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ON REALIZABILITY OF A GRAPH AS THE PRIME GRAPH OF A FINITE GROUP

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Abstract. The problem of the realizability of a graph as the prime graph (the Gruenberg–Kegel graph) of a finite group is considered. This problem is completely solved for graphs with at most five vertices.

Keywords: finite group, prime graph (Gruenberg–Kegel graph), realizability of a graph.

1. Introduction

In the finite group theory many researchers are interested in various problems of the study of finite groups by their arithmetical properties. One of such problems is the problem of the study of a finite group by its prime graph.

We use the term "group" while meaning "finite group".

Let G be a group. Denote by $\pi(G)$ the set of all prime divisors of the order of G. If $|\pi(G)| = n$ then G is called *n-primary*. Denote by $\omega(G)$ the *spectrum* of the group G, i.e., the set of its element orders. The set $\omega(G)$ defines the *prime graph* (or the *Gruenberg–Kegel graph*) $\Gamma(G)$ of the group G; in this graph, the vertex set is $\pi(G)$ and two different vertices p and q are adjacent if and only if $pq \in \omega(G)$. Denote the number of connected components of $\Gamma(G)$ by $s(G)$ and the

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set of connected components of $\Gamma(G)$ by $\{\pi_i(G) \mid 1 \geq i \leq s(G)\}$; for a group G of even order, we assume that $2 \in \pi_1(G)$.

Denote by $F(m, n)$ a Frobenius group with the kernel of order m and a complement of order n, where m and n are coprime.

For a prime p and a positive integer n, by p^n denote an elementary abelian p-group of the order p^n. By $\mathbb{C}_n$ denote a cyclic group of order n.

We need some definitions from the graph theory. Through the paper the term "graph" always means an undirected graph without loops and multiple edges.

A *clique* (resp. *coclique*) is a graph in which all vertices are pairwise adjacent (resp. nonadjacent). A clique (resp. coclique) with n vertices is called n-*clique* (resp. n-*coclique).*

The prime graph of a group G can be considered as a graph with $|\pi(G)|$ vertices, where the vertices are marked by different primes and two different vertices p and q are adjacent if and only if there is an element $g \in G$ such that the order of g is pq. The following problem arises by means of such prime graph consideration.

Problem 1. *Let Γ be a graph with vertices marked by different primes. Is there exist a group such that Γ is its prime graph*?

We say that a graph Γ *is realizable as the prime graph of a group* if there exists a marking of the vertices of Γ by different primes such that Γ becomes the prime graph of an appropriate group.

As a generalization of Problem 1, we obtain the following

Problem 2. *Let Γ be a graph. Is Γ realizable as the prime graph of a group*?

There are not many works devoted to these interesting problems. In unpublished graduate work of I. N. Zharkov [15], who was a student of V. D. Mazurov, it was proved that a chain is realizable as the prime graph of a group if and only if its length is at most 4.

The same problems were considered by H.P. Tong-Viet [11] for the graph $\Delta(G)$ whose vertex set is the set all primes dividing irreducible character degrees of a finite group G and two distinct vertices p and q are adjacent if and only if the product pq divides some irreducible character degree of G.

In general, both problems have negative solutions, we prove it in Section 3. So, it is interesting to consider cases when the problems have positive solution. In the present paper, we solve completely Problem 2 for graphs with at most five vertices.

The main result of the paper is the following

Theorem. *Let Γ be a graph with at most five vertices. Then*

(1) *If Γ is* 5*-coclique then Γ is not realizable as the prime graph of a finite group.*

(2) *If Γ is not* 5*-coclique then Γ is realizable as the prime graph of a finite group.*

2. Auxiliary Results

Our notation and terminology are mostly standard and can be found in [1, 2, 5, 6]. In particular, if A and B are groups, p is a prime and n is a natural number then we use the following notation: $\mathbb{C}_n$ (or simply n) for a cyclic group of order n, p^n for the elementary abelian p-group of order p^n, $A.B$ for an extension of A by B, $A : B$ ($A \leftthreetimes B$) for a split extension (semi-direct product) of A by (with, on) B.

Let Γ be always a graph. The vertex set and the edge set of the graph Γ are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively.

We give some results that will be used in the proof of the theorem.

Lemma 1 (Gruenberg–Kegel theorem [13, Theorem A]). *If G is a finite group with disconnected prime graph, then one of the following statements holds*:

(1) *G is a Frobenius group*;

(2) *G is a 2-Frobenius group*;

(3) *G is an extension of a nilpotent $\pi_1(G)$-group by a group A, where $Inn(P) \leq A \leq \mathrm{Aut}(P)$, P is a simple nonabelian group with $s(G) \leq s(P)$, and $A/Inn(P)$ is a $\pi_1(G)$-group.*

The following two lemmas provide the structure of the prime graphs of Frobenius groups and 2-Frobenius groups.

Lemma 2 ([14]). *Let G be a Frobenius group or 2-Frobenius group. Then $\Gamma(G)$ has exactly two connected components. If G is solvable, then each connected component of $\Gamma(G)$ is a clique.*

Lemma 3 ([16, Proposition 1]). *Let π_1 and π_2 be some non-intersecting non-empty sets of primes. Then there exists a solvable Frobenius group G such that π_1 and π_2 are connected components of $\Gamma(G)$.*

By Lemmas 2 and 3 the union of two non-intersecting cliques is realizable as the prime graph of a solvable Frobenius group.

The next lemma is obvious.

Lemma 4. *Let G_1 and G_2 be a finite groups such that $\pi(G_1) \cap \pi(G_2) = \emptyset$ and $G = G_1 \times G_2$. Then $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{\{u, v\} \mid u \in V(G_1), v \in V(G_2)\}$*

As a corollary of Gruenberg—Kegel theorem (Lemma 1) and of the description of connected components of prime graphs of finite simple groups obtained in [7, 13] we have the following

Lemma 5. *If G is a finite group then $s(G) \leq 6$.*

The following useful result is well known (see, for example, [3, Lemma 4]).

Lemma 6. *Suppose that G is a finite simple group, F is a field of characteristic $p > 0$, V is an absolutely irreducible FG-module, and β is a Brauer character of the module V. If g is an element of G of a prime order different from p, then*

$$dim\ C_V(g) = (\beta|_{\langle g \rangle}, 1|_{\langle g \rangle}) = \frac{1}{|g|} \sum_{x \in \langle g \rangle} \beta(x).$$

3. Negative solutions of Problems 1 and 2 in general case

It is easy to prove that Problems 1 and 2 have negative solutions in the general case.

Let's consider a 3-coclique Γ with $V(\Gamma) = \{p, q, r\}$, where p, q and r are odd primes. There are no groups with such prime graph. Assume the converse. Let G be a group with $\Gamma(G) = \Gamma$. Then by Feit-Thompson theorem G is solvable and by Gruenberg—Kegel theorem it is a Frobenius group or a 2-Frobenius group. Prime graphs of Frobenius groups and 2-Frobenius groups have exactly two connected components by Lemma 2. But $\Gamma(G)$ has three connected components. This contradiction proves that there is no group G such that $\Gamma(G) = \Gamma$. So, Problem 1 has negative solution in the general case.

Let's consider a graph Γ with at least 7 connected components. By Lemma 5, the graph Γ is not realizable as the prime graph of a group. So, the Problem 2 has negative solution in the general case.

4. Proof of Theorem

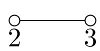
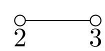
First of all, we prove that 5-coclique is not realizable as the prime graph of a group. By Lemma 1 and [7, 13], a group G with $s(G) \geq 5$ has a composition factor isomorphic to J_4 or $E_8(q)$ for $q \equiv 0, 1, 4(5)$, consequently, $|\pi(G)| > 5$.

To prove the rest of the theorem we must enumerate all graphs with at most 5 vertices (see, for example, [4, Appendix 1]) and find at least one realization for any such graph different from 5-clique.

The graph with one vertex is realizable as the prime graph of any p-group.

4.1. **Graphs with two vertices.** There are only two graphs with two vertices. Their realizations are listed in Table 1.

Table 1. Realizations of graphs with 2 vertices

Graph	Group G	Graph $\Gamma(G)$
	S_3	
	$\mathbb{C}_6$	

4.2. **Graphs with three vertices.** In [8], a description of 3-primary groups was obtained. From this description it is easy to see that the theorem holds for a graph with three vertices. Realizations for this case are listed in Table 2.

Table 2. Realizations of graphs with 3 vertices

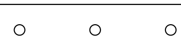
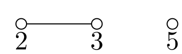
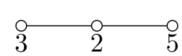
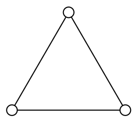
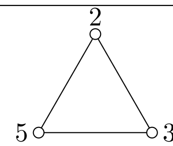
Graph	Group G	Graph $\Gamma(G)$
	A_5	
	S_6	
	$Aut(A_6)$	

Table 2 (continued).

Graph	Group G	Graph $\Gamma(G)$
	$\mathbb{C}_{30}$	

4.3. **Graphs with four vertices.** In [9], a description of 4-primary groups was obtained. From this description and Lemmas 2, 3 and 4 it is easy to see that the theorem holds for a graph with four vertices. Realizations for this case are listed in Table 3. The definition of the group G_1 provided after Table 3.

Table 3. Realizations of graphs with 4 vertices

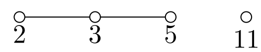
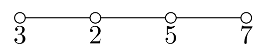
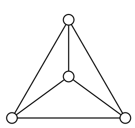
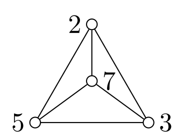
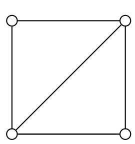
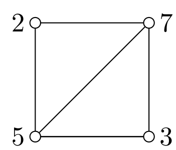
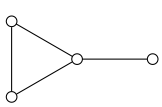
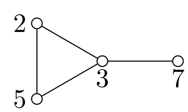
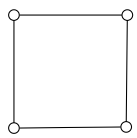
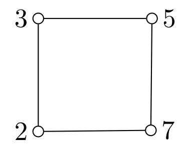
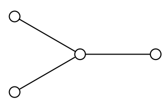
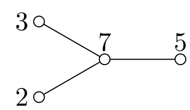
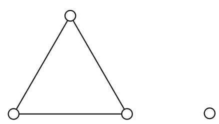
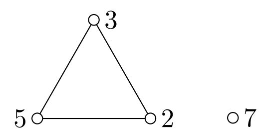
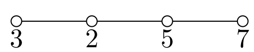
Graph	Group G	Graph $\Gamma(G)$
	$L_3(4)$	
	$L_2(16)$	
	$U_5(2)$	
	G_1	
	$\mathbb{C}_{210}$	
	$S_3 \times \mathbb{C}_{35}$	
	A_{10}	
	$F(31 \cdot 61, 15)$	

Table 3 (continued).

Graph	Group G	Graph $\Gamma(G)$
	$D_{10} \times F(7,3)$	
	$A_5 \times \mathbb{C}_7$	
	A_9	

A construction of the group G_1 can be found in [10, Example 3]. Consider $F = \langle c\rangle \times \langle d\rangle$, where $|c| = 5$, $|d| = 7$ and $H = \langle a\rangle \times \langle b\rangle$, where $|a| = 2$, $|b| = 3$, and a inverts F, b acts on F centralizing $O_5(F)$ and fixed point freely on $O_7(F)$. If we consider the natural semidirect product $G_1 = F \rtimes H$ then $\Gamma(G_1)$ is the following



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4.4. **Graphs with five vertices.** From [1], Lemmas 2, 3 and 4, [12, Propositions 2.1, 3.1, 4.1] and [5, Theorem 4.5.1, Proposition 4.9.1] we obtain the realizations of the most part of graphs with 5 vertices, beside graphs which are realizable as $\Gamma(G_i)$, where $2 \le i \le 7$. In Table 4 we provide the realizations of the graphs with 5-vertices. The definition of groups G_i $(2 \le i \le 7)$ is given after Table 4.

In Table 4 we denote by f an outer field automorphism of a group of Lie type.

Table 4. Realizations of graphs with 5 vertices

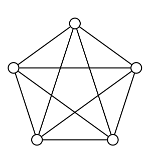
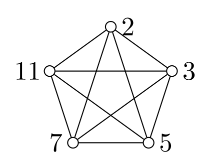
Graph	Group G	Graph $\Gamma(G)$
	$\mathbb{C}_{2310}$	
	M_{22}	

Table 4 (continued)

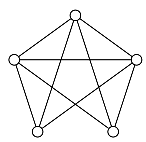
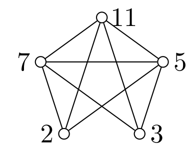
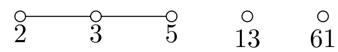
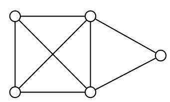
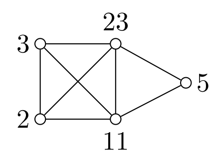
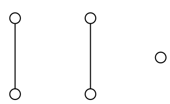
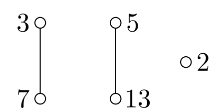
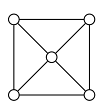
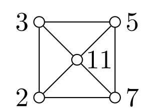
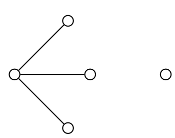
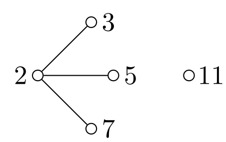
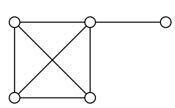
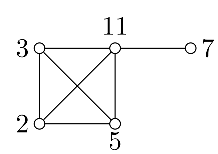
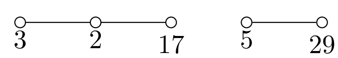
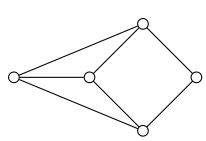
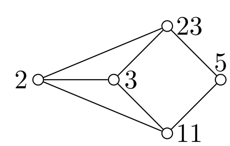
Graph	Group G	Graph $\Gamma(G)$
	$S_3 \times \mathbb{C}_{385}$	
	$Aut(Sz(8))$	
	$S_6 \times \mathbb{C}_{253}$	
	$L_2(64)$	
	$D_{10} \times \mathbb{C}_{11} \times F(7,3)$	
	$Aut(M_{22})$	
	$A_8 \times \mathbb{C}_{11}$	
	$L_2(17^2) : \langle f \rangle$	
	$S_6 \times F(23,11)$	

Table 4 (continued)

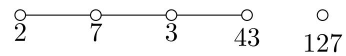
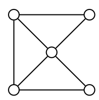
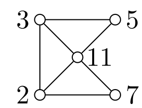
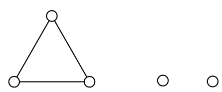
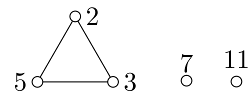
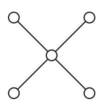
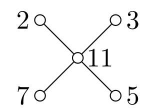
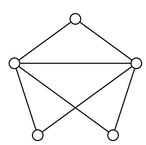
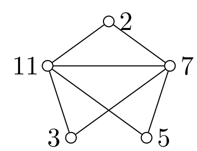
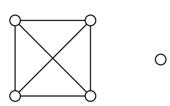
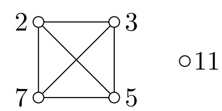
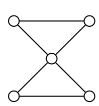
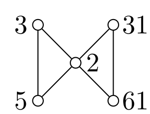
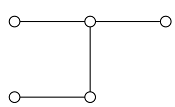
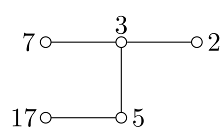
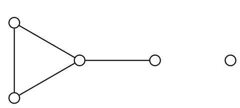
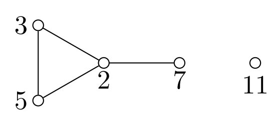
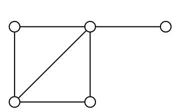
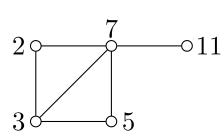
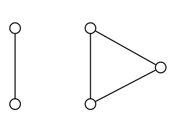
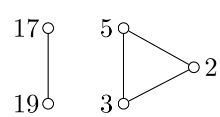
Graph	Group G	Graph $\Gamma(G)$
	$L_2(2^7):\langle f\rangle$	
	$G_1 \times \mathbb{C}_{11}$	
	$U_6(2)$	
	$L_3(4) \times \mathbb{C}_{11}$	
	$A_5 \times \mathbb{C}_{77}$	
	A_{12}	
	$\mathbb{C}_2 \times F(31 \cdot 61, 15)$	
	$L_3(3^2):\langle f\rangle$	
	McL	
	$U_5(2) \times \mathbb{C}_7$	
	$F(17 \cdot 19, 2 \cdot 3 \cdot 5)$	

Table 4 (continued)

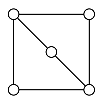
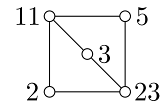
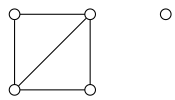
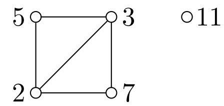
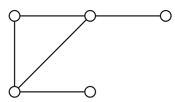
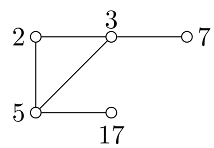
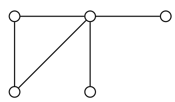
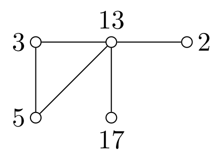
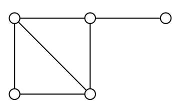
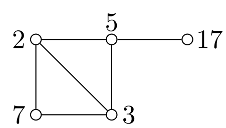
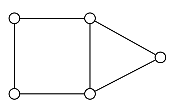
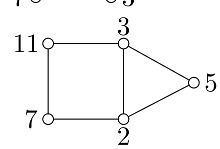
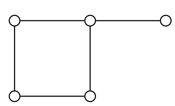
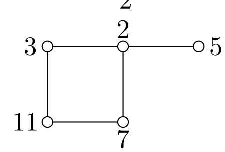
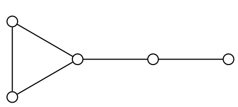
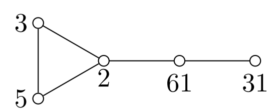
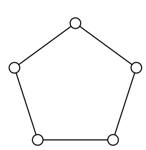
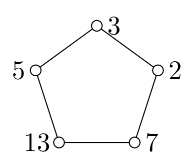
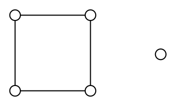
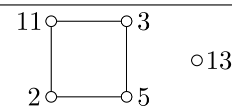
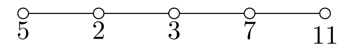
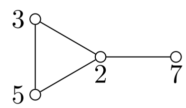
Graph	Group G	Graph $\Gamma(G)$
	$A_5 \times F(23,11)$	
	A_{11}	
	$L_4(2^2)$	
	$L_2(16) \times \mathbb{C}_{13}$	
	$L_4(2^2) : \langle f \rangle$	
	G_2	
	G_3	
	G_4	
	G_5	

Table 4 (continued)

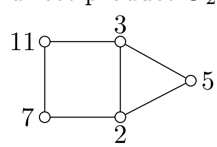
Graph	Group G	Graph $\Gamma(G)$
	G_6	
	G_7	

Let $H = 2^{\cdot}A_8$. Then $\Gamma(H)$ is the following



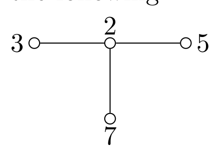
.

By [1], there exists a faithful irreducible 8-dimensional $GF(11)H$ module V. By Lemma 6 and [1], an element of order 5 from H acts on V fixed point freely, but some elements of orders 3 and 7 from H have non-trivial fixed points in V. So, the prime graph of the natural semidirect product $G_2 = V \leftthreetimes H$ is the following



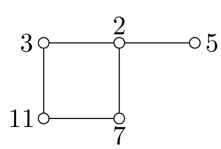
.

Let $H = 2^{\cdot}A_7$. Then $\Gamma(H)$ is the following



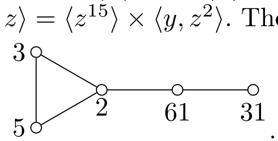
.

By [1], there exists a faithful 4-dimensional irreducible $GF(11)H$ module V. By Lemma 6 and [1], the prime graph of the natural semidirect product $G_3 = V \leftthreetimes H$ is the following



.

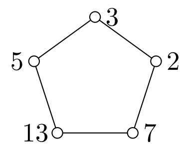
Let $G_4 = (\langle x\rangle \times \langle y\rangle) \leftthreetimes \langle z\rangle$, where $|x| = 31$, $|y| = 61$, $|z| = 30$, $\langle x, z\rangle$ and $\langle y, z^2\rangle$ are Frobenius groups and $\langle y, z\rangle = \langle z^{15}\rangle \times \langle y, z^2\rangle$. Then $\Gamma(G_4)$ is the following



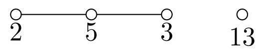
.

By [1], there exists a faithful 6-dimensional irreducible $GF(5)SL_2(7)$-module V. In $SL_2(7)$ there is a subgroup H isomorphic to $\mathbb{C}_2 \times F(7,3)$. Then by Lemma 6 and [1], elements of orders 2 and 7 from H act fixed point freely of V, and an element of order 3 from H has a non-trivial fixed point in V. Consider 1-dimensional

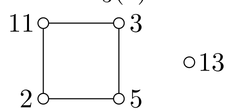
$GF(13)H$-module $W = \langle w \rangle$ such that $O_7(H)$ is the kernel of this module, and an element $g \in H$ of order 6 acts on W as $g(w) = 6w$. Then the prime graph of the natural semidirect product $G_5 = (V \times W) \leftthreetimes H$ is the following



Let $H = U_3(4)$. By [9] $\Gamma(H)$, is the following

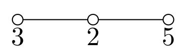


By [1], there exists a faithful irreducible 12-dimensional $GF(p)U_3(4)$-module V, where p is a prime and $p \notin \pi(H)$. By Lemma 6 and [1], we obtain that elements of orders 5 and 13 from H act fixed point freely on V, but some elements of orders 2 and 3 from H have non-trivial fixed points in V. Therefore, the prime graph of the natural semidirect product $G_6 = V \leftthreetimes U_3(4)$ is the following

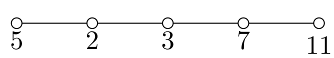


In [15], it was proved that the chain

is realizable as the prime graph of a group of a form $(11^2 \times 7^4) \leftthreetimes SL_2(5)$. We provide here an independent and shorter proof of this fact. The group $A = SL_2(5)$ has the following prime graph



It is well known by [1] that A embeds into $SL_2(11)$, therefore there exists a faithful irreducible 2-dimensional $GF(11)A$-module V. Also by Lemma 6 and [1], any non-trivial element from A acts on V fixed point freely. Moreover, there exists an embedding A into $SL_4(7)$. Therefore, there exists a faithful irreducible 4-dimensional $GF(7)A$-module W. By Lemma 6 and [1], elements of orders 2 and 5 from A act on W fixed point freely, and an element of order 3 from A has a non-trivial fixed point in W. Hence, the prime graph of the natural semidirect product $(V \times W) \leftthreetimes A$ is the following



The theorem is proved.

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