let us define a complex inner product on $\mathbf{C}^{V \times V}$ by

$$\langle M, N \rangle = \mathrm{tr}(M^* N) = \mathrm{sum}(\overline{M} \circ N),$$

where $\mathrm{sum}(M)$ denotes the sum of the entries of M, and $\circ$ denotes the Schur multiplication of matrices. Then the basis $A_0, A_1, \ldots, A_d$ of the Bose – Mesner algebra $\mathbf{C}[\mathcal{A}]$ is orthogonal with respect to the inner product.

Further, for a matrix $M \in \mathbf{C}^{V\times V}$, let $\hat{M}$ denote its orthogonal projection onto $\mathbf{C}[\mathcal{A}]$, i.e.,

$$\hat{M} = \sum_{i=0}^{d} \frac{\langle A_i, M\rangle}{\langle A_i, A_i\rangle} A_i = \sum_{j=0}^{d} \frac{\langle E_j, M\rangle}{\langle E_j, E_j\rangle} E_j.$$

It follows from [6, Theorem 3.2.1] that

$$\hat{M} = \sum_{i=0}^{d} \frac{\langle A_i, M\rangle}{vv_i} A_i = \sum_{j=0}^{d} \frac{\langle E_j, M\rangle}{f_j} E_j. \tag{3}$$

Let F be a $v \times v$ permutation matrix. Then F is an automorphism of $\mathcal{A}$ if it commutes with each matrix of $\mathbf{C}[\mathcal{A}]$. G. Higman derived the following necessary condition for F to be an automorphism of $\mathcal{A}$, see also [3].

Let F be an automorphism of $\mathcal{A}$ and σ denote the corresponding permutation associated with F. Define $\alpha_i(\sigma)$ to be the number of vertices x of the vertex set of $\mathcal{A}$ such that $(x, \sigma(x)) \in R_i$. Note that $\alpha_i(\sigma) = \langle A_i, F\rangle$. Then using equality (3):

$$\hat{F} = \sum_{i=0}^{d} \frac{\alpha_i(\sigma)}{vv_i} A_i = \sum_{i=0}^{d} \frac{\langle F, E_i\rangle}{f_i} E_i,$$

and, further, exploiting this equation, one can show (see [6]) that a number

$$\langle F, E_j\rangle = \frac{f_j}{v} \sum_{i=0}^{d} \frac{F_{ji}}{v_i} \alpha_i(\sigma)$$

must be an algebraic integer.

This condition is widely used in a study of feasible automorphisms of distance-regular graphs [8].

Now let F be a projection matrix (i.e., $F^2 = F$) that commutes with $\mathbf{C}[\mathcal{A}]$. In his monograph [6], C. Godsil noticed that the arguments that prove the Higman condition also yield that

$$\langle F, E_j\rangle = \frac{f_j}{v} \sum_{i=0}^{d} \frac{P_{ji}}{v_i} \langle F, A_i\rangle \tag{4}$$

must be a non-negative integer and this observation “could be used to show that certain equitable partitions do not exist”.

Let us define the projection matrix associated to an equitable partition. Let π be a partition of $\mathcal{A}$ with characteristic matrix H and the cells $C_1, \ldots, C_t$. Then $D = H^{\mathrm{T}}H$ is the diagonal matrix such that $D_{i,i} = |C_i|$. Now the matrix

$$F = HD^{-1}H^{\mathrm{T}}$$

represents the orthogonal projection onto the column space of H. We call F the *projection matrix* of equitable partition π.

One can see that $F_{x,y}$ is equal to $1/|C_i|$ if x and y are both in C_i and zero otherwise. So, for an appropriate ordering of X, F has the following block form:

$$F = \begin{pmatrix} \frac{1}{|C_1|}J_{|C_1|} & O_{|C_1|\times|C_2|} & \cdots & O_{|C_1|\times|C_t|} \\ O_{|C_2|\times|C_1|} & \frac{1}{|C_2|}J_{|C_2|} & \cdots & O_{|C_2|\times|C_t|} \\ \cdots & \cdots & \cdots & \cdots \\ O_{|C_t|\times|C_1|} & \cdots & O_{|C_t|\times|C_{t-1}|} & \frac{1}{|C_t|}J_{|C_t|} \end{pmatrix},$$

where J is the all ones square matrix of proper order. Note that π is equitable if and only if F commutes with $\mathbf{C}[\mathcal{A}]$ [7].

Theorem 2. *(Godsil condition for equitable partitions) Let π be an equitable partition of a symmetric association scheme $\mathcal{A}$ with the projection matrix F. Then*

$$\langle F, E_j\rangle = \frac{f_j}{v}\sum_{i=0}^{d}\frac{P_{ji}}{v_i}\langle F, A_i\rangle$$

is a nonnegative integer for any $j \in \{0, \dots, d\}$.

First of all, we note that the quotient matrices of A_i share the same eigenspaces, that implies the following fact on their eigenvalues. By m_i we denote the dimension of the space W_iH, where W_i is the eigenspace of $\mathcal{A}$, see (1).

Lemma 1. *Let π be an equitable partition of a symmetric association scheme $\mathcal{A}$ with the charachteristic matrix H:*

$$A_iH = HN_i,$$

$m_i = dim(W_iH)$ for any $i \in \{0, \dots, d\}$. Then for any $i \in \{0, \dots, d\}$ the spectrum of N_i consists of numbers $P_{0i}, \dots, P_{di}$ with multiplicities $m_0, \dots, m_d$ respectively.

Proof. For a left eigenvector $\overline{w}$ of A_i with eigenvalue P_{ji} we have $\overline{w}A_iH = P_{ji}\overline{w}H = \overline{w}HN_i$ so that $\overline{w}H$ is a left eigenvector of N_i with the same eigenvalue P_{ji} iff $\overline{w}H \neq \overline{0}$. Thus, if the space W_jH is non-zero then it is a subspace of an eigenspace of N_i.

Note that H has rank t, so $\cup_{j\in\{0,\dots,d\}}W_jH$ is the whole eigenspace of N_i. Since the considerations above do not depend on the choice of i, we get the desired property.

Theorem 3. *Let π be an equitable partition of a symmetric association scheme $\mathcal{A}$ with the projection matrix F. Then the following equality holds:*

$$\langle F, E_j\rangle = m_j,$$

where m_j is the multiplicity of the eigenvalue P_{ji} as an eigenvalue of N_i on the eigenspace W_jH, for every $0 \le i \le d$.

Proof. First of all, we note that, for $0 \le i \le d$,

$$\langle F, A_i\rangle = \mathrm{tr}(FA_i) = n_{i1}^1 + n_{i2}^2 + \ldots + n_{it}^t = \mathrm{tr}(N_i),$$

and therefore $\langle F, A_i\rangle$ is equal to the sum of all eigenvalues of N_i.

We now have

$$\langle F, E_j\rangle = \frac{f_j}{v}\sum_{i=0}^{d}\frac{P_{ji}}{v_i}\langle F, A_i\rangle = \frac{f_j}{v}\sum_{i=0}^{d}\frac{P_{ji}}{v_i}\mathrm{tr}(N_i) = \frac{f_j}{v}\sum_{i=0}^{d}\frac{P_{ji}}{v_i}\sum_{k=0}^{d}m_kP_{ki}.$$

Using equality (2), we obtain

$$\langle F, E_j\rangle = \frac{1}{v}\sum_{i=0}^{d}Q_{ij}\sum_{k=0}^{d}P_{ki}m_k = \sum_{k=0}^{d}m_k\sum_{i=0}^{d}\frac{Q_{ij}P_{ki}}{v} = \sum_{k=0}^{d}m_k\delta_{j,k} = m_j,$$

which proves the theorem.

Therefore it follows from Theorem 2 that, for a putative equitable partition of association scheme, condition (4) cannot say more than Theorem 1. Moreover, the following example shows that the condition may be even weaker than an obvious condition of integrality of the elements of the quotient matrix.

Consider a perfect matching $M = \{(i,i') : i \in \{0,\dots,4\}\}$. Let C be a labeled cycle with vertex set $\{0,\dots,4\}$, and C' be its complement with vertex set $\{0',\dots,4'\}$ so that $i' \sim_{C'} j'$ if and only if $i \not\sim_C j$.

Define a graph Γ with vertex set $\{i : i \in \{0,\dots,4\}\} \cup \{i' : i \in \{0,\dots,4\}\}$ and edge set consisting of edges C, C', and M. It is easy to see that Γ is the Petersen graph, which is known to be *distance-regular* with diameter 2, i.e., the distance relations on its vertex set form an association scheme with two classes.

Let us consider a 5-partition consisting of a cell with a pair of adjacent vertices Γ (say, $C_1 = \{0,0'\}$) and of four cells of pairs of non-adjacent vertices (say, $C_2 = \{1,2'\}$, $C_3 = \{2,1'\}$, $C_4 = \{3,4'\}$, $C_5 = \{4,3'\}$). Clearly, the partition is not equitable, because, for example, the vertex $2'$ has a neighbour in C_4, whereas the vertex 1 does not. In terms of the quotient matrices this is equivalent to the fact that the quotient matrix B of A_1 has a noninteger entry $B_{2,4} = 1/2$.

However, the partition is feasible with respect to (4). It is easy to see that $\langle F, A_0\rangle$ is always the number of cells in the partition, i.e. $\langle F, A_0\rangle = 5$, and, further, $\langle F, A_1\rangle = 1, \langle F, A_2\rangle = 4$.

Using equality (4), we obtain:

$$\langle F, E_0\rangle = 1, \langle F, E_1\rangle = 2, \langle F, E_2\rangle = 2,$$

which completes our example.

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