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# NORMAL FAMILIES OF LIGHT MAPPINGS OF THE SPHERE ONTO ITSELF

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Abstract. Considering the class $\mathcal{D}$ of all continuous light mappings of the Riemann sphere $\bar{\mathbf{C}}$ onto itself, we introduce the notion of $\mathcal{D}$-normal family and prove that every mapping $f$ from a given Möbius invariant and $\mathcal{D}$-normal family $\mathcal{F} \subset \mathcal{D}$ is a composition of a $K$-quasiconformal automorphism of $\bar{\mathbf{C}}$ with the mapping, realized by a meromorphic function on $\bar{\mathbf{C}}$, where a constant $K$ is common for all $f \in \mathcal{F}$.

**Keywords:** quasiconformal mapping, mapping of bounded distortion, quasimeromorphic mapping, graph convergence, normal family, Möbius mapping, Möbius invariant family, Stoilov theorem, light mapping, open mapping.

**1.** It was shown in ([1], §5, Theorem 7, p. 28) that for any family $\mathcal{F}$ of homeomorphisms of domains in $\mathbf{C}$ which is normal and invariant under compositions with similarity transformations there exists a finite constant $q \geq 1$ such that each mapping $f \in \mathcal{F}$ is $q$-quasiconformal.

Our goal in this paper is to extend the criterion of quasiconformality mentioned above to the case of non necessary univalent mappings of the extended complex plane $\bar{\mathbf{C}}$ onto itself. Such quasiconformal mappings are known as *mappings with bounded distortion* (see [2]) or *quasimeromorphic mappings* (see [3], 10.6, p. 128). For that purpose we modify the classical concept of normal family of mappings using the notion of *graph convergence* for the mappings of compact sets (see, for example, [4] or [5], §2, p. 384–391).

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**Definition 1.** *([6], Chap. 7, (4.4), p. 130). A continuous mapping $f : X \to Y$ between metric spaces is called light if there is no one non-degenerate continuum in $X$ on which the mapping $f$ is constant. If $X$ is compact then this condition is equivalent to the fact that for any point $y \in Y$ the set $f^{-1}(y)$ has topological dimension* 0.

Throughout the paper the family of all continuous light mappings of the extended complex plane $\bar{\mathbf{C}}$ onto itself will be noted by $\mathcal{D}$.

**Definition 2.** *A non-empty family $\mathcal{F} \subset \mathcal{D}$ is called $\mathcal{D}$-normal if for every graphically convergent sequence $\{f_n\} \subset \mathcal{F}$ the topological limit*

$$\Gamma = \lim_{n\to\infty} \Gamma f_n \ ,$$

*where $\Gamma f_n \subset \bar{\mathbf{C}} \times \bar{\mathbf{C}}$ stands for the graph of $f_n$, is of the form*

$$\Gamma = (Z \times \bar{\mathbf{C}}) \cup \Gamma f_0 \ ,$$

*where $Z$ is a compact zero-dimensional set in $\bar{\mathbf{C}}$ (possibly, empty), and the mapping $f_0 : \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ is either constant or $\in \mathcal{D}$.*

**Definition 3.** *A non-empty family $\mathcal{F} \subset \mathcal{D}$ is called Möbius invariant if for all Möbius transformations $\mu, \nu : \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ and for every $f \in \mathcal{F}$ the composition $\nu \circ f \circ \mu$ is also in $\mathcal{F}$.*

The main result of the paper is the following

**Theorem 1.** *Let $\mathcal{F}$ be a Möbius invariant $\mathcal{D}$-normal family of mappings of $\bar{\mathbf{C}}$ onto itself. Then there exists a constant $K$ such that every mapping $f \in \mathcal{F}$ is of the form $f = \varphi \circ \Phi$, where $\Phi$ is a $K$-quasiconformal automorphism of a sphere $\bar{\mathbf{C}}$ and $\varphi$ is the mapping realized by a meromorphic function on $\bar{\mathbf{C}}$ ($\equiv$ rational function).*

**2.** First of all, we show that every mapping $\in \mathcal{F}$ is open. Then Stoilov theorem gives the representation $f = \varphi \circ \Phi$, where $\Phi$ is a homeomorphism of the sphere onto itself, and $\varphi$ is a meromorphic function. Finally, we prove the existence of some upper bound for the coefficient of quasiconformality of $\Phi$ which is common for all $f \in \mathcal{F}$.

**Lemma 1.** *Every mapping of a Möbius invariant $\mathcal{D}$-normal family $\mathcal{F} \subset \mathcal{D}$ is open.*

*Proof.* Suppose, on the contrary, that some mapping $g \in \mathcal{F}$ is not open. Then there exist a point $z_0 \in \bar{\mathbf{C}}$, and an open neighborhood $U'$ of $z_0$ such that $g(z_0)$ is not interior point in the set $g(U')$. Since $\mathcal{D}$ contains no constant mappings, there exists such a point $z_1 \in \bar{\mathbf{C}}$ that $g(z_1) \neq g(z_0)$. Choose two Möbius mappings $\nu, \mu : \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ such that $\mu(z_0) = 0 = \nu(g(z_0))$ and $\mu(z_1) = \infty = \nu(g(z_1))$. Then $f = \nu \circ g \circ \mu^{-1} \in \mathcal{F}$, $f(0) = 0$, $f(\infty) = \infty$, and the point 0 has the open neighborhood $U'' = \mu(U')$ such that $f(0)$ is not interior point in $f(U'')$. By continuity of $f$, there exists an open neighborhood $U \subset U''$ of the point 0 such that $f(z) \neq \infty$ for all $z \in U$. Since the set $f^{-1}(0)$ is zero-dimensional, we may select the open connected neighborhood $V$ of the point 0 such that $\bar{V} \subset U \setminus \{\infty\}$ and

$$f(z) \neq \infty \text{ for all } z \in \bar{V} \ ; \quad f(t) \neq 0 \text{ for all } t \in \partial V. \tag{2.1.1}$$

The set $f(\partial V)$ is compact (the image of the compact set $\partial V$ under continuous mapping) and $0 \notin f(V)$. Thus for some $r > 0$ holds

$$f(\partial V) \cap \bar{B}(0, r) = \emptyset. \tag{2.1.2}$$

The equality

$$f(\bar V)\cap \bar B(0,r) = f(V\cup \partial V)\cap B(\bar 0,r) = (f(V)\cup f(\partial V))\cap \bar B(0,r) = \\ = (f(V)\cap \bar B(0,r))\cup (f(\partial V)\cap \bar B(0,r)) = f(V)\cap \bar B(0,r)$$

shows that

$$f(V)\cap \bar B(0,r) \text{ is compact .} \tag{2.1.3}$$

*Step 1.* Constructing the sequences $\{a_n^*\}$ and $\{a_n\}$.

Since $0 = f(0)$ is not the interior point of the set $f(V)$, there exists a sequence $\{a_n^*\}\subset B(0,r/3)$ such that $a_n^*\to 0$ as $n\to\infty$ and

$$a_n^*\notin f(V)\cap \bar B(0,r) \ \text{ for all } n. \tag{2.1.4}$$

By surjectivity of $f$, the set $f^{-1}(a_n^*)$ is non-empty for all $n$, and it follows from (2.1.4) that $a_n^*\notin f(V)$, i.e. $f^{-1}(a_n^*)\cap V=\emptyset$. By (2.1.2) $a_n^*\notin \partial V$, therefore $f^{-1}(a_n^*)\subset \bar{\mathbf C}\setminus \bar V$. Taking some points $a_n\in f^{-1}(a_n^*)$ we get the sequence $\{a_n\}\subset \bar{\mathbf C}\setminus \bar V$. By passing to a subsequence, if necessary, we can assume that $a_n\to a_0$ as $n\to\infty$. In this situation we obtain $f(a_0)=\lim f(a_n)=\lim a_n^*=0$, and since (2.1.2) $a_0\notin\partial V$, that is, $a_0\in \bar{\mathbf C}\setminus \bar V$. Because of $f(\infty)=\infty$, we have $a_0\neq\infty$, that is, $a_0\in \mathbf C\setminus \bar V$. Hence, for the sequence $\{a_n\}$ the following relations hold

$$f(a_n)=a_n^* \ ; \ a_n\in \mathbf C\setminus \bar V \ ; \ a_n\to a_0\in \mathbf C\setminus \bar V \ ; \ f(a_0)=0\ . \tag{2.1.5}$$

*Step 2.* Constructing the sequences $\{b_n^*\}$ and $\{b_n\}$.

For each $n$ put $r_n=\mathrm{dist}(a_n^*, f(V))>0$. Since $0\in f(V)$, we have $r_n\le |a_n^*|<r/3$ and $\bar B(a_n^*,r_n)\subset \bar B(0,2r/3)\subset B(0,r)$. Thus, there exists a point $b_n^*\in f(V)\cap \partial B(a_n^*,r_n)$ and the open disk $B(a_n^*,r_n)$ has an empty intersection with $f(\bar V)$.

In view of $|b_n^*|\le |a_n^*|+r_n\le 2|a_n^*|$, we have $r_n\to 0$ and $b_n^*\to 0$ as $n\to\infty$. Consequently, we get the sequence $\{b_n^*\}$ satisfying

$$b_n^*\in f(V)\cap B(0,r) \ ; \ b_n^*\to 0 \ ; \ r_n=|b_n^*-a_n^*|\to 0 \ ; \ B(a_n^*,r_n)\cap f(\bar V)=\emptyset\ . \tag{2.1.6}$$

The former relation implies that there is a point $b_n\in V\cap f^{-1}(b_n^*)$. Passing to a subsequence, if necessary, we may assume that $b_n\to b_0\in \bar V$ as $n\to\infty$. Then $f(b_0)=\lim f(b_n)=\lim b_n^*=0$, and therefore $b_0\notin\partial V$, i.e. $b_0\in V$. Hence, we have a sequence $\{b_n\}$ such that

$$b_n\in V \ ; \ f(b_n)=b_n^* \ ; \ b_n\to b_0\in V \ \ f(b_0)=0\ . \tag{2.1.7}$$

*Step 3.* Constructing the sequences $\{c_n^*\}$ and $\{c_n\}$.

Notice that $\bar B(b_n^*,r_n)\subset \bar B(0,|a_n^*|+2r_n)\subset \bar B(0,3|a_n^*|)\subset B(0,r)$, that is

$$\bar B(b_n^*,r_n)\subset B(0,r)\ . \tag{2.1.8}$$

Let $S_n=\partial B(b_n^*,r_n)$. Since the continuum $f(\bar V)$ joins the point $b_n^*$ to the set $f(\partial V)$ which lies outside of the disk $B(0,r)$, we obtain $f(\bar V)\cap S_n\neq\emptyset$, and since $S_n\subset B(0,r)$, the set $f(V)\cap S_n=f(\bar V)\cap S_n$ is non-empty compact in $f(V)$. Therefore, $f^{-1}(S_n)\cap \bar V$ is non-empty for all $n$ and $\varepsilon_n := \mathrm{dist}(b_n, \bar V\cap f^{-1}(S_n))>0$. Let $\alpha=(1/2)\mathrm{dist}(b_0,\partial V)$.

Suppose that $\limsup_{n\to\infty}\varepsilon_n\ge\alpha_0$ for some $0<\alpha_0<\alpha$. Passing, if necessary, to a subsequence, we may assume that $|b_n-b_0|<\alpha_0/2$ and $\varepsilon_n>\alpha_0/2$ for all $n$. Then

$$\bar B(b_n,\alpha_0/2)\subset B(b_0,\alpha)\subset V \ ; \ \bar B(b_n,\alpha_0/2)\cap (f^{-1}(S_n)\cap V)=\emptyset\ .$$

Hence, the continuum $f(\bar B(b_n,\alpha_0/2))$ is contained in the disk $B(b_n^*,r_n)$ for all $n$.

For all sufficiently large $n$ the inclusion $\bar{B}(b_0, \alpha_0/4) \subset B(b_n, \alpha_0/2)$ holds, and consequently, the inclusion $f(\bar{B}(b_0, \alpha_0/4)) \subset B(b_n^*, r_n)$ is true for all sufficiently large $n$. In view of $b_n^* \to 0$ and $r_n \to 0$ as $n \to \infty$, we obviously get the inclusion $f(\bar{B}(b_0, \alpha_0/4)) \subset \{0\}$, which means that $f(z) \equiv 0$ in $\bar{B}(b_0, \alpha_0/4)$. However, it contradicts the lightness of the mapping $f$. Thus,

$$\varepsilon_n = \operatorname{dist}(b_n, \bar{V} \cap f^{-1}(S_n)) \to 0 \quad \text{as } n \to \infty \ . \tag{2.1.9}$$

Therefore, for each $n$ there is a point $c_n \in V$ such that $c_n^* = f(c_n) \in S_n$, and so we obtain the sequences $\{c_n\}$ and $\{c_n^*\}$ with the following properties:

$$c_n^* = f(c_n) \in f(V) \cap \bar{B}(0, r) \ \ ; \ \ |c_n^* - b_n^*| = r_n \to 0 \ \ ; |c_n - b_n| \to 0 \ . \tag{2.1.10}$$

*Step 4.* Constructing the sequence of mappings $\{g_n\}$.

The Möbius mappings

$$\nu_n(w) = i\frac{w - b_n^*}{a_n^* - b_n^*} \ \ ; \ \ \mu_n(z) = b_n + \frac{c_n - b_n}{a_n - b_n}(z - b_n)$$

are satisfying

$$\begin{gathered} \nu_n(\infty) = \infty \ \ ; \ \ \nu_n(a_n^*) = i \ \ ; \ \nu_n^*(b_n^*) = 0 \\ |\nu_n(c_n^*)| = 1 \ \ ; \ \ \nu_n(B(a_n^*, r_n)) = B(i, 1) \ . \\ \mu_n(\infty) = \infty \ \ ; \ \ \mu_n(b_n) = b_n \ \ ; \ \ \mu_n(a_n) = c_n. \end{gathered} \tag{2.1.11}$$

Thus the mapping $g_n := \nu_n \circ f \circ \mu_n$ belongs to the family $\mathcal{F}$ and

$$g_n(\infty) = \infty \ \ ; \ \ g_n(b_n) = \nu_n(b_n^*) = 0 \ \ ; \ \ |g_n(a_n)| = |\nu_n(c_n^*)| = 1 \ . \tag{2.1.12}$$

According to the generalized Bolzano-Weierstrass theorem ([7], §29. VIII, p. 348), we can find a topologically convergent subsequence in the sequence $\{\Gamma g_n\}$ of compact sets in $\bar{\mathbf{C}} \times \bar{\mathbf{C}}$. Hence, without loss of generality, we may assume that there exists a graphical limit $\Gamma = \lim_{n\to\infty} \Gamma g_n$, which, in view of $\mathcal{D}$-normality of the family $\mathcal{F}$, is of the form

$$\Gamma = (Z \times \bar{\mathbf{C}}) \cup \Gamma g_0 \ ,$$

where $Z \subset \bar{\mathbf{C}}$ is a zero-dimensional compact set, and $g_0$ is either constant, or a mapping of a sphere $\bar{\mathbf{C}}$ onto itself.

*Step 5.* Getting the contradiction.

Since $b_0$ is an interior point of $V$ and (see (2.1.7)) $f(b_0) = 0$, there is $\delta > 0$ such that

$$\bar{B}(b_0, \delta) \subset V \ \ ; \ \ f(\bar{B}(b_0, \delta)) \subset B(0, r) \ ,$$

and therefore, the following relation holds for all $n$ (see (2.1.6)):

$$f(\bar{B}(b_0, \delta)) \cap B(a_n^*, r_n) = \emptyset \ . \tag{2.1.13}$$

In view of the convergence $b_n \to b_0$ the estimate $|b_n - b_0| < \delta/2$ is valid for all sufficiently large numbers $n$. Since $a_0 \neq b_0$, (see (2.1.5), (2.1.7)) we have $|a_0 - b_0| > 0$. The mapping $\mu_n$ is similarity with the coefficient

$$k_n = \frac{|c_n - b_n|}{|a_n - b_n|} \to \frac{0}{|a_0 - b_0|} = 0$$

(see (2.1.10)). Consequently, for every $R > 0$ there exists a number $n_1$ such that for all $n > n_1$ the estimate $\operatorname{diam}(\mu_n(\bar{B}(0, R))) < \delta/2$ holds. Hence $\mu_n(\bar{B}(0, R)) \subset B(b_0, \delta)$ for all $n > n_1$ and therefore $(f \circ \mu_n)(\bar{B}(0, R)) \cap B(a_n^*, r_n) = \emptyset$ (see (2.1.13)).

Then (see (2.1.13), (2.1.11))

$$g_n(\bar{B}(0,R)) \cap B(i,1) = \emptyset \ .$$

This means that the set $\Gamma g_n$ does not meet $\bar{B}(0,R) \times B(i,1)$ for all $n > n_1$. Passing to the limit as $n \to \infty$, we obtain the relation

$$\Gamma \cap (\bar{B}(0,R) \times B(i,1)) = \emptyset.$$

This implies that the set $Z$ has empty intersection with the disk $\bar{B}(0,R)$ and, in view of its validity for all $R > 0$, we have $Z \cap \mathbf{C} = \emptyset$. Thus $Z \subset \{\infty\}$ and $\Gamma \subset (\{\infty\} \times \bar{\mathbf{C}}) \cup \Gamma g_0$. Therefore, the set $\Gamma$ over $\mathbf{C}$ coincides with the graph of a mapping $g_0 : \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ which is either constant, or surjective. But the latter case cannot be realized because of

$$g_0(\bar{\mathbf{C}}) = \{g_0(\infty)\} \cup g_0(\mathbf{C}) \subset \{g_0(\infty)\} \cup (\bar{\mathbf{C}} \setminus B(i,1)) \neq \bar{\mathbf{C}} \ .$$

Consequently, $g_0 \equiv const$. However, $(b_n, 0) \in \Gamma g_n$ for all $n$, and thus $(b_0, 0) \in \Gamma g_0$. By (2.1.12), passing to a subsequence, we may assume that there is a convergence $g_n(a_n) \to e^{i\alpha}$ as $n \to \infty$. Then $(a_n, g_n(a_n)) \in \Gamma g_n$, and therefore $(a_0, e^{i\alpha}) \in \Gamma g_0$. This means that $g_0(a_0) = e^{i\alpha} \neq 0 = g_0(b_0)$, and the mapping $g_0$ is not constant.

This contradiction means that for any given $f \in \mathcal{F}$ the image $f(U)$ of an open set $U \subset \bar{\mathbf{C}}$ must be also an open set. The lemma is proven.

**3.** The fundamental *Stoilov theorem* ([8], 49, p. 341) for the continuous light open mappings of two-dimensional manifolds can be formulated in our case as follows:

**Theorem 2.** *If $f : \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ is continuous light open mapping then there exist a homeomorphism $\Phi : \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ and a meromorphic function $\varphi(z)$ on $\bar{\mathbf{C}}$ such that $f = \varphi \circ \Phi$.*

Based on lemma 1, directly from the Stoilov theorem we get the following statement.

**Corollary 1.** *If $\mathcal{F} \subset \mathcal{D}$ is some Möbius invariant $\mathcal{D}$-normal family of mappings of $\bar{\mathbf{C}}$ onto itself then every mapping $f \in \mathcal{F}$ can be represented as $f = \varphi \circ \Phi$, where $\Phi$ is a homeomorphism of a sphere $\bar{\mathbf{C}}$, and $\varphi(z)$ is a meromorphic (non-constant) function on $\bar{\mathbf{C}}$.*

In particular, it means that every mapping $f \in \mathcal{F}$ has a finite multiplicity, the set $B_f$ of its branch points is finite, $f$ is a local homeomorphism in the domain $D_f = \bar{\mathbf{C}} \setminus B_f$ and is *covering mapping* over $\bar{\mathbf{C}} \setminus f^{-1}(f(B_f))$ (see [2], Chapter 2, §10.2, p. 191).

**4.** We shall denote the distance between the points $a, b \in \bar{\mathbf{C}}$ in chordal metric (*chordal distance*) by symbol $h(a,b)$. If mapping $f : \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ is a homeomorphism in a neighborhood of a point $z \in \bar{\mathbf{C}}$ then the value (finite or infinite)

$$k_f(z) := \limsup_{\varepsilon \to 0} \frac{\max_{\{\zeta : h(\zeta,z)=\varepsilon\}} h(f(\zeta), f(z))}{\min_{\{\zeta : h(\zeta,z)=\varepsilon\}} h(f(\zeta), f(z))}$$

is called by *quasiconformality characteristic* of a mapping $f$ at the point $z$.

**Lemma 2.** *Let $\mathcal{F} \subset \mathcal{D}$ be some Möbius invariant $\mathcal{D}$-normal family of the mappings $\bar{\mathbf{C}}$ onto itself. Then there exists a finite constant $K \geq 1$ such that for every mapping $f \in \mathcal{F}$ the estimate*

$$k_f(z) \leq K \tag{4.1.1}$$

*is valid at every point $z \in D_f = \bar{\mathbf{C}} \setminus B_f$, where $B_f$ is a set of all branch points for a mapping $f$.*

*Proof.* If, on the contrary, there is no such finite constant $K$, then

(A1): there exists a sequence of mappings $\{f_n\} \subset \mathcal{F}$ and a sequence of points $z_n \in D_{f_n}$ such that $k_{f_n}(z_n) \to \infty$ as $n \to \infty$.

For every $n$ we choose some point $b_n \notin f_n^{-1}(f_n(z_n))$ and construct two Möbius mappings $\nu_n, \mu_n : \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ such that

$$\mu_n(z_n) = 0 = \nu_n(f_n(z_n)) \quad ; \quad \mu_n(b_n) = \infty \quad ; \quad \nu_n(f_n(b_n)) = \infty .$$

Then for $g_n = \nu_n \circ f_n \circ \mu_n^{-1} \in \mathcal{F}$ the following relations are valid:

$$0 \in D_{g_n} = \mu_n(D_{f_n}) \quad ; \quad g_n(0) = 0 \quad ; \quad g_n(\infty) = \infty \quad ; \quad k_{g_n}(0) = k_{f_n}(z_n) \to \infty$$

as $n \to \infty$. Because of $0 \in D_{g_n}$, there is a closed disk $\bar{B}(0, r_n) \subset D_{g_n}$ in which $g_n$ is injective. Using the auxiliary similarity $\tau_n : z \mapsto r_n z$ which transforms $\bar{B}(0,1)$ into $\bar{B}(0, r_n)$, we get the mapping $g_n \circ \tau_n \in \mathcal{F}$ being injective in $\bar{B}(0,1)$, with the fixed points 0, $\infty$, and for which $k_{g_n \circ \tau_n}(0) = k_{g_n}(0) \to \infty$ as $n \to \infty$.

Hence, in assertion (A1) a sequence $\{f_n\} \subset \mathcal{F}$ can be chosen so that

$$\bar{B}(0,1) \subset D_{f_n} \quad ; \quad f_n(0) = 0 \quad ; \quad f_n(\infty) = \infty \quad ; \quad k_n = k_{f_n}(0) \to \infty$$

as $n \to \infty$, and the restriction $f_n|\bar{B}(0,1)$ is a homeomorphism.

Using the definition of quasiconformality and passing, if necessary, to a subsequence, we can find $\varepsilon_n > 0$ for every number $n$ such that $\varepsilon_n < 1/(2n)$ and

$$\frac{\max_{|\zeta|=\varepsilon_n} |f_n(\zeta)|}{\min_{|\zeta|=\varepsilon} |f_n(\zeta)|} \geq \min\{n, k_n/2\} .$$

On the circle $\{|\zeta| = \varepsilon_n\}$ pick two distinct points $a_n$ and $b_n$ such that

$$|f_n(a_n)| = \max_{|\zeta|=\varepsilon_n} |f_n(\zeta)| \quad ; \quad |f_n(b_n)| = \min_{|\zeta|=\varepsilon_n} |f_n(\zeta)|$$

and, consequently,

$$\frac{|f_n(a_n)|}{|f_n(b_n)|} \geq \min\{n, k_n/2\} . \tag{4.1.2}$$

By the fact that $f_n$ is a homeomorphism on $\bar{B}(0,1)$, we have the inclusions (4.1.3):

$$f_n(\bar{B}(0,1) \setminus B(0,\varepsilon_n)) \subset \bar{\mathbf{C}} \setminus B(0, |f_n(b_n)|) \quad ; \quad f_n(\bar{B}(0,\varepsilon_n)) \subset \bar{B}(0, |f_n(a_n)|) .$$

For the Möbius transformations $\mu_n(z) = b_n z$ and $\nu_n(z) = z/f_n(b_n)$ the sequence of mappings $\{g_n = \nu_n \circ f_n \circ \mu_n\} \subset \mathcal{F}$ has the following properties:

$$g_n(0) = 0 \quad ; \quad g_n(\infty) = \infty \quad ; \quad g_n(1) = 1 \quad ; c_n := a_n/b_n \in \{|\zeta| = 1\} \quad ;$$

$$|g_n(c_n)| \geq \min\{n, k_n/2\} \quad ; \quad g_n|\bar{B}(0, 1/\varepsilon_n) \text{ is a homeomorphism.} \tag{4.1.4}$$

Moreover, (4.1.3) yields the relations

$$g_n(\bar{B}(0,1/\varepsilon_n) \setminus B(0,1)) \subset \bar{\mathbf{C}} \setminus B(0,1) \quad ; \quad g_n(\bar{B}(0,1)) \subset \bar{B}(0, |g_n(c_n)|) . \tag{4.1.5}$$

Since $\bar{B}(0,2) \subset \bar{B}(0, 1/\varepsilon_n)$ for all $n$, every $g_n$ maps $B(0,2)$ onto the domain $g_n(B(0,2))$ which contains the points 0, 1 and $g_n(c_n)$. The set $\{\zeta : |\zeta| \geq |g_n(c_n)|\} \cap g_n(\bar{B}(0,2))$ contains some non-degenerate continuum $l_n$ joining the points $g_n(c_n)$

and boundary of the domain $g_n(B(0,2))$. Consequently, its inverse image $g_n^{-1}(l_n)$ is a non-degenerate continuum in $\bar{B}(0,2)$ joining the point $c_n$ and the circle $\partial B(0,2)$. Therefore, it contains the continuum $\gamma_n \subset g_n^{-1}(l_n)$ which is outside of $B(0,1)$ and joins the circles $\{|\zeta| = 1\}$ and $\{|\zeta| = 2\}$. So we have the following inequality for every $\zeta \in \gamma_n$

$$|g_n(\zeta)| \geq |g_n(c_n)| \geq \min\{n, k_n/2\} \ . \qquad (4.1.6)$$

Passing, if necessary, to a subsequence in $\{g_n\}$, we may assume that there is topological convergence $\gamma_n \to \gamma$ and $\Gamma g_n \to \Gamma$ as $n \to \infty$, where $\gamma$ is a continuum in the ring $\{1 \leq |\zeta| \leq 2\}$ which joins the boundary circles of this ring. In view of $\mathcal{D}$-normality of the family $\mathcal{F}$, the graphical limit $\Gamma$ must be of the form $\Gamma = (Z \times \bar{\mathbf{C}}) \cup \Gamma f_0$, where either $f_0 \in \mathcal{D}$, or $f_0 \equiv \mathit{const}$ on $\bar{\mathbf{C}}$.

Applying (4.1.5), the relation $g_n(K) \cap B(0,1) = \emptyset$ is valid for each compact set $K$ in the ring $G = \{1 < |\zeta| < 2\}$. Thus, for every point $z_0 \in G$ the set $\Gamma \cap (\{z_0\} \times \bar{\mathbf{C}})$ does not meet $\{z_0\} \times \bar{B}(0,1/2)$. Hence $Z \cap G = \emptyset$ and $\Gamma \cap (G \times \bar{\mathbf{C}}) = \Gamma(f_0|G)$, i.e. the set $\Gamma$ over the ring $G$ coincides with the graph of the mapping $f_0$, and the convergence $g_n \to f_0$ is uniform on the compact sets in $G$.

Thus, for every point $z_0 \in \gamma \cap G$ there is a sequence of points $\zeta_n \in \gamma_n$ converging to $z_0$. It follows from continuous convergence $g_n \to f_0$ in $G$ (which is equivalent to the uniform convergence on the compact subsets) that $f_0(z_0) = \lim_{n\to\infty} g_n(z_n)$, and the estimate (4.1.6) yields the equality $f_0(z_0) = \infty$. Thus, $f_0 \equiv \infty$ on the continuum $\gamma$. Consequently, $f_0 \notin \mathcal{D}$, and therefore $f_0 \equiv \infty$ on $\bar{\mathbf{C}}$.

Next, consider a straight segment $L$ with the endpoints 0 and 1. The mapping $g_n$ is homeomorphism in the disk $\bar{B}(0,1/\varepsilon_n)$ (see (4.1.4)), so each set $\sigma_n = g_n^{-1}(L) \cap \bar{B}(0,1/\varepsilon_n)$ is a continuum in $\bar{B}(0,1)$ joining the points 0 and 1. Moreover,

$$g_n(\sigma_n) = L \subset \bar{B}(0,1) \ . \qquad (4.1.7)$$

Passing, if necessary, to a subsequence, we may assume that there is topological convergence $\sigma_n \to \sigma_0$ as $n \to \infty$, where $\sigma_0$ is non-degenerate continuum in the disk $\bar{B}(0,1)$ joining the points 0 and 1. Then the set $\sigma_0 \cap B(0,1)$ contains some non-degenerate continuum and, in view of the set $Z$ being zero-dimensional, contains some point $p \in \sigma_0 \setminus (Z \cup \{0\})$. The set $Z \cup \{0\}$ is closed, hence there is an open neighborhood $W$ of the point $p$ such that $\bar{W} \subset B(0,1) \setminus Z$. Then the set $\Gamma$ over $\bar{W}$ must coincide with the graph of the mapping $f_0$. It means the uniform convergence $g_n \to \infty$ on $\bar{W}$ (in chordal metric). But then, for all sufficiently large $n$, there exists a point $p_n \in \bar{W} \cap \sigma_n$ such that $|g_n(p_n)| > 2$, and it contradicts (4.1.7).

Lemma 2 is proven.

*Proof of Theorem 1.* By the corollary 1, for every $f \in \mathcal{F}$ the set $B_f$ of branch points is finite. Lemma 2 shows that in the representation $f = \varphi \circ \Phi$ a homeomorphism $\Phi : \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ is $K$-quasiconformal in the domain $\bar{\mathbf{C}} \setminus B_f$. By the fact of removability of one-point singularity for quasiconformal mapping (see, for example [10], Theorem 17.3, p. 52), a homeomorphism $\Phi$ is $K$-quasiconformal on the whole sphere $\bar{\mathbf{C}}$, the constant $K$ not depending on $f \in \mathcal{F}$. This completes the proof of Theorem 1.

**5.** As the conclusion, we shall show the $\mathcal{D}$-normality of the family $\mathcal{F}(p,K)$ of all $K$-quasimeromorphic mappings $f : \bar{\mathbf{C}} \to \bar{\mathbf{C}}$, such that for every $w \in \bar{\mathbf{C}}$ the number of points in the inverse image $f^{-1}(w)$ is less than or equal to $p$.

We remind that a *quasimeromorphic* mapping ($\equiv$ mapping with bounded distortion) $f$ of the Riemann sphere $\bar{\mathbf{C}}$ onto itself is open, and the inverse image $f^{-1}(w)$ of every point $w$ consists of finite number of points, so that $f \in \mathcal{D}$. According to Stoilov theorem, every quasimeromorphic mapping $f$ of the Riemann sphere onto itself may be represented as the composition $f = \varphi \circ \Phi$ where $\Phi : \bar{\mathbf{C}} \to \bar{\mathbf{C}}$ is a quasiconformal homeomorphism, and $\varphi(z)$ is a non-constant meromorphic function on $\bar{\mathbf{C}}$. Applying the Möbius transformation $\mu$, which maps the points $\Phi(0)$, $\Phi(1)$, $\Phi(\infty)$ into the points 0, 1, $\infty$, respectively, we may get a representation $f = (\varphi \circ \mu^{-1}) \circ (\mu \circ \Phi) = \varphi_0 \circ \Phi_0$, where the quasiconformal homeomorphism $\Phi_0$ has the fixed points 0, 1, $\infty$ and $\varphi_0(z)$ is a non-constant meromorphic function on $\bar{\mathbf{C}}$.

In proof of the following lemma we apply the classical Montel theorem (see, for example, [9], Chapter 2, §7, Theorem 1, p. 68). Here is the rewording of this theorem for our case of meromorphic functions: *Given a domain $G \subset \bar{\mathbf{C}}$, let $\mathcal{M}$ be a family of meromorphic functions in $G$ which do not attain three fixed values from $\bar{\mathbf{C}}$. Then every sequence $\{f_n\} \subset \mathcal{M}$ contains a subsequence which uniformly converges (in chordal metric) on the compact sets in $G$ to either a constant, or meromorphic function.*

**Lemma 3.** *The family $\mathcal{F} = \mathcal{F}(p, 1)$ of all mappings realized by meromorphic functions on $\bar{\mathbf{C}}$ with multiplicity $\leq p$, is $\mathcal{D}$-normal.*

*Proof.* Consider an arbitrary sequence $\{f_n\} \subset \mathcal{F}$ which has a graphical limit $\Gamma = \lim_{n\to\infty} \Gamma f_n$. Assume that the set $\Gamma$ over a point $z_0$ does not coincide with $\bar{\mathbf{C}}$. Since $\Gamma$ is closed in the space $\bar{\mathbf{C}} \times \bar{\mathbf{C}}$, there exist a neighborhood $G$ of a point $z_0$ and a non-empty open set $W \subset \bar{\mathbf{C}}$ such that $G \times W$ does not meet $\Gamma$. Taking three distinct points in $W$, one obtains a sequence $\{f_n|G\}$ of the meromorphic functions which do not attain three fixed values. Then, by Montel theorem, the set $\Gamma$ over $G$ coincides with a graph of some function $f_0$, which is either a constant, or meromorphic function.

Hence, for every point $z_0$ we have either $\{z_0\} \times \bar{\mathbf{C}} \subset \Gamma$ (such point $z_0$ is called to be *the point of singularity* for the sequence $\{f_n\}$), or the graphical limit $\Gamma$ over some open neighborhood of $z_0$ coincides with a graph of meromorphic function.

Suppose that the set $Z$ of all points of singularity contains $m$ pairwise distinct points $a_1, a_2, ..., a_m$. Construct pairwise disjoint neighborhoods $G_1, G_2, ..., G_m$ of these points and fix three-point set $T = \{0, 1, \infty\}$. Then for given neighborhood $G_1$ of the point $a_1$ there are a point $b_1 \in T$ and a subsequence $\{f_{n1}\} \subset \{f_n\}$ such that $f_{n1}^{-1}(b_1) \cap G_1 \neq \emptyset$ for all $n1$. By fact that passing to a subsequence has no influence on the graphical limit, there is a subsequence $\{f_{n2}\}$ in $\{f_{n1}\}$ such that $f_{n2}^{-1}(b_2) \cap G_2 \neq \emptyset$ for all $n2$ and some $b_2 \in T$. Continuing the construction, we finally get a subsequence $\{g_k\} \subset \{f_n\}$ such that $g_k^{-1}(b_j) \cap G_j \neq \emptyset$ for all $k$ and $j = 1, ..., m$. In the set $\{b_1, ..., b_m\}$ each point of $\{0, 1, \infty\}$ can be met not more than $p$ times (recall that each mapping $g_k$ has multiplicity $\leq p$). Hence, $m \leq 3p$.

Consequently, the set $Z$ consists of finite number of points. Then the graphical limit $\Gamma$ over the domain $D = \bar{\mathbf{C}} \setminus Z$ coincides with a graph of function $f_0 : D \to \bar{\mathbf{C}}$ which is either a constant, or meromorphic function with multiplicity $\leq p$. Moreover, each $z_0 \in Z$ is the singularity point for the analytic function $f_0$. According to Picard theorem, and since $f_0$ has multiplicity $\leq p$, the point $z_0 \in Z$ cannot be essential singularity for the function $f_0$. Thus, there is a limit $f_0(z_0) := \lim_{z\to z_0} f_0(z)$, and the function $f_0(z)$ has an extension to a meromorphic function on the whole Riemann sphere $\bar{\mathbf{C}}$.

Therefore, $\Gamma = (Z \times \bar{\mathbf{C}}) \cup \Gamma f_0$, where $Z$ is finite set, and the mapping $f_0$ is either a constant or a meromorpic function on $\bar{\mathbf{C}}$.

In view of the arbitrariness of the graphically converging sequence $\{f_n\} \subset \mathcal{F}$ to choose, this means $\mathcal{D}$-normality of the family $\{F\}$. The lemma is proven.

**Theorem 3.** *The family $\mathcal{F} = \mathcal{F}(p, K)$ of all $K$-quasimeromorphic mappings of the Riemann sphere $\bar{\mathbf{C}}$ onto itself with multiplicity $\leq p$ is $\mathcal{D}$-normal family.*

*Proof.* Let us given the graphically converging sequence $\{f_n\} \subset \mathcal{F}$ with the graphical limit $\Gamma$. As was noticed above, every mapping $f_n$ can be represented as the composition $f_n = \varphi_n \circ \Phi_n$, where $\Phi_n$ is a $K$-quasiconformal automorphism of the sphere $\bar{\mathbf{C}}$ with the fixed points $0, 1, \infty$, and $\varphi_n$ is a non-constant meromorphic function on $\bar{\mathbf{C}}$ with multiplicity $\leq p$.

By the fact of the well known compactness principle for quasiconformal mappings (see, for example, [10], Corollary 19.5, p. 67; Theorem 20.5, p. 69), there exists a subsequence $\{\Phi_{n.k}\} \subset \{\Phi_n\}$ uniformly converging (in chordal metric) on $\bar{\mathbf{C}}$ to a $K$-quasiconformal homeomorphism $\Phi_0 : \bar{\mathbf{C}} \to \bar{\mathbf{C}}$. Then the sequence of mappings $\{\varphi_{n.k} = f_{n.k} \circ \Phi_{n.k}^{-1}\}$ is $\mathcal{D}$-normal family by Lemma 3. Passing to a subsequence, if necessary, we may assume that it has a graphical limit $\Gamma' = \lim_{k\to\infty} \Gamma\varphi_{n.k}$ of the form $\Gamma' = (Z' \times \bar{\mathbf{C}}) \cup \Gamma\varphi_0$, where $Z'$ is a finite set and $\varphi_0$ is either a constant, or some meromorphic function on $\bar{\mathbf{C}}$ with multiplicity $\leq p$. Hence, by the fact of uniform convergence $\Phi_{n.k} \to \Phi_0$ on $\bar{\mathbf{C}}$, we have the equality $\Gamma = \Theta(\Gamma')$ where the mapping $\Theta$ in the space $\bar{\mathbf{C}} \times \bar{\mathbf{C}}$ is given by the formula $\Theta(z, w) = (\Phi_0(z), w)$. Consequently,

$$\Gamma = (\Phi_0(Z') \times \bar{\mathbf{C}}) \cup \Gamma(\varphi_0 \circ \Phi_0) \ .$$

The set $Z = \Phi_0(Z')$ does not contain continua, and the function $f_0 = \psi_0 \circ \Phi_0$ is either a constant, or belongs to the class $\mathcal{D}$.

In view of the arbitrariness of graphically convergent sequence $\{f_n\} \subset \mathcal{F}$, the family $\mathcal{F}$ is $\mathcal{D}$-normal. The theorem is proven.

Given a homeomorphism $f : \bar{\mathbf{C}} \to \bar{\mathbf{C}}$, the family $M[f]$ consisting of compositions $\nu \circ f \circ \mu$ with all possible Möbius transformations of the sphere $\bar{\mathbf{C}}$, is called by *Möbius envelope* of the mapping $f$. From Theorems 1 and 3, we immediately obtain

**Corollary 2.** *The light continuous mapping $f$ of the Riemann sphere onto itself is quasimeromorphic if and only if its Möbius envelope $M[f]$ is $\mathcal{D}$-normal family.*

**6.** The condition in Theorem 3 of bounded multiplicity $\leq p$ for all mappings under consideration is essential one. For example, we can take an analytic function $\Lambda(z)$ in the disk $B(0, 1)$ which does not have an extension outside of this disk. Using the Runge's theorem, construct the sequence of the polynomials $P_n(z)$ which uniformly converges to the function $\Lambda(z)$ on the compact sets in $B(0, 1)$ . Then for every graphically converging subsequence $\{P_n\}$ the set $Z$ of all point of singularity contains the whole circle $\partial B(0, 1)$. This means that the set of all polynomials on $\bar{\mathbf{C}}$ is not $\mathcal{D}$-normal family.

In this connection two following questions may be set:

(1) Can one state in conditions of Theorem 1 that all mappings from a $\mathcal{D}$-normal family $\mathcal{F}$ have the multiplicity $\leq p$ with some common $p$?

(2) Can one state that for every graphically converging sequence from a $\mathcal{D}$-normal family given by Theorem 1, the set $Z$ of singularity consists of finite number of points?

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