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MSC 35B27, 35D30, 74F10**SMALL PERTURBATIONS OF TWO-PHASE FLUID IN PORES:
EFFECTIVE MACROSCOPIC MONOPHASIC
VISCOELASTIC BEHAVIOR**

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ABSTRACT. The linearized model of joint motion of an elastic porous body and a two-phase viscous compressible liquid in pores is considered. The reciprocal deformation of liquid phases is governed by Rakhmatullin's scheme. It is assumed that the porous body has a periodic geometry and that the ratio of the pattern periodic cell and the diameter of the entire mechanical system is a small parameter in the model. The homogenization procedure, i.e. a limiting passage as the small parameter tends to zero, is fulfilled. As the result, we find that the limiting distributions of displacements of the media serve as a solution of a well-posed initial-boundary value problem for the model of linear monophasic viscoelastic material with memory of shape. Moreover, coefficients of this newly constructed model arise from microstructure, more precisely, they are uniquely defined by data in the original model. Homogenization procedure is based on the method of two-scale convergence and is mathematically rigorously justified.

Keywords: two-phase fluid in pores, homogenization of periodic structure, two-scale convergence, viscoelastic body.

1. INTRODUCTION

We consider a linearized isothermal model of joint motion of an elastic porous ground and a two-phase Newtonian viscous compressible liquid. Interaction between

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the two liquid phases is subjected to the condition that the pressure is the same in the both liquid phases at any point [1, Chapter 1, Section 1], [2]. This condition is called Rakhmatullin's scheme. The contact discontinuity on the boundary between solid and liquid phases obeys the classical Rankine–Hugoniot conditions and the condition of continuity of displacements. In the article, this model has the name *Problem A*. Earlier, its formulation was derived in the work [3] as the result of a systematical study of the most general nonlinear non-isothermal dynamical model of motion of a two-phase thermofluid filling in an elastic porous ground [1, 4, 5]. In [3] the well-posedness of Problem A was established in the class of generalized solutions and the basic estimates for solutions were obtained. In the present article, these results are outlined in Sections 2.1, 2.3, and 2.4.

We supplement the porous space with a periodical geometry (see in Section 2.2). The small parameter is introduced into consideration, which is the ratio between the minimal period of microstructure and the diameter of the whole porous body.

In Sections 3–5 the homogenization procedure is fulfilled, i.e. a limiting passage as the small parameter tends to zero is worked out. In our consideration, we suppose that the physical characteristics (i.e., coefficients in the model) of distinct phases do not depend on the small parameter. As the result we construct a monophasic model, which incorporates a system of integro-differential equations of dynamics of a viscoelastic body with the shape memory of previous mechanical states and a set of boundary and initial conditions. Homogenization is fulfilled by means of the Allaire–Nguetseng method [6, 7] on the rigorous mathematical level.

The present article can be regarded to as a continuation of the works [3, 8]. As well, we can notice that this our work belongs to the mainstream of the present theory of homogenization of periodic structures. This theory ascends to the proceedings of E. Sanchez-Palencia [9], R. Burridge and J.B. Keller [10], S.N. Bakhvalov and G.P. Panasenko [11], R.P. Gilbert and A. Mikelić [12], and A.M. Meirmanov [13].

2. THE ISOTHERMAL MODEL ON THE MICROSCOPIC LEVEL

2.1. Statement of the problem on the microscopic level. We consider a linearized isothermal model of joint motion of an elastic porous ground and a two-phase fluid. The fluid fills in the whole porous space. We suppose that the mechanical interaction between fluid phases obeys Rakhmatullin's scheme. The contact discontinuity between the liquid and the elastic components is subjected to the standard Rankine–Hugoniot conditions and to the conditions of continuity of displacements. The well-posedness of this model in the class of weak generalized solutions was proved in [3].

Problem A. (The linearized isothermal model of joint motion of a viscous porous ground and a two-phase Newtonian viscous compressible fluid.) In a space-time domain $Q = \Omega \times (0, T)$, where $T = \text{const} > 0$ is a fixed time moment and Ω is the unit cube in \mathbb{R}^3 (that is, $\Omega = (0, 1)^3$), which consists of two disjoint components Ω_s and Ω_f , and the immovable interface $\Gamma_0 = \partial\Omega_f \cap \partial\Omega_s$ between them, it is necessary to find the following:

the displacement fields \mathbf{w}_1 and \mathbf{w}_2 in the liquid phases, and the displacement field \mathbf{w} in the elastic component, satisfying the following equations in Ω_f and Ω_s and relations on Γ_0 :

the Stokes equations in the liquid phases:

$$(2.1a) \quad \rho_{if}^0 m_i \frac{\partial^2 \mathbf{w}_i}{\partial t^2} = c_{\rho i} \rho_{if}^0 \alpha_i^0 \nabla_x (m_1 \operatorname{div}_x \mathbf{w}_1 + m_2 \operatorname{div}_x \mathbf{w}_2) \\ + \operatorname{div}_x \left[m_i \left(\nu_i - \frac{2}{3} \mu_i \right) \left(\operatorname{div}_x \frac{\partial \mathbf{w}_i}{\partial t} \right) \mathbb{I} + 2m_i \mu_i \mathbb{D} \left(x, \frac{\partial \mathbf{w}_i}{\partial t} \right) \right] \\ + \rho_{if}^0 m_i \mathbf{g} + (-1)^i \mathbf{F}_{12}, \quad (\mathbf{x}, t) \in \Omega_f \times (0, T), \quad i = 1, 2,$$

where the force of interface interaction is defined by the law

$$(2.1b) \quad \mathbf{F}_{12} = K_F (\partial(\mathbf{w}_1 - \mathbf{w}_2) / \partial t);$$

Lamé's equations in the solid elastic phase:

$$(2.1c) \quad \rho_s \frac{\partial^2 \mathbf{w}}{\partial t^2} = \left(\eta - \frac{2}{3} \lambda \right) \nabla_x \operatorname{div}_x \mathbf{w} + \operatorname{div}_x (2\lambda \mathbb{D}(x, \mathbf{w})) + \rho_s \mathbf{g}, \quad (\mathbf{x}, t) \in \Omega_s \times (0, T);$$

the conditions of continuity of displacements on Γ_0 :

$$(2.1d) \quad \mathbf{w}_1 = \mathbf{w}, \quad \mathbf{w}_2 = \mathbf{w}, \quad (\mathbf{x}, t) \in \Gamma_0 \times (0, T);$$

and the standard Rankine–Hugoniot conditions of continuity of stress on Γ_0 :

$$(2.1e) \quad (c_{\rho 1} \rho_{1f}^0 \alpha_1^0 + c_{\rho 2} \rho_{2f}^0 \alpha_2^0) (m_1 \operatorname{div}_x \mathbf{w}_1 + m_2 \operatorname{div}_x \mathbf{w}_2) \mathbf{n} \\ + m_1 \left(\nu_1 - \frac{2}{3} \mu_1 \right) \left(\operatorname{div}_x \frac{\partial \mathbf{w}_1}{\partial t} \right) \mathbf{n} + m_2 \left(\nu_2 - \frac{2}{3} \mu_2 \right) \left(\operatorname{div}_x \frac{\partial \mathbf{w}_2}{\partial t} \right) \mathbf{n} \\ + 2m_1 \mu_1 \mathbb{D} \left(x, \frac{\partial \mathbf{w}_1}{\partial t} \right) \mathbf{n} + 2m_2 \mu_2 \mathbb{D} \left(x, \frac{\partial \mathbf{w}_2}{\partial t} \right) \mathbf{n} \\ = \left(\eta - \frac{2}{3} \lambda \right) (\operatorname{div}_x \mathbf{w}) \mathbf{n} + 2\lambda \mathbb{D}(x, \mathbf{w}) \mathbf{n}, \quad (\mathbf{x}, t) \in \Gamma_0 \times (0, T).$$

The model is endowed by the homogeneous conditions on $\partial\Omega \times (0, T)$:

$$(2.1f) \quad \mathbf{w}_1 = \mathbf{w}_2 = 0 \quad \text{on } (\partial\Omega_f \cap \partial\Omega) \times (0, T), \quad \mathbf{w} = 0 \quad \text{on } (\partial\Omega_s \cap \partial\Omega) \times (0, T)$$

and initial data

$$(2.1g) \quad \mathbf{w}_1|_{t=0} = \mathbf{w}_1^0, \quad \mathbf{w}_2|_{t=0} = \mathbf{w}_2^0 \quad \text{for } \mathbf{x} \in \Omega_f, \quad \mathbf{w}|_{t=0} = \mathbf{w}^0 \quad \text{for } \mathbf{x} \in \Omega_s,$$

$$(2.1h) \quad \frac{\partial \mathbf{w}_1}{\partial t} \Big|_{t=0} = \mathbf{v}_1^0, \quad \frac{\partial \mathbf{w}_2}{\partial t} \Big|_{t=0} = \mathbf{v}_2^0 \quad \text{for } \mathbf{x} \in \Omega_f, \quad \frac{\partial \mathbf{w}}{\partial t} \Big|_{t=0} = \mathbf{v}^0 \quad \text{for } \mathbf{x} \in \Omega_s.$$

The given functions satisfy the continuity and Rankine–Hugoniot conditions (2.1d) and (2.1e) at the initial time moment $t = 0$. They also meet the following regularity requirements:

$$(2.1i) \quad \mathbf{w}_1^0, \mathbf{w}_2^0 \in H^1(\Omega_f), \quad \mathbf{w}^0 \in H^1(\Omega_s), \\ \mathbf{v}_1^0, \mathbf{v}_2^0 \in L^2(\Omega_f), \quad \mathbf{v}^0 \in L^2(\Omega_s), \quad \mathbf{g} \in L^2(\Omega \times (0, T)).$$

Remark 1. In the above formulation of Problem A there are merely three sought vector-functions \mathbf{w}_1 , \mathbf{w}_2 , and \mathbf{w} . From the physical viewpoint we can add also the distributions of volumetric saturations of the liquid phases α_i , the genuine densities of the liquid phases ρ_i^0 ($i = 1, 2$), and the pressure p to the set of the sought

functions. According to [3, formulas (5.1e)–(5.1h)], the values of α_i , ρ_i^0 , and p are expressed explicitly in terms of values of \mathbf{w}_1 , \mathbf{w}_2 , and \mathbf{w} by the formulas:

$$(2.1j) \quad \alpha_1 = \frac{m_1 m_2}{c_{\rho 1} \rho_{1f}^0 m_2 + c_{\rho 2} \rho_{2f}^0 m_1} \left[c_{\rho 1} \rho_{1f}^0 (1 - \operatorname{div}_x \mathbf{w}_1) - c_{\rho 2} \rho_{2f}^0 (1 - \operatorname{div}_x \mathbf{w}_2) + \frac{c_{\rho 2} \rho_{2f}^0}{m_2} \right], \quad (\mathbf{x}, t) \in \Omega_f \times (0, T),$$

$$(2.1k) \quad \alpha_2 = \frac{m_1 m_2}{c_{\rho 1} \rho_{1f}^0 m_2 + c_{\rho 2} \rho_{2f}^0 m_1} \left[c_{\rho 2} \rho_{2f}^0 (1 - \operatorname{div}_x \mathbf{w}_2) - c_{\rho 1} \rho_{1f}^0 (1 - \operatorname{div}_x \mathbf{w}_1) + \frac{c_{\rho 1} \rho_{1f}^0}{m_1} \right], \quad (\mathbf{x}, t) \in \Omega_f \times (0, T),$$

(remark that $\alpha_1 + \alpha_2 = 1$),

$$(2.1l) \quad \rho_1^0 = \rho_{1f}^0 \left[1 - \frac{\alpha_1^0}{m_1} (m_1 \operatorname{div}_x \mathbf{w}_1 + m_2 \operatorname{div}_x \mathbf{w}_2) \right], \quad (\mathbf{x}, t) \in \Omega_f \times (0, T),$$

$$(2.1m) \quad \rho_2^0 = \rho_{2f}^0 \left[1 - \frac{\alpha_2^0}{m_2} (m_1 \operatorname{div}_x \mathbf{w}_1 + m_2 \operatorname{div}_x \mathbf{w}_2) \right], \quad (\mathbf{x}, t) \in \Omega_f \times (0, T),$$

$$(2.1n) \quad p = p_* - c_{\rho 1} \rho_{1f}^0 \frac{\alpha_1^0}{m_1} (m_1 \operatorname{div}_x \mathbf{w}_1 + m_2 \operatorname{div}_x \mathbf{w}_2), \quad (\mathbf{x}, t) \in \Omega_f \times (0, T).$$

In (2.1a), (2.1c), (2.1e), and further in the article by $\mathbb{D}(x, \boldsymbol{\varphi})$ we denote the symmetric part of the gradient of any rather smooth vector-function $\boldsymbol{\varphi}(\mathbf{x})$:

$$\mathbb{D}_{ij}(x, \boldsymbol{\varphi}) = (1/2)(\partial_{x_i} \varphi_j + \partial_{x_j} \varphi_i), \quad i, j = 1, 2, 3,$$

and by $\mathbb{D}(y, \boldsymbol{\varphi})$ we denote the symmetric part of the gradient of any rather smooth vector-function $\boldsymbol{\varphi}(\mathbf{y})$:

$$\mathbb{D}_{ij}(y, \boldsymbol{\varphi}) = (1/2)(\partial_{y_i} \varphi_j + \partial_{y_j} \varphi_i), \quad i, j = 1, 2, 3.$$

Functions of \mathbf{y} will appear as a result of homogenization procedure: \mathbf{y} is the microscopic (fast) variable. By \mathbb{I} we denote the identity map in \mathbb{R}^3 : $\mathbb{I} = (\delta_{ij})$, where δ_{ij} is Kronecker's symbol.

The coefficients α_1^0 , α_2^0 , \mathbf{g} , ν_1 , ν_2 , μ_1 , μ_2 , K_F , ρ_s , η , λ , p_* , m_1 , m_2 , ρ_{1f}^0 , ρ_{2f}^0 , $c_{\rho 1}$, and $c_{\rho 2}$ are constant. They do not depend on the small parameter ε that was mentioned in Section 1 (Introduction). Here α_1^0 and α_2^0 are volumetric saturations of liquid phases, \mathbf{g} is the density of distributed mass forces, namely, the acceleration of free fall, ν_1 and ν_2 are the bulk viscosity coefficients in fluid phases, μ_1 and μ_2 are the shear viscosity coefficients in the fluid phases, η and λ are the bulk and shear elastic coefficients in the solid phases, respectively; ρ_s is the mean constant density of the solid elastic phase at the natural rest state, p_* is the mean constant pressure at the natural rest state in the solid phase, m_1 and m_2 are the mean constant values of density in the solid phase. The coefficients ρ_{1f}^0 and ρ_{2f}^0 are the genuine mean densities of the corresponding liquid phases, $c_{\rho 1}$ and $c_{\rho 2}$ are the squares of the speed of sound. Vector \mathbf{n} is the unit normal to Γ_0 pointing into Ω_f . The conditions $\nu_1 - (2/3)\mu_1 > 0$, $\nu_2 - (2/3)\mu_2 > 0$, and $\eta - (2/3)\lambda > 0$ hold (in consistency with the thermodynamics foundations). By \mathbf{v}_1 and \mathbf{v}_2 we denote velocities in the liquid phases and by K_F we denote the coefficient of interphase friction in the fluid component.

2.2. The periodical geometry of microstructure. Geometry of the domains Ω_s and Ω_f is given. It depends on a small parameter $\varepsilon > 0$. Therefore, further we denote $\Omega_s = \Omega_s^\varepsilon$ and $\Omega_f = \Omega_f^\varepsilon$. The domain Ω_f^ε is filled by a two-phase viscous compressible fluid and the domain Ω_s^ε is occupied by an elastic material. The parameter $\varepsilon > 0$ is the ratio of the characteristic sizes l_0 and L_0 of the micro- and macro-structures, respectively. Since Ω is the unit cube $(0, 1)^3$, we have $l_0 = \varepsilon$ and $L_0 = 1$.

We give the formal description of the geometry following [12]. Firstly, we postulate the structure inside the pattern cell $\mathcal{Y} = (0, 1)^3$. We suppose that *the solid phase* \mathcal{Y}_s is some open set in \mathcal{Y} and that *the liquid phase* \mathcal{Y}_f is the complement to the closure of \mathcal{Y}_s , i.e., $\mathcal{Y}_f = \mathcal{Y} \setminus \overline{\mathcal{Y}_s}$. Then we construct the periodic repetition of \mathcal{Y}_s over the whole space \mathbb{R}^3 . In this line we set $\mathcal{Y}_s^k = \mathcal{Y}_s + \mathbf{k}$ with all $\mathbf{k} \in \mathbb{R}^3$. Clearly, $E_s := \cup_{\mathbf{k} \in \mathbb{R}^3} \mathcal{Y}_s^k$ and $E_f := \mathbb{R}^3 \setminus \overline{E_s}$ are the open sets in \mathbb{R}^3 . We impose the following demands on \mathcal{Y}_s and E_s :

- \mathcal{Y}_s is a simply connected set of strictly positive Lebesgue measure with a Lipschitz boundary $\partial\mathcal{Y}_s$; the set \mathcal{Y}_f has a strictly positive Lebesgue measure in \mathcal{Y} , as well;
- E_f and E_s are open sets in \mathbb{R}^3 with a Lipschitz boundary between them; each of these two sets is locally situated on the one side of its boundary; E_s is a simply connected set.

Using this construction we introduce a regular ε -net covering Ω such that any cell $\mathcal{Y}_i^\varepsilon$ is a cube with the edge equal to ε . For simplicity, consider that $1/\varepsilon$ is a natural number. Each cube $\mathcal{Y}_i^\varepsilon$, $1/\varepsilon = 1, 2, 3, \dots, N$, is obtained from the pattern cell \mathcal{Y} by means of the linear homeomorphism Π_i^ε . This homeomorphism is the composition of a $1/\varepsilon$ -times compression and a delation. Now define $\mathcal{Y}_{s_i}^\varepsilon = \Pi_i^\varepsilon(\mathcal{Y}_s)$, $\mathcal{Y}_{f_i}^\varepsilon = \Pi_i^\varepsilon(\mathcal{Y}_f)$, and

$$\Omega_s^\varepsilon = \bigcup_{1 \leq k \leq 1/\varepsilon^3} \mathcal{Y}_{s_k}^\varepsilon, \quad \Omega_f^\varepsilon = \bigcup_{1 \leq k \leq 1/\varepsilon^3} \mathcal{Y}_{f_k}^\varepsilon, \quad \Gamma^\varepsilon = \overline{\Omega_s^\varepsilon} \cap \Omega_f^\varepsilon.$$

Clearly, $\Omega_f^\varepsilon = \varepsilon E_f \cap \Omega$ and $\Omega_s^\varepsilon = \varepsilon E_s \cap \Omega$.

2.3. Notion of a weak generalized solution of Problem A. Following [3, Section 5.2], let us introduce some necessary notation and formulate a definition of generalized solution of Problem A. At first, let us substitute the vector-functions \mathbf{w}_i^ε in the equations (2.1a) by the new sought functions

$$(2.2a) \quad \mathbf{w}_c^\varepsilon := \frac{\rho_{1f}^0 m_1}{\rho_f^0} \mathbf{w}_1^\varepsilon + \frac{\rho_{2f}^0 m_2}{\rho_f^0} \mathbf{w}_2^\varepsilon,$$

$$(2.2b) \quad \mathbf{w}_r^\varepsilon := \mathbf{w}_2^\varepsilon - \mathbf{w}_1^\varepsilon,$$

which are the mean velocity of the two-phase fluid and the relative velocity of the fluid phases, respectively. In (2.2a)–(2.2b) and further in the article by ρ_f^0 we denote the mean density of the two-phase fluid $\rho_f^0 := \rho_{1f}^0 m_1 + \rho_{2f}^0 m_2$.

By $\bar{\chi}(\mathbf{y})$ denote the characteristic function of the set E_f in \mathbb{R}^3 and by $\chi^\varepsilon(\mathbf{x})$ denote the characteristic function of the subdomain Ω_f^ε in Ω :

$$(2.3) \quad \bar{\chi}(\mathbf{y}) := \begin{cases} 1, & \mathbf{y} \in E_f, \\ 0, & \mathbf{y} \in E \setminus E_f. \end{cases} \quad \chi^\varepsilon(\mathbf{x}) := \begin{cases} 1, & \mathbf{x} \in \Omega_f^\varepsilon, \\ 0, & \mathbf{x} \in \Omega \setminus \Omega_f^\varepsilon. \end{cases}$$

Then, due to the structure of the sets E_f and Ω_f^ε , the identity $\chi^\varepsilon(\mathbf{x}) = \bar{\chi}\left(\frac{\mathbf{x}}{\varepsilon}\right)$ holds true. Introduce into consideration the functions

$$(2.4a) \quad \mathbf{u}_c^\varepsilon := \chi^\varepsilon \mathbf{w}_c^\varepsilon + (1 - \chi^\varepsilon) \mathbf{w}^\varepsilon, \quad (\mathbf{x}, t) \in \Omega \times (0, T),$$

$$(2.4b) \quad \mathbf{u}_r^\varepsilon := \chi^\varepsilon \mathbf{w}_r^\varepsilon, \quad (\mathbf{x}, t) \in \Omega \times (0, T),$$

and the ‘‘uniform’’ density $\rho^{0\varepsilon} := \chi^\varepsilon \rho_f^0 + (1 - \chi^\varepsilon) \rho_s$.

Definition 1. A pair of vector-functions $\{\mathbf{u}_c^\varepsilon(\mathbf{x}, t), \mathbf{u}_r^\varepsilon(\mathbf{x}, t)\}$ is called a weak generalized solution of Problem A, if these functions satisfy

(1) the regularity conditions

$$(2.5a) \quad \mathbf{u}_c^\varepsilon, \mathbf{u}_r^\varepsilon \in L^\infty(0, T; H_0^1(\Omega)),$$

$$(2.5b) \quad \frac{\partial \mathbf{u}_c^\varepsilon}{\partial t}, \frac{\partial \mathbf{u}_r^\varepsilon}{\partial t} \in L^\infty(0, T; L^2(\Omega)),$$

$$(2.5c) \quad \frac{\partial \mathbf{u}_r^\varepsilon}{\partial t} \in L^2(0, T; H_0^1(\Omega)),$$

$$(2.5d) \quad \chi^\varepsilon \operatorname{div}_x \frac{\partial \mathbf{u}_c^\varepsilon}{\partial t}, \chi^\varepsilon \mathbb{D}\left(x, \frac{\partial \mathbf{u}_c^\varepsilon}{\partial t}\right) \in L^2(0, T; L^2(\Omega));$$

(2) the initial data

$$(2.6a) \quad \mathbf{u}_c^\varepsilon|_{t=0} = \mathbf{u}_c^{0\varepsilon} := \chi^\varepsilon \left(\frac{\rho_{1f}^0 m_1}{\rho_f^0} \mathbf{w}_1^{0\varepsilon} + \frac{\rho_{2f}^0 m_2}{\rho_f^0} \mathbf{w}_2^{0\varepsilon} \right) + (1 - \chi^\varepsilon) \mathbf{w}^{0\varepsilon},$$

$$(2.6b) \quad \mathbf{u}_r^\varepsilon|_{t=0} = \mathbf{u}_r^{0\varepsilon} := \chi^\varepsilon (\mathbf{w}_2^{0\varepsilon} - \mathbf{w}_1^{0\varepsilon}),$$

$$(2.6c) \quad \frac{\partial \mathbf{u}_c^\varepsilon}{\partial t} \Big|_{t=0} = \mathbf{v}_c^{0\varepsilon} := \chi^\varepsilon \left(\frac{\rho_{1f}^0 m_1}{\rho_f^0} \mathbf{v}_1^{0\varepsilon} + \frac{\rho_{2f}^0 m_2}{\rho_f^0} \mathbf{v}_2^{0\varepsilon} \right) + (1 - \chi^\varepsilon) \mathbf{v}^{0\varepsilon},$$

$$(2.6d) \quad \frac{\partial \mathbf{u}_r^\varepsilon}{\partial t} \Big|_{t=0} = \mathbf{v}_r^{0\varepsilon} := \chi^\varepsilon (\mathbf{v}_2^{0\varepsilon} - \mathbf{v}_1^{0\varepsilon});$$

(3) the integral equality

$$(2.7a) \quad \int_0^\tau \int_\Omega \left\{ \rho^{0\varepsilon} \frac{\partial \mathbf{u}_c^\varepsilon}{\partial t} \cdot \frac{\partial \Phi}{\partial t} - \chi^\varepsilon \alpha_c^0 \operatorname{div}_x \mathbf{u}_c^\varepsilon \operatorname{div}_x \Phi \right. \\ \left. - \chi^\varepsilon \left[m_1 \left(\nu_1 - \frac{2}{3} \mu_1 \right) + m_2 \left(\nu_2 - \frac{2}{3} \mu_2 \right) \right] \operatorname{div}_x \frac{\partial \mathbf{u}_c^\varepsilon}{\partial t} \operatorname{div}_x \Phi \right. \\ \left. - (1 - \chi^\varepsilon) \left(\eta - \frac{2}{3} \lambda \right) \operatorname{div}_x \mathbf{u}_c^\varepsilon \operatorname{div}_x \Phi \right. \\ \left. - 2\chi^\varepsilon (m_1 \mu_1 + m_2 \mu_2) \mathbb{D}\left(x, \frac{\partial \mathbf{u}_c^\varepsilon}{\partial t}\right) : \mathbb{D}(x, \Phi) - 2(1 - \chi^\varepsilon) \lambda \mathbb{D}(x, \mathbf{u}_c^\varepsilon) : \mathbb{D}(x, \Phi) \right. \\ \left. - \chi^\varepsilon \frac{m_1 m_2}{\rho_f^0} \left[\alpha_c^0 (\rho_{1f}^0 - \rho_{2f}^0) \operatorname{div}_x \mathbf{u}_r^\varepsilon \operatorname{div}_x \Phi \right. \right. \\ \left. \left. + \left(\rho_{1f}^0 \left(\nu_2 - \frac{2}{3} \mu_2 \right) - \rho_{2f}^0 \left(\nu_1 - \frac{2}{3} \mu_1 \right) \right) \operatorname{div}_x \frac{\partial \mathbf{u}_r^\varepsilon}{\partial t} \operatorname{div}_x \Phi \right. \right. \\ \left. \left. + 2(\rho_{1f}^0 \mu_2 - \rho_{2f}^0 \mu_1) \mathbb{D}\left(x, \frac{\partial \mathbf{u}_r^\varepsilon}{\partial t}\right) : \mathbb{D}(x, \Phi) \right] + \rho^0 \mathbf{g} \cdot \Phi \right\} dx dt = \\ \int_\Omega \rho^0 \frac{\partial \mathbf{u}_c^\varepsilon}{\partial t}(\mathbf{x}, \tau) \cdot \Phi(\mathbf{x}, \tau) dx - \int_\Omega \rho^0 \mathbf{v}_c^{0\varepsilon} \cdot \Phi(\mathbf{x}, 0) dx$$

for any time moment $\tau \in [0, T]$ and for any test function $\Phi \in C^1(\Omega \times [0, T])$ such that $\Phi|_{\partial\Omega} = 0$;

(4) the integral equality

$$(2.7b) \quad \int_0^\tau \int_{\Omega_f} \left\{ 2\rho_{1f}^0 \rho_{2f}^0 \frac{m_1 m_2}{\rho_f^0} \frac{\partial \mathbf{u}_r^\varepsilon}{\partial t} \cdot \frac{\partial \Psi}{\partial t} + (\rho_{2f}^0 m_2 - \rho_{1f}^0 m_1) \frac{\partial \mathbf{u}_c^\varepsilon}{\partial t} \cdot \frac{\partial \Psi}{\partial t} \right. \\ \left. - \frac{m_1 m_2}{\rho_f^0} \left[\alpha_r^0 (\rho_{1f}^0 - \rho_{2f}^0) \operatorname{div}_x \mathbf{u}_r^\varepsilon \operatorname{div}_x \Psi \right. \right. \\ \left. \left. + \left(\rho_{2f}^0 \left(\nu_1 - \frac{2}{3} \mu_1 \right) + \rho_{1f}^0 \left(\nu_2 - \frac{2}{3} \mu_2 \right) \right) \operatorname{div}_x \frac{\partial \mathbf{u}_r^\varepsilon}{\partial t} \operatorname{div}_x \Psi \right. \right. \\ \left. \left. + 2(\rho_{2f}^0 \mu_1 + \rho_{1f}^0 \mu_2) \mathbb{D} \left(x, \frac{\partial \mathbf{u}_r^\varepsilon}{\partial t} \right) : \mathbb{D}(x, \Psi) \right] \right. \\ \left. - 2K_F \frac{\partial \mathbf{u}_r^\varepsilon}{\partial t} \cdot \Psi - \alpha_r^0 \operatorname{div}_x \mathbf{u}_c^\varepsilon \operatorname{div}_x \Psi \right. \\ \left. - \left(m_2 \left(\nu_2 - \frac{2}{3} \mu_2 \right) - m_1 \left(\nu_1 - \frac{2}{3} \mu_1 \right) \right) \operatorname{div}_x \frac{\partial \mathbf{u}_c^\varepsilon}{\partial t} \operatorname{div}_x \Psi \right. \\ \left. - 2(m_2 \mu_2 - m_1 \mu_1) \mathbb{D} \left(x, \frac{\partial \mathbf{u}_c^\varepsilon}{\partial t} \right) : \mathbb{D}(x, \Psi) + (\rho_{2f}^0 m_2 - \rho_{1f}^0 m_1) \mathbf{g} \cdot \Psi \right\} dx dt = \\ \int_{\Omega_f} \left\{ 2\rho_{1f}^0 \rho_{2f}^0 \frac{m_1 m_2}{\rho_f^0} \frac{\partial \mathbf{u}_r^\varepsilon}{\partial t}(\mathbf{x}, \tau) \cdot \Psi(\mathbf{x}, \tau) + (\rho_{2f}^0 m_2 - \rho_{1f}^0 m_1) \frac{\partial \mathbf{u}_c^\varepsilon}{\partial t}(\mathbf{x}, \tau) \cdot \Psi(\mathbf{x}, \tau) \right\} dx \\ - \int_{\Omega_f} \left\{ 2\rho_{1f}^0 \rho_{2f}^0 \frac{m_1 m_2}{\rho_f^0} \mathbf{v}_r^{0\varepsilon} \cdot \Psi(\mathbf{x}, 0) + (\rho_{2f}^0 m_2 - \rho_{1f}^0 m_1) \mathbf{v}_c^{0\varepsilon} \cdot \Psi(\mathbf{x}, 0) \right\} dx$$

for any time moment $\tau \in [0, T]$ and for any test function $\Psi \in C^1(\Omega_f \times [0, T])$ such that $\Psi|_{\partial\Omega_f} = 0$;

(5) the equality $(1 - \chi^\varepsilon) \mathbf{u}_r^\varepsilon = 0$ a.e. in Ω , $\operatorname{supp}(\mathbf{u}_r^\varepsilon) \subset \Omega_f$.

In this definition and further in the article we denote $\alpha_c^0 := c_{\rho 2} \rho_{2f}^0 \alpha_2^0 + c_{\rho 1} \rho_{1f}^0 \alpha_1^0$ and $\alpha_r^0 := c_{\rho 2} \rho_{2f}^0 \alpha_2^0 - c_{\rho 1} \rho_{1f}^0 \alpha_1^0$.

Remark 2. Following [3, Remark 4] note that if there exists a classical solution of Problem A, then this solution is a weak generalized solution in the sense of Definition 1. The inverse assertion holds true as well: if $\mathbf{g} \in C(\Omega \times [0, T])$ and there exists a weak generalized solution of Problem A in the sense of Definition 1 such that $\frac{\partial \mathbf{u}_c^\varepsilon}{\partial t}, \frac{\partial \mathbf{u}_r^\varepsilon}{\partial t} \in C([0, T]; C^2(\Omega_f))$, $\frac{\partial^2 \mathbf{u}_c^\varepsilon}{\partial t^2}, \frac{\partial^2 \mathbf{u}_r^\varepsilon}{\partial t^2} \in C([0, T]; C(\Omega_f))$, $\mathbf{u}_c^\varepsilon \in C([0, T]; C^2(\Omega_s))$, $\frac{\partial^2 \mathbf{u}_c^\varepsilon}{\partial t^2} \in C([0, T]; C(\Omega_s))$, then introducing $\mathbf{w}_1^\varepsilon := \mathbf{u}_c^\varepsilon - \frac{m_2 \rho_{2f}^0}{\rho_f^0} \mathbf{u}_r^\varepsilon$

and $\mathbf{w}_2^\varepsilon := \mathbf{u}_c^\varepsilon + \frac{m_1 \rho_{1f}^0}{\rho_f^0} \mathbf{u}_r^\varepsilon$ for $\mathbf{x} \in \Omega_f$ and $\mathbf{w} := \mathbf{u}_c^\varepsilon$ for $\mathbf{x} \in \Omega_s$ we see that the triple

$(\mathbf{w}_1^\varepsilon, \mathbf{w}_2^\varepsilon, \mathbf{w}^\varepsilon)$ solves the equations (2.1a) and (2.1c) and the contact discontinuity conditions (2.1d) and (2.1e), that is, the triple $(\mathbf{w}_1^\varepsilon, \mathbf{w}_2^\varepsilon, \mathbf{w}^\varepsilon)$ is a classical solution of Problem A.

2.4. Well-posedness of Problem A. The following results on existence, uniqueness and qualitative properties of solutions to Problem A were established in [3].

Proposition 1. Fix $\varepsilon > 0$ arbitrarily. Whenever $\mathbf{g} \in L^2(\Omega \times (0, T))$, $\mathbf{u}_c^{0\varepsilon}, \mathbf{u}_r^{0\varepsilon} \in H_0^1(\Omega)$, and $\mathbf{v}_c^{0\varepsilon}, \mathbf{v}_r^{0\varepsilon} \in L^2(\Omega)$, there exists a unique weak generalized solution $\{\mathbf{u}_c^\varepsilon, \mathbf{u}_r^\varepsilon\}$ of Problem A in the sense of Definition 1.

Furthermore, the solution satisfies the energy identity

$$\begin{aligned}
(2.8a) \quad & \frac{1}{2} \rho_{1f}^0 m_1 \int_{\Omega} \chi^\varepsilon \left| \frac{\partial \mathbf{w}_1^\varepsilon}{\partial t}(\tau) \right|^2 d\mathbf{x} \\
& + \frac{1}{2} \rho_{2f}^0 m_2 \int_{\Omega} \chi^\varepsilon \left| \frac{\partial \mathbf{w}_2^\varepsilon}{\partial t}(\tau) \right|^2 d\mathbf{x} + \frac{1}{2} \rho_s \int_{\Omega} (1 - \chi^\varepsilon) \left| \frac{\partial \mathbf{u}_c^\varepsilon}{\partial t}(\tau) \right|^2 d\mathbf{x} \\
& + \frac{1}{2} c_{\rho 1} \rho_{1f}^0 \alpha_1^0 m_1 \int_{\Omega} \chi^\varepsilon |\operatorname{div}_x \mathbf{w}_1^\varepsilon(\tau)|^2 d\mathbf{x} + \frac{1}{2} c_{\rho 2} \rho_{2f}^0 \alpha_2^0 m_2 \int_{\Omega} \chi^\varepsilon |\operatorname{div}_x \mathbf{w}_2^\varepsilon(\tau)|^2 d\mathbf{x} \\
& + \frac{1}{2} \left(\eta - \frac{2}{3} \lambda \right) \int_{\Omega} (1 - \chi^\varepsilon) |\operatorname{div}_x \mathbf{u}_c^\varepsilon(\tau)|^2 d\mathbf{x} + \frac{1}{2} \lambda \int_{\Omega} (1 - \chi^\varepsilon) |\mathbb{D}(x, \mathbf{u}_c^\varepsilon(\tau))|^2 d\mathbf{x} \\
& + m_1 \left(\nu_1 - \frac{2}{3} \mu_1 \right) \int_0^\tau \int_{\Omega} \chi^\varepsilon \left| \operatorname{div}_x \frac{\partial \mathbf{w}_1^\varepsilon}{\partial t} \right|^2 d\mathbf{x} dt \\
& + m_2 \left(\nu_2 - \frac{2}{3} \mu_2 \right) \int_0^\tau \int_{\Omega} \chi^\varepsilon \left| \operatorname{div}_x \frac{\partial \mathbf{w}_2^\varepsilon}{\partial t} \right|^2 d\mathbf{x} dt \\
& + m_1 \mu_1 \int_0^\tau \int_{\Omega} \chi^\varepsilon \left| \mathbb{D} \left(x, \frac{\partial \mathbf{w}_1^\varepsilon}{\partial t} \right) \right|^2 d\mathbf{x} dt \\
& + m_2 \mu_2 \int_0^\tau \int_{\Omega} \chi^\varepsilon \left| \mathbb{D} \left(x, \frac{\partial \mathbf{w}_2^\varepsilon}{\partial t} \right) \right|^2 d\mathbf{x} dt + K_F \int_0^\tau \int_{\Omega} \chi^\varepsilon \left| \frac{\partial \mathbf{u}_c^\varepsilon}{\partial t} \right|^2 d\mathbf{x} dt \\
& = \frac{1}{2} \rho_{1f}^0 m_1 \int_{\Omega} \chi^\varepsilon |\mathbf{v}_1^{0\varepsilon}|^2 d\mathbf{x} + \frac{1}{2} \rho_{2f}^0 m_2 \int_{\Omega} \chi^\varepsilon |\mathbf{v}_2^{0\varepsilon}|^2 d\mathbf{x} + \frac{1}{2} \int_{\Omega} (1 - \chi^\varepsilon) |\mathbf{v}_c^{0\varepsilon}|^2 d\mathbf{x} \\
& + \frac{1}{2} c_{\rho 1} \rho_{1f}^0 \alpha_1^0 m_1 \int_{\Omega} \chi^\varepsilon |\operatorname{div}_x \mathbf{w}_1^{0\varepsilon}|^2 d\mathbf{x} + \frac{1}{2} c_{\rho 2} \rho_{2f}^0 \alpha_2^0 m_2 \int_{\Omega} \chi^\varepsilon |\operatorname{div}_x \mathbf{w}_2^{0\varepsilon}|^2 d\mathbf{x} \\
& + \frac{1}{2} \left(\eta - \frac{2}{3} \lambda \right) \int_{\Omega} (1 - \chi^\varepsilon) |\operatorname{div}_x \mathbf{u}_c^{0\varepsilon}|^2 d\mathbf{x} + \frac{1}{2} \lambda \int_{\Omega} (1 - \chi^\varepsilon) |\mathbb{D}(x, \mathbf{u}_c^{0\varepsilon})|^2 d\mathbf{x} \\
& - \int_0^\tau \int_{\Omega} \chi^\varepsilon c_{\rho 1} \rho_{1f}^0 \alpha_1^0 m_2 \operatorname{div}_x \mathbf{w}_2^\varepsilon \operatorname{div}_x \frac{\partial \mathbf{w}_1^\varepsilon}{\partial t} d\mathbf{x} dt \\
& - \int_0^\tau \int_{\Omega} \chi^\varepsilon c_{\rho 2} \rho_{2f}^0 \alpha_2^0 m_1 \operatorname{div}_x \mathbf{w}_1^\varepsilon \operatorname{div}_x \frac{\partial \mathbf{w}_2^\varepsilon}{\partial t} d\mathbf{x} dt \\
& + \int_0^\tau \int_{\Omega} \chi^\varepsilon \rho_{1f}^0 m_1 \mathbf{g} \cdot \frac{\partial \mathbf{w}_1^\varepsilon}{\partial t} d\mathbf{x} dt + \int_0^\tau \int_{\Omega} \chi^\varepsilon \rho_{2f}^0 m_2 \mathbf{g} \cdot \frac{\partial \mathbf{w}_2^\varepsilon}{\partial t} d\mathbf{x} dt \\
& + \int_0^\tau \int_{\Omega} (1 - \chi^\varepsilon) \rho_s \mathbf{g} \cdot \frac{\partial \mathbf{u}_c^\varepsilon}{\partial t} d\mathbf{x} dt
\end{aligned}$$

and the energy inequality

$$\begin{aligned}
(2.8b) \quad & \frac{1}{4} \rho_{1f}^0 m_1 \int_{\Omega} \chi^\varepsilon \left| \frac{\partial \mathbf{w}_1^\varepsilon}{\partial t}(\tau) \right|^2 d\mathbf{x} + \frac{1}{4} \rho_{2f}^0 m_2 \int_{\Omega} \chi^\varepsilon \left| \frac{\partial \mathbf{w}_2^\varepsilon}{\partial t}(\tau) \right|^2 d\mathbf{x} \\
& + \frac{1}{4} \rho_s \int_{\Omega} (1 - \chi^\varepsilon) \left| \frac{\partial \mathbf{u}_c^\varepsilon}{\partial t}(\tau) \right|^2 d\mathbf{x} \\
& + \frac{1}{2} c_{\rho 1} \rho_{1f}^0 \alpha_1^0 m_1 \int_{\Omega} \chi^\varepsilon |\operatorname{div}_x \mathbf{w}_1^\varepsilon(\tau)|^2 d\mathbf{x} + \frac{1}{2} c_{\rho 2} \rho_{2f}^0 \alpha_2^0 m_2 \int_{\Omega} \chi^\varepsilon |\operatorname{div}_x \mathbf{w}_2^\varepsilon(\tau)|^2 d\mathbf{x} \\
& + \frac{1}{2} \left(\eta - \frac{2}{3} \lambda \right) \int_{\Omega} (1 - \chi^\varepsilon) |\operatorname{div}_x \mathbf{u}_c^\varepsilon(\tau)|^2 d\mathbf{x} + \frac{1}{2} \lambda \int_{\Omega} (1 - \chi^\varepsilon) |\mathbb{D}(x, \mathbf{u}_c^\varepsilon(\tau))|^2 d\mathbf{x} \\
& + \frac{1}{2} m_1 \left(\nu_1 - \frac{2}{3} \mu_1 \right) \int_0^\tau \int_{\Omega} \chi^\varepsilon \left| \operatorname{div}_x \frac{\partial \mathbf{w}_1^\varepsilon}{\partial t} \right|^2 d\mathbf{x} dt
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2}m_2\left(\nu_2 - \frac{2}{3}\mu_2\right) \int_0^\tau \int_\Omega \chi^\varepsilon \left| \operatorname{div}_x \frac{\partial \mathbf{w}_2^\varepsilon}{\partial t} \right|^2 dx dt \\
& \quad + m_1\mu_1 \int_0^\tau \int_\Omega \chi^\varepsilon \left| \mathbb{D}\left(x, \frac{\partial \mathbf{w}_1^\varepsilon}{\partial t}\right) \right|^2 dx dt \\
& \quad + m_2\mu_2 \int_0^\tau \int_\Omega \chi^\varepsilon \left| \mathbb{D}\left(x, \frac{\partial \mathbf{w}_2^\varepsilon}{\partial t}\right) \right|^2 dx dt + K_F \int_0^\tau \int_\Omega \chi^\varepsilon \left| \frac{\partial \mathbf{u}_r^\varepsilon}{\partial t} \right|^2 dx dt \\
& \leq \frac{1}{2}\rho_{1f}^0 m_1 \int_\Omega \chi^\varepsilon |\mathbf{v}_1^{0\varepsilon}|^2 dx + \frac{1}{2}\rho_{2f}^0 m_2 \int_\Omega \chi^\varepsilon |\mathbf{v}_2^{0\varepsilon}|^2 dx + \frac{1}{2} \int_\Omega (1 - \chi^\varepsilon) |\mathbf{v}_c^{0\varepsilon}|^2 dx \\
& \quad + \frac{1}{2}c_{\rho 1}\rho_{1f}^0 \alpha_1^0 m_1 \int_\Omega \chi^\varepsilon |\operatorname{div}_x \mathbf{w}_1^{0\varepsilon}|^2 dx + \frac{1}{2}c_{\rho 2}\rho_{2f}^0 \alpha_2^0 m_2 \int_\Omega \chi^\varepsilon |\operatorname{div}_x \mathbf{w}_2^{0\varepsilon}|^2 dx \\
& \quad + \frac{1}{2}\left(\eta - \frac{2}{3}\lambda\right) \int_\Omega (1 - \chi^\varepsilon) |\operatorname{div}_x \mathbf{u}_c^{0\varepsilon}|^2 dx + \frac{1}{2}\lambda \int_\Omega (1 - \chi^\varepsilon) |\mathbb{D}(x, \mathbf{u}_c^{0\varepsilon})|^2 dx \\
& + \rho_{1f}^0 m_1 \int_0^\tau \int_\Omega \chi^\varepsilon |\mathbf{g}|^2 dx dt + \rho_{2f}^0 m_2 \int_0^\tau \int_\Omega \chi^\varepsilon |\mathbf{g}|^2 dx dt + \rho_s \int_0^\tau \int_\Omega (1 - \chi^\varepsilon) |\mathbf{g}|^2 dx dt \\
& \quad + (e^{c_0\tau} - 1) \left(\frac{1}{2}\rho_{1f}^0 m_1 \int_\Omega \chi^\varepsilon |\mathbf{v}_1^{0\varepsilon}|^2 dx \right. \\
& \quad \left. + \frac{1}{2}\rho_{2f}^0 m_2 \int_\Omega \chi^\varepsilon |\mathbf{v}_2^{0\varepsilon}|^2 dx + \frac{1}{2} \int_\Omega (1 - \chi^\varepsilon) |\mathbf{v}_c^{0\varepsilon}|^2 dx \right. \\
& \quad \left. + \frac{1}{2}c_{\rho 1}\rho_{1f}^0 \alpha_1^0 m_1 \int_\Omega \chi^\varepsilon |\operatorname{div}_x \mathbf{w}_1^{0\varepsilon}|^2 dx + \frac{1}{2}c_{\rho 2}\rho_{2f}^0 \alpha_2^0 m_2 \int_\Omega \chi^\varepsilon |\operatorname{div}_x \mathbf{w}_2^{0\varepsilon}|^2 dx \right. \\
& \quad \left. + \frac{1}{2}\left(\eta - \frac{2}{3}\lambda\right) \int_\Omega (1 - \chi^\varepsilon) |\operatorname{div}_x \mathbf{u}_c^{0\varepsilon}|^2 dx + \frac{1}{2}\lambda \int_\Omega (1 - \chi^\varepsilon) |\mathbb{D}(x, \mathbf{u}_c^{0\varepsilon})|^2 dx \right) \\
& \quad + c_0 \int_0^\tau e^{c_0 t} \left(\int_0^t \int_\Omega \rho^0 |\mathbf{g}(\mathbf{x}, s)|^2 e^{-c_0 s} ds \right) dt \quad \forall \tau \in [0, T].
\end{aligned}$$

In (2.8a) and (2.8b) the following notation is in use:

$$(2.9a) \quad \mathbf{w}_1^\varepsilon := \mathbf{u}_c^\varepsilon - \frac{\rho_{2f}^0 m_2}{\rho_f^0} \mathbf{u}_r^\varepsilon, \quad \mathbf{w}_2^\varepsilon := \mathbf{u}_c^\varepsilon + \frac{\rho_{1f}^0 m_1}{\rho_f^0} \mathbf{u}_r^\varepsilon,$$

$$(2.9b) \quad \mathbf{w}_1^{0\varepsilon} := \mathbf{u}_c^{0\varepsilon} - \frac{\rho_{2f}^0 m_2}{\rho_f^0} \mathbf{u}_r^{0\varepsilon}, \quad \mathbf{w}_2^{0\varepsilon} := \mathbf{u}_c^{0\varepsilon} + \frac{\rho_{1f}^0 m_1}{\rho_f^0} \mathbf{u}_r^{0\varepsilon},$$

$$(2.9c) \quad \mathbf{v}_1^{0\varepsilon} := \mathbf{v}_c^{0\varepsilon} - \frac{\rho_{2f}^0 m_2}{\rho_f^0} \mathbf{v}_r^{0\varepsilon}, \quad \mathbf{v}_2^{0\varepsilon} := \mathbf{v}_c^{0\varepsilon} + \frac{\rho_{1f}^0 m_1}{\rho_f^0} \mathbf{v}_r^{0\varepsilon},$$

$$(2.9d) \quad c_0 := \max \left\{ \frac{(c_{\rho 1}\rho_{1f}^0 \alpha_1^0 m_2)^2}{c_{\rho 2}\rho_{2f}^0 \alpha_2^0 m_1 m_2 \left(\nu_1 - \frac{2}{3}\mu_1\right)}; \frac{(c_{\rho 2}\rho_{2f}^0 \alpha_2^0 m_1)^2}{c_{\rho 1}\rho_{1f}^0 \alpha_1^0 m_1 m_2 \left(\nu_2 - \frac{2}{3}\mu_2\right)} \right\}.$$

3. HOMOGENIZED MACROSCOPIC MODEL

The following theorem is the main result of the article.

Theorem 1. *Let vector-functions $\mathbf{u}_c^{0\varepsilon}, \mathbf{u}_r^{0\varepsilon} \in H_0^1(\Omega)$ and $\mathbf{v}_c^{0\varepsilon}, \mathbf{v}_r^{0\varepsilon} \in L^2(\Omega)$ satisfy the limiting relations*

$$(3.1) \quad \mathbf{u}_c^{0\varepsilon} \rightarrow \mathbf{u}_c^*, \quad \mathbf{u}_r^{0\varepsilon} \rightarrow 0 \text{ weakly in } H_0^1(\Omega),$$

$$(3.2) \quad \mathbf{v}_c^{0\varepsilon} \rightarrow \mathbf{v}_c^*, \quad \mathbf{v}_r^{0\varepsilon} \rightarrow 0 \text{ weakly in } L^2(\Omega)$$

as $\varepsilon \searrow 0$, with some vector-functions $\mathbf{u}_c^{0*} \in H_0^1(\Omega)$ and $\mathbf{v}_c^{0*} \in L^2(\Omega)$. Let $\{\mathbf{u}_c^\varepsilon, \mathbf{u}_r^\varepsilon\}$ be the weak generalized solution of Problem A corresponding to the data $\mathbf{u}_c^{0\varepsilon}, \mathbf{u}_r^{0\varepsilon}, \mathbf{v}_c^{0\varepsilon}$, and $\mathbf{v}_r^{0\varepsilon}$ for any fixed $\varepsilon > 0$ (such that $\varepsilon^{-1} \in \mathbb{N}$). Then, as $\varepsilon \searrow 0$ ($\varepsilon^{-1} \in \mathbb{N}$), the sequence $\{\mathbf{u}_c^\varepsilon, \mathbf{u}_r^\varepsilon\}$ converges weakly in $H^1(Q)$ to the pair $\{\mathbf{u}_c^*, 0\}$, where \mathbf{u}_c^* is the weak generalized solution of Problem B stated below. In the formulation of Problem B, the constant fourth-order tensors \mathbb{E} and \mathbb{M} , and the fourth-order-tensor-valued function $t \mapsto \mathbb{G}(t)$ depend merely on geometry of the sets \mathcal{Y}_f and \mathcal{Y}_s and on values of $\alpha_c^0, \alpha_r^0, \rho_{1f}^0, \rho_{2f}^0, \rho_s, \nu_1, \mu_1, m_1, \nu_2, \mu_2$, and m_2 . More precisely, \mathbb{E}, \mathbb{M} , and $\mathbb{G}(t)$ are uniquely defined by the equalities (5.10a)–(5.10c) (in Section 5).

The notion of two-scale convergence is introduced below in Section 4.

Problem B. In the space-time domain $Q = \Omega \times (0, T)$, where $T > 0$ is an arbitrarily given time moment and $\Omega = (0, 1)^3$, it is necessary to find a displacement field $\mathbf{u}_c^*(\mathbf{x}, t)$, satisfying the momentum equation

$$(3.3) \quad (\rho_f^0 |\mathcal{Y}_f| + \rho_s |\mathcal{Y}_s|) \frac{\partial^2 \mathbf{u}_c^*}{\partial t^2} - \operatorname{div}_x \left(\mathbb{E} : \nabla_x \mathbf{u}_c^* + \mathbb{M} : \nabla_x \frac{\partial \mathbf{u}_c^*}{\partial t} + \int_0^t \mathbb{G}(t-s) : \nabla_x \mathbf{u}_c^*(\mathbf{x}, s) ds \right) = (\rho_f^0 |\mathcal{Y}_f| + \rho_s |\mathcal{Y}_s|) \mathbf{g},$$

the initial data

$$(3.4) \quad \mathbf{u}_c^*(\mathbf{x}, 0) = \mathbf{u}_c^{0*}(\mathbf{x}), \quad \frac{\partial \mathbf{u}_c^*}{\partial t}(\mathbf{x}, 0) = \mathbf{v}_c^{0*}(\mathbf{x}),$$

and the boundary condition

$$(3.5) \quad \mathbf{u}_c^* = 0 \text{ on } \partial\Omega \times (0, T).$$

Notation 1. In the equation (3.3) and further in the paper by $|\mathcal{Y}_f|$ and $|\mathcal{Y}_s|$ we denote the volumes of fluid and solid components in the pattern unit cell, i.e. $|\mathcal{Y}_f| = \int_{\mathcal{Y}_f} d\mathbf{y}$ and $|\mathcal{Y}_s| = \int_{\mathcal{Y}_s} d\mathbf{y}$.

Definition 2. Function \mathbf{u}_c^* is called a weak generalized solution of Problem B, if it meets the regularity requirement $\mathbf{u}_c^* \in L^2(0, T; W_2^1(\Omega))$, and satisfies the boundary condition (3.5) in the trace sense and the integral equality

$$(3.6) \quad \int_0^\tau \int_\Omega \left\{ (\rho_f^0 |\mathcal{Y}_f| + \rho_s |\mathcal{Y}_s|) \frac{\partial \mathbf{u}_c^*}{\partial t} \cdot \frac{\partial \boldsymbol{\varphi}}{\partial t} - \left(\mathbb{E} : \nabla_x \mathbf{u}_c^* + \mathbb{M} : \nabla_x \frac{\partial \mathbf{u}_c^*}{\partial t} + \int_0^t \mathbb{G}(t-s) : \nabla_x \mathbf{u}_c^*(\mathbf{x}, s) ds \right) : \nabla_x \boldsymbol{\varphi} + (\rho_f^0 |\mathcal{Y}_f| + \rho_s |\mathcal{Y}_s|) \mathbf{g} \cdot \boldsymbol{\varphi} \right\} d\mathbf{x} dt = (\rho_f^0 |\mathcal{Y}_f| + \rho_s |\mathcal{Y}_s|) \int_\Omega \frac{\partial \mathbf{u}_c^*}{\partial t}(\mathbf{x}, \tau) \cdot \boldsymbol{\varphi}(\mathbf{x}, \tau) d\mathbf{x} - (\rho_f^0 |\mathcal{Y}_f| + \rho_s |\mathcal{Y}_s|) \int_\Omega \mathbf{v}_c^{0*}(\mathbf{x}) \cdot \boldsymbol{\varphi}(\mathbf{x}, 0) d\mathbf{x},$$

for any smooth test vector-function $\boldsymbol{\varphi}(\mathbf{x}, t)$ vanishing near the boundary $\partial\Omega$ and in the neighborhood of the plane $t = T$.

Strictly keeping the track of the considerations in [12, Section 2.4] and [8, Theorem 2] we can make the following important conclusions:

Remark 3. 1. *The tensors \mathbb{E} , \mathbb{M} , and $\mathbb{G}(t)$ are symmetric, i.e., their components satisfy the equalities*

$$(3.7) \quad \begin{aligned} \mathbb{E}^{ijkl} &= \mathbb{E}^{jikl} = \mathbb{E}^{jilk} = \mathbb{E}^{klij}, & \mathbb{M}^{ijkl} &= \mathbb{M}^{jikl} = \mathbb{M}^{jilk} = \mathbb{M}^{klij}, \\ \mathbb{G}^{ijkl}(t) &= \mathbb{G}^{jikl}(t) = \mathbb{G}^{jilk}(t) = \mathbb{G}^{klij}(t) & (i, j, k, l = 1, 2, 3). \end{aligned}$$

2. *The fourth-rank tensor $\mathcal{A}^\gamma \stackrel{def}{=} \gamma\mathbb{M} + \mathbb{E} + \hat{\mathbb{G}}(\gamma)$ is strictly positively defined for $\gamma > 0$. By $\hat{\mathbb{G}}(\gamma)$ the Laplace transform of $\mathbb{G}(t)$ is denoted and at the same time it is assumed that $\mathbb{G}(t) = 0$ for $t > T$. Recall that the Laplace transform of an arbitrary locally integrable and not fast-increasing on the semi-axis $(0, \infty)$ function $\varphi(t)$ is defined by the formula*

$$\hat{\varphi}(\gamma) = \mathcal{L}[\varphi](\gamma) = \int_0^\infty \varphi(t)e^{-\gamma t} dt, \quad \gamma > 0.$$

3. *If the both sets \mathcal{Y}_f and E_f are connected then \mathbb{M} is strictly positively defined.*

4. *If the set $\partial\mathcal{Y} \cap \partial\mathcal{Y}_f$ is empty, in other words, the porous space Ω_f^ε consists only of trapped pores, then \mathbb{M} is zero tensor and \mathbb{E} is strictly positively defined.*

In view of these properties of symmetry and positive definiteness of the tensors \mathbb{E} , \mathbb{M} , and $\mathbb{G}(t)$, Problem B is identified as an initial-boundary value problem for a model of linear viscoelasticity with memory of shape, except for the case of the trapped pores (see item 4 of Remark 3), in which the homogenized model takes the form of a model of linear elasticity. Comparing with the well-known formulations in the linear theory of poroviscoelasticity (see, for example, [14, Chapter 9]), we conclude that \mathbb{M} is the effective viscosity tensor of the averaged medium, \mathbb{E} is the effective instantaneous elasticity tensor, and $\mathbb{G}(t)$ is the tensor-valued relaxation function determining influence of mechanical history of the medium during the period $(0, t)$ on the current state at the moment t .

On the strength of Theorem 1, Problem B is solvable in the sense of Definition 2, provided with the condition that the coefficients of the equation (3.3) admit certain relations with data of the microstructure, since some solution of Problem B can be constructed as a limit of solutions of Problem A, as $\varepsilon \searrow 0$. At the same time, it should be noticed that, if we assume that the coefficients of the equation (3.3) a priori satisfy the properties in Remark 3, then the conclusion about well-posedness of Problem B is correct independently of whether problem B is connected with the microstructure, or not. More precisely, the following assertion holds true:

Theorem 2. *Assume that the tensors in the equation (3.3) have the properties, stated in Remark 3, and \mathbb{G} satisfies the regularity demand $\mathbb{G}^{ijkl} \in L^2(0, T)$ ($i, j, k, l = 1, 2, 3$). Let all of these tensors be, in general, irrelevant to the data given for Problem A. Then for any given initial distributions $\mathbf{u}_c^{0*} \in H_0^1(\Omega)$, $\mathbf{v}_c^{0*} \in L^2(\Omega)$ and the right-hand side $\mathbf{g} \in L^2(Q)$ of the equation (3.3), there exists a unique generalized solution of Problem B, in the sense of Definition 2.*

PROOF of this theorem can be found in [12, Section 2.4].

Remark 4. *Using the solution of Problem B, we can calculate the averaged mechanical quantities α_i , ρ_i^0 ($i = 1, 2$), and p by the formulas:*

$$(3.8) \quad \alpha_1 = a_1 + \sum_{i,j=1}^3 \mathbb{K}_1^{ij} \frac{\partial u_{ci}^*}{\partial x_j} + \int_0^t \sum_{i,j=1}^3 \frac{\partial u_{ci}^*}{\partial x_j} \mathbb{H}_1^{ij}(t-s) ds,$$

$$(3.9) \quad \alpha_2 = a_2 + \sum_{i,j=1}^3 \mathbb{K}_2^{ij} \frac{\partial u_{ci}^*}{\partial x_j} + \int_0^t \sum_{i,j=1}^3 \frac{\partial u_{ci}^*}{\partial x_j} \mathbb{H}_2^{ij}(t-s) ds,$$

$$(3.10) \quad \rho_i^0 = \rho_{if}^0 \left[1 - \frac{\alpha_i^0}{m_i} \left(\sum_{l,j=1}^3 \mathbb{K}_3^{lj} \frac{\partial u_{cl}^*}{\partial x_j} + \int_0^t \sum_{l,j=1}^3 \frac{\partial u_{cl}^*}{\partial x_j} \mathbb{H}_3^{lj}(t-s) ds \right) \right],$$

$$(3.11) \quad p = p_* - c_{\rho 1} \rho_{1f}^0 \frac{\alpha_1^0}{m_1} \left(\sum_{l,j=1}^3 \mathbb{K}_3^{lj} \frac{\partial u_{cl}^*}{\partial x_j} + \int_0^t \sum_{l,j=1}^3 \frac{\partial u_{cl}^*}{\partial x_j} \mathbb{H}_3^{lj}(t-s) ds \right).$$

Here the constants a_1 and a_2 , the constant matrices \mathbb{K}_r , $r = 1, 2, 3$, and the 3×3 -matrix-valued functions $t \mapsto \mathbb{H}_r(t)$, $r = 1, 2, 3$ depend merely on geometry of the sets \mathcal{Y}_f and \mathcal{Y}_s and on values of ρ_{1f}^0 , ρ_{2f}^0 , m_1 , ν_2 , μ_2 , m_2 , $c_{\rho 1}$, and $c_{\rho 2}$. More precisely, a_1 , a_2 , \mathbb{K}_r , and $\mathbb{H}_r(t)$ are defined by the equalities (5.11a)–(5.11h) (in Section 5).

4. PROOF OF THEOREM 1. PRELIMINARIES: THE ALLAIRE–NGUETSENG TWO-SCALE CONVERGENCE METHOD

In this section we outline the notion of two-scale convergent sequences and then pass to the limit in Problem A, as $\varepsilon \searrow 0$, with the help of this notion. As a result, we derive a system of limiting two-scale equations. This limiting procedure is the first step in the proof of Theorem 1.

Definition 3. (*G. Nguetseng [6].*) Sequence $\{\varphi^\varepsilon\}_{\varepsilon \searrow 0} \subset L^2(Q)$ is called two-scale convergent to a limit $\varphi \in L^2(Q \times \mathcal{Y})$, if and only if for any 1-periodic and in \mathbf{y} function $\sigma \in L^2(Q; C(\mathcal{Y}))$ the limiting relation

$$\lim_{\varepsilon \rightarrow 0} \int_Q \varphi^\varepsilon(\mathbf{x}, t) \sigma(\mathbf{x}, t, \mathbf{x}/\varepsilon) d\mathbf{x} dt = \int_{Q \times \mathcal{Y}} \varphi(\mathbf{x}, t, \mathbf{y}) \sigma(\mathbf{x}, t, \mathbf{y}) d\mathbf{x} d\mathbf{y} dt$$

holds.

Existence of two-scale convergent sequences and their main properties are established in the following fundamental theorem [6, 7].

Theorem TS.

- 1) Each bounded in $L^2(Q)$ sequence contains a subsequence, which two-scale converges to some limit belonging to $L^2(Q \times \mathcal{Y})$;
- 2) If a sequence in $L^2(Q)$ two-scale converges to two functions $\varphi_1, \varphi_2 \in L^2(Q \times \mathcal{Y})$ simultaneously then $\varphi_1 = \varphi_2$ a.e. in $Q \times \mathcal{Y}$;
- 3) Let sequences $\{\varphi^\varepsilon\}$ and $\{\nabla_x \varphi^\varepsilon\}$ be bounded in $L^2(Q)$. Then there exist functions $\varphi \in L^2(0, T; H^1(\Omega))$ and $\psi \in L^2(Q; H^1(\mathcal{Y}))$, ψ is 1-periodic in \mathbf{y} , and a subsequence of $\{\varphi^\varepsilon\}$ such that the subsequences $\{\varphi^\varepsilon\}$ and $\{\nabla_x \varphi^\varepsilon\}$ two-scale converge to ϕ and to $\nabla_x \varphi(\mathbf{x}, t) + \nabla_y \psi(\mathbf{x}, t, \mathbf{y})$, respectively.

Remark 5. Let $\sigma \in L^\infty(\mathcal{Y})$. Extend σ from \mathcal{Y} onto the whole space \mathbb{R}^3 by the periodic repetition. Let $\sigma^\varepsilon(\mathbf{x}) = \sigma(\mathbf{x}/\varepsilon)$ ($\mathbf{x} \in \Omega$) and a sequence $\{\varphi^\varepsilon\} \subset L^2(Q)$ two-scale converge to some limit $\varphi \in L^2(Q \times \mathcal{Y})$. Then the sequence $\{\sigma^\varepsilon \varphi^\varepsilon\}$ two-scale converges to the limit $\sigma(\mathbf{y}) \varphi(\mathbf{x}, t, \mathbf{y})$.

Now turn to consideration of the limiting transition in the equations of Problem A, as $\varepsilon \searrow 0$ ($\varepsilon^{-1} \in \mathbb{N}$). On the strength of the energy inequality (2.8b), the sequences

$\{\mathbf{u}_c^\varepsilon\}$, $\{\mathbf{u}_r^\varepsilon\}$, $\left\{\frac{\partial \mathbf{u}_c^\varepsilon}{\partial t}\right\}$, $\left\{\frac{\partial \mathbf{u}_r^\varepsilon}{\partial t}\right\}$, $\{\nabla_x \mathbf{u}_c^\varepsilon\}$, $\{\nabla_x \mathbf{u}_r^\varepsilon\}$, $\left\{\nabla_x \frac{\partial \mathbf{u}_r^\varepsilon}{\partial t}\right\}$, $\{(1 - \chi^\varepsilon)\mathbb{D}(x, \mathbf{u}_c^\varepsilon)\}$, $\left\{\chi^\varepsilon \mathbb{D}\left(x, \frac{\partial \mathbf{u}_c^\varepsilon}{\partial t}\right)\right\}$, and $\left\{\mathbb{D}\left(x, \frac{\partial \mathbf{u}_r^\varepsilon}{\partial t}\right)\right\}$ are uniformly bounded in $L^2(\Omega \times [0, T])$ with respect to ε . From this fact, Theorem TS, and Remark 5, it follows that there exist a subsequence of $\{\varepsilon > 0 \mid \varepsilon^{-1} \in \mathbb{N}\}$ and the functions \mathbf{u}_c^* , $\mathbf{u}_r^* \in L^2(0, T; H_0^1(\Omega))$ and \mathbf{U}_c , $\mathbf{U}_r \in L^2(\Omega \times [0, T] \times \mathcal{Y})$ such that

$$(4.1) \quad \nabla_y \mathbf{U}_c, \nabla_y \mathbf{U}_r \in L^2(\Omega \times [0, T] \times \mathcal{Y}),$$

$$(4.2) \quad (1 - \bar{\chi}(\mathbf{y}))(\mathbb{D}(x, \mathbf{u}_c^*) + \mathbb{D}(y, \mathbf{U}_c)) \in L^2(\Omega \times [0, T] \times \mathcal{Y}),$$

$$(4.3) \quad \mathbb{D}\left(x, \frac{\partial \mathbf{u}_r^*}{\partial t}\right) + \mathbb{D}\left(y, \frac{\partial \mathbf{U}_r}{\partial t}\right) \in L^2(\Omega \times [0, T] \times \mathcal{Y}),$$

$$(4.4) \quad \mathbf{U}_c, \mathbf{U}_r \text{ are 1-periodic in } \mathbf{y},$$

and the following limiting relations hold true, as $\varepsilon \searrow 0$:

$$(4.5) \quad \mathbf{u}_c^\varepsilon \rightharpoonup \mathbf{u}_c^* \text{ weakly in } L^2(0, T; H_0^1(\Omega)),$$

$$(4.6) \quad \mathbf{u}_r^\varepsilon \rightharpoonup \mathbf{u}_r^* \text{ weakly in } L^2(0, T; H_0^1(\Omega)),$$

$$(4.7) \quad \nabla_x \mathbf{u}_c^\varepsilon \rightharpoonup \nabla_x \mathbf{u}_c^*(\mathbf{x}, t) + \nabla_y \mathbf{U}_c(\mathbf{x}, t, \mathbf{y}),$$

$$(4.8) \quad \nabla_x \mathbf{u}_r^\varepsilon \rightharpoonup \nabla_x \mathbf{u}_r^*(\mathbf{x}, t) + \nabla_y \mathbf{U}_r(\mathbf{x}, t, \mathbf{y}),$$

$$(4.9) \quad \nabla_x \frac{\partial \mathbf{u}_r^\varepsilon}{\partial t} \rightharpoonup \nabla_x \frac{\partial \mathbf{u}_r^*}{\partial t}(\mathbf{x}, t) + \nabla_y \frac{\partial \mathbf{U}_r}{\partial t}(\mathbf{x}, t, \mathbf{y}),$$

$$(4.10) \quad (1 - \chi^\varepsilon)\mathbb{D}(x, \mathbf{u}_c^\varepsilon) \rightharpoonup (1 - \bar{\chi}(\mathbf{y}))(\mathbb{D}(x, \mathbf{u}_c^*) + \mathbb{D}(y, \mathbf{U}_c))$$

$$(4.11) \quad \mathbb{D}\left(x, \frac{\partial \mathbf{u}_r^\varepsilon}{\partial t}\right) \rightharpoonup \mathbb{D}\left(x, \frac{\partial \mathbf{u}_r^*}{\partial t}\right) + \mathbb{D}\left(y, \frac{\partial \mathbf{U}_r}{\partial t}\right).$$

Relations (4.7)–(4.11) are understood in the two-scale sense.

Remark 6. Multiply the equation $(1 - \chi^\varepsilon)\mathbf{u}_r^\varepsilon = 0$ by $\Phi = \varphi_1(\mathbf{x}, t)\varphi_2(\mathbf{x}/\varepsilon)$, where $\varphi_1(\mathbf{x}, t)$ is smooth, $\varphi_2(\mathbf{y})$ is 1-periodic and $\text{supp } \varphi_2 \subset \mathcal{Y}_s$. Then upon integrating over $\Omega \times [0, \tau]$ we get

$$\int_0^\tau \int_\Omega \mathbf{u}_r^\varepsilon \cdot \varphi_1(\mathbf{x}, t)\varphi_2\left(\frac{\mathbf{x}}{\varepsilon}\right) d\mathbf{x}dt = 0.$$

Passing to the limit as $\varepsilon \searrow 0$ we deduce the identity

$$\int_0^\tau \int_\Omega \mathbf{u}_r^* \cdot \varphi_1(\mathbf{x}, t) \int_{\mathcal{Y}} \varphi_2(\mathbf{y}) d\mathbf{y} d\mathbf{x}dt = 0.$$

Due to arbitrariness of $\varphi_1(\mathbf{x}, t)$ and $\varphi_2(\mathbf{y})$ we get that $\mathbf{u}_r^* = 0$ in Q .

Thus we proved the assertion in Theorem 1 that $\mathbf{u}_r^\varepsilon \rightarrow 0$ weakly in $H^1(Q)$.

Remark 7. Using the same arguments, as in Remark 6, on the strength of representations (2.2b) and (2.4b), notice that if families of initial data $\{\mathbf{u}_r^{0\varepsilon}\}_{\varepsilon>0}$ and $\{\mathbf{v}_r^{0\varepsilon}\}_{\varepsilon>0}$ are uniformly bounded with respect to ε then the limiting relations (3.1)₂ and (3.2)₂ hold true. In this sense, the hypotheses (3.1)₂ and (3.2)₂ in the formulation of Theorem 1 can be slightly weakened.

Let us substitute the test function of the form

$$\Phi = \varphi_1(\mathbf{x}, t) + \varepsilon \varphi_2\left(\mathbf{x}, t, \frac{\mathbf{x}}{\varepsilon}\right)$$

into the integral equality (2.7a) and the test function of the form

$$\Psi = \varepsilon \psi_1(\mathbf{x}, t) \psi_2\left(\frac{\mathbf{x}}{\varepsilon}\right),$$

into the integral equality (2.7b). Here $\varphi_1(\mathbf{x}, t)$, $\varphi_2(\mathbf{x}, t, \mathbf{y})$, and $\psi_1(\mathbf{x}, t)$ are arbitrary smooth functions vanishing in the neighborhood of the boundary $\partial\Omega$, $\mathbf{y} \mapsto \varphi_2(\mathbf{x}, t, \mathbf{y})$ is a 1-periodic function in \mathbf{y} , and $\psi = \psi_2(\mathbf{y})$ is an arbitrary smooth 1-periodic function such that $\text{supp } \psi_2 \subset \mathcal{Y}_f$. Working out the limiting transitions in these integral equalities, as $\varepsilon \searrow 0$, and choosing a proper subsequence of $\{\varepsilon > 0, \varepsilon^{-1} \in \mathbb{N}\}$ (if necessary), on the strength of the relations (3.1), (3.2), (4.5)–(4.11), and the equality $\mathbf{u}_r^* = 0$, we derive the system consisting of the three homogenized two-scale integral equalities.

From the equation (2.7a) we deduce the two following integral equalities:

$$\begin{aligned} (4.12a) \quad & \int_0^\tau \int_\Omega \left\{ (\rho_f^0 |\mathcal{Y}_f| + \rho_s |\mathcal{Y}_s|) \frac{\partial \mathbf{u}_c^*}{\partial t} \cdot \frac{\partial \varphi_1}{\partial t} - \alpha_c^0 (|\mathcal{Y}_f| \text{div}_x \mathbf{u}_c^* + \langle \bar{\chi} \text{div}_y \mathbf{U}_c \rangle_y) \text{div}_x \varphi_1 \right. \\ & - \left(\left(\nu_1 - \frac{2}{3} \mu_1 \right) m_1 + \left(\nu_2 - \frac{2}{3} \mu_2 \right) m_2 \right) \left(|\mathcal{Y}_f| \text{div}_x \frac{\partial \mathbf{u}_c^*}{\partial t} + \left\langle \bar{\chi} \text{div}_y \frac{\partial \mathbf{U}_c}{\partial t} \right\rangle_y \right) \text{div}_x \varphi_1 \\ & - \left(\eta - \frac{2}{3} \lambda \right) (|\mathcal{Y}_s| \text{div}_x \mathbf{u}_c^* + \langle (1 - \bar{\chi}) \text{div}_y \mathbf{U}_c \rangle_y) \text{div}_x \varphi_1 \\ & - 2(\mu_1 m_1 + \mu_2 m_2) \left(|\mathcal{Y}_f| \mathbb{D} \left(x, \frac{\partial \mathbf{u}_c^*}{\partial t} \right) + \left\langle \bar{\chi} \mathbb{D} \left(y, \frac{\partial \mathbf{U}_c}{\partial t} \right) \right\rangle_y \right) : \mathbb{D}(x, \varphi_1) \\ & - 2\lambda (|\mathcal{Y}_s| \mathbb{D}(x, \mathbf{u}_c^*) + \langle (1 - \bar{\chi}) \mathbb{D}(y, \mathbf{U}_c) \rangle_y) : \mathbb{D}(x, \varphi_1) \\ & - \frac{m_1 m_2}{\rho_f^0} \left[\alpha_c^0 (\rho_{1f}^0 - \rho_{2f}^0) \langle \bar{\chi} \text{div}_y \mathbf{U}_r \rangle_y \text{div}_x \varphi_1 \right. \\ & \left. + \left(\rho_{1f}^0 \left(\nu_2 - \frac{2}{3} \mu_2 \right) - \rho_{2f}^0 \left(\nu_1 - \frac{2}{3} \mu_1 \right) \right) \left\langle \bar{\chi} \text{div}_y \frac{\partial \mathbf{U}_r}{\partial t} \right\rangle_y \text{div}_x \varphi_1 \right. \\ & \left. + 2(\rho_{1f}^0 \mu_2 - \rho_{2f}^0 \mu_1) \left\langle \bar{\chi} \mathbb{D} \left(y, \frac{\partial \mathbf{U}_r}{\partial t} \right) \right\rangle_y : \mathbb{D}(x, \varphi_1) \right] + (\rho_f^0 |\mathcal{Y}_f| + \rho_s |\mathcal{Y}_s|) \mathbf{g} \cdot \varphi_1 \Big\} dx dt \\ & = (\rho_f^0 |\mathcal{Y}_f| + \rho_s |\mathcal{Y}_s|) \int_\Omega \frac{\partial \mathbf{u}_c^*}{\partial t}(\mathbf{x}, \tau) \cdot \varphi_1(\mathbf{x}, \tau) d\mathbf{x} \\ & \quad - (\rho_f^0 |\mathcal{Y}_f| + \rho_s |\mathcal{Y}_s|) \int_\Omega \mathbf{v}_c^{0*}(\mathbf{x}) \cdot \varphi_1(\mathbf{x}, 0) d\mathbf{x}, \end{aligned}$$

$$\begin{aligned} (4.12b) \quad & \int_0^\tau \int_\Omega \left\langle \bar{\chi} \alpha_c^0 (\text{div}_x \mathbf{u}_c^* + \text{div}_y \mathbf{U}_c) \text{div}_y \varphi_2 \right. \\ & + \bar{\chi} \left(\left(\nu_1 - \frac{2}{3} \mu_1 \right) m_1 + \left(\nu_2 - \frac{2}{3} \mu_2 \right) m_2 \right) \left(\text{div}_x \frac{\partial \mathbf{u}_c^*}{\partial t} + \text{div}_y \frac{\partial \mathbf{U}_c}{\partial t} \right) \text{div}_y \varphi_2 \\ & + (1 - \bar{\chi}) \left(\eta - \frac{2}{3} \lambda \right) (\text{div}_x \mathbf{u}_c^* + \text{div}_y \mathbf{U}_c) \text{div}_y \varphi_2 \\ & + 2\bar{\chi} (\mu_1 m_1 + \mu_2 m_2) \left(\mathbb{D} \left(x, \frac{\partial \mathbf{u}_c^*}{\partial t} \right) + \mathbb{D} \left(y, \frac{\partial \mathbf{U}_c}{\partial t} \right) \right) : \mathbb{D}(y, \varphi_2) \\ & \left. + 2(1 - \bar{\chi}) \lambda (\mathbb{D}(x, \mathbf{u}_c^*) + \mathbb{D}(y, \mathbf{U}_c)) : \mathbb{D}(y, \varphi_2) \right\} \end{aligned}$$

$$\begin{aligned}
& + \bar{\chi} \frac{m_1 m_2}{\rho_f^0} \left[\alpha_c^0 (\rho_{1f}^0 - \rho_{2f}^0) \operatorname{div}_y \mathbf{U}_r \operatorname{div}_y \boldsymbol{\varphi}_2 \right. \\
& \quad + \left(\rho_{1f}^0 \left(\nu_2 - \frac{2}{3} \mu_2 \right) - \rho_{2f}^0 \left(\nu_1 - \frac{2}{3} \mu_1 \right) \right) \operatorname{div}_y \frac{\partial \mathbf{U}_r}{\partial t} \operatorname{div}_y \boldsymbol{\varphi}_2 \\
& \quad \left. + 2(\rho_{1f}^0 \mu_2 - \rho_{2f}^0 \mu_1) \mathbb{D} \left(y, \frac{\partial \mathbf{U}_r}{\partial t} \right) : \mathbb{D}(y, \boldsymbol{\varphi}_2) \right] \Big|_y d\mathbf{x} dt = 0.
\end{aligned}$$

From the equation (2.7b) we deduce

$$\begin{aligned}
(4.12c) \quad & \int_0^\tau \int_\Omega \psi_1 \left\langle \frac{m_1 m_2}{\rho_f^0} \left[\alpha_r^0 (\rho_{1f}^0 - \rho_{2f}^0) \operatorname{div}_y \mathbf{U}_r \operatorname{div}_y \boldsymbol{\psi}_2 \right. \right. \\
& \quad + \left(\rho_{2f}^0 \left(\nu_1 - \frac{2}{3} \mu_1 \right) + \rho_{1f}^0 \left(\nu_2 - \frac{2}{3} \mu_2 \right) \right) \operatorname{div}_y \frac{\partial \mathbf{U}_r}{\partial t} \operatorname{div}_y \boldsymbol{\psi}_2 \\
& \quad \left. + 2(\rho_{2f}^0 \mu_1 + \rho_{1f}^0 \mu_2) \mathbb{D} \left(y, \frac{\partial \mathbf{U}_r}{\partial t} \right) : \mathbb{D}(y, \boldsymbol{\psi}_2) \right] \\
& \quad + \alpha_r^0 (\operatorname{div}_x \mathbf{u}_c^* + \operatorname{div}_y \mathbf{U}_c) \operatorname{div}_y \boldsymbol{\psi}_2 \\
& \quad + \left(\left(\nu_2 - \frac{2}{3} \mu_2 \right) m_2 - \left(\nu_1 - \frac{2}{3} \mu_1 \right) m_1 \right) \left(\operatorname{div}_x \frac{\partial \mathbf{u}_c^*}{\partial t} + \operatorname{div}_y \frac{\partial \mathbf{U}_c}{\partial t} \right) \operatorname{div}_y \boldsymbol{\psi}_2 \\
& \quad \left. + 2(m_2 \mu_2 - m_1 \mu_1) \left(\mathbb{D} \left(x, \frac{\partial \mathbf{u}_c^*}{\partial t} \right) + \mathbb{D} \left(y, \frac{\partial \mathbf{U}_c}{\partial t} \right) \right) : \mathbb{D}(y, \boldsymbol{\psi}_2) \right\rangle_{y_f} d\mathbf{x} dt = 0.
\end{aligned}$$

In the sense of the theory of distributions, the system (4.12a)–(4.12c) is equivalent to the system of nine scalar differential equations for the nine unknown functions $u_{ci}^*(\mathbf{x}, t)$, $U_{ci}(\mathbf{x}, t, \mathbf{y})$, $U_{ri}(\mathbf{x}, t, \mathbf{y})$ ($i = 1, 2, 3$). Hence, this is a closed system.

Proposition 2. *For any given $\mathbf{u}_c^*|_{t=0} \in W_2^1(\Omega)$, $\mathbf{v}_c^0 \in L^2(Q)$, the system (4.12a)–(4.12c) has a unique weak generalized solution $\mathbf{u}_c^* \in W_2^1(Q)$, $\mathbf{U}_c, \mathbf{U}_r \in L^2(Q \times \mathcal{Y})$ satisfying the regularity conditions (4.1)–(4.3).*

PROOF. Existence of solutions has already been proved by means of the limiting transition, as $\varepsilon \searrow 0$, in Problem A. Uniqueness of solutions is proved quite analogously to [12, Lemma 5], [15, Lemma 6]. \square

5. PROOF OF THEOREM 1: ASYMPTOTIC DECOMPOSITION AND DEDUCTION OF THE EFFECTIVE HOMOGENIZED EQUATIONS

Let us resolve the equations (4.12b) and (4.12c) with respect to the functions $U_c(\mathbf{x}, t, \mathbf{y})$ and $U_r(\mathbf{x}, t, \mathbf{y})$. In this calculation we think of the function $\mathbf{u}_c^*(\mathbf{x}, t)$ as to be given. We implement the asymptotic decomposition as follows.

Firstly, we change the sought functions by the formulas

$$(5.1) \quad U_c(\mathbf{x}, t, \mathbf{y}) = \bar{\chi} \left(\frac{m_1 \rho_{1f}^0}{\rho_f^0} U_1(\mathbf{x}, t, \mathbf{y}) + \frac{m_2 \rho_{2f}^0}{\rho_f^0} U_2(\mathbf{x}, t, \mathbf{y}) \right) + (1 - \bar{\chi}) U_3(\mathbf{x}, t, \mathbf{y}),$$

$$(5.2) \quad U_r(\mathbf{x}, t, \mathbf{y}) = \bar{\chi} \left(U_2(\mathbf{x}, t, \mathbf{y}) - U_1(\mathbf{x}, t, \mathbf{y}) \right).$$

Secondly, we seek the functions $U_1(\mathbf{x}, t, \mathbf{y})$, $U_2(\mathbf{x}, t, \mathbf{y})$, and $U_3(\mathbf{x}, t, \mathbf{y})$ in the forms

$$U_1(\mathbf{x}, t, \mathbf{y}) = \sum_{i,j=1}^3 \left[\frac{\partial u_{ci}^*}{\partial x_j}(\mathbf{x}, t) \mathbf{Z}_1^{ij}(\mathbf{y}) + \int_0^t \frac{\partial u_{ci}^*}{\partial x_j}(\mathbf{x}, \tau) \mathbf{Z}_2^{ij}(\mathbf{y}, t - \tau) d\tau \right],$$

$$\mathbf{U}_2(\mathbf{x}, t, \mathbf{y}) = \sum_{i,j=1}^3 \left[\frac{\partial u_{ci}^*}{\partial x_j}(\mathbf{x}, t) \mathbf{Z}_3^{ij}(\mathbf{y}) + \int_0^t \frac{\partial u_{ci}^*}{\partial x_j}(\mathbf{x}, \tau) \mathbf{Z}_4^{ij}(\mathbf{y}, t - \tau) d\tau \right],$$

$$\mathbf{U}_3(\mathbf{x}, t, \mathbf{y}) = \sum_{i,j=1}^3 \frac{\partial u_{ci}^*}{\partial x_j}(\mathbf{x}, t) \mathbf{Z}_5^{ij}(\mathbf{y}),$$

where the vector-functions \mathbf{Z}_k^{ij} , $k = 1, 2, 3, 4, 5$ are unknown and should be found.

We substitute (5.1) and (5.2) into (4.12b) and (4.12c). After very much technical but rather straightforward calculations we arrive at the following integral equalities:

$$(5.3)$$

$$\begin{aligned} & \int_0^\tau \int_\Omega \left\langle \sum_{i,j=1}^3 \frac{\partial u_{ci}^*}{\partial x_j}(\mathbf{x}, t) \left\{ \bar{\chi}(\mathbf{y}) \left[\left(\alpha_c^0 \delta_{ij} + \alpha_c^0 \frac{m_1 \rho_{1f}^0}{\rho_f^0} \operatorname{div}_y \mathbf{Z}_1^{ij}(\mathbf{y}) + \alpha_c^0 \frac{m_2 \rho_{2f}^0}{\rho_f^0} \operatorname{div}_y \mathbf{Z}_3^{ij}(\mathbf{y}) \right. \right. \right. \right. \\ & \quad + \left(m_1 \left(\nu_1 - \frac{2}{3} \mu_1 \right) + m_2 \left(\nu_2 - \frac{2}{3} \mu_2 \right) \right) \frac{m_1 \rho_{1f}^0}{\rho_f^0} \operatorname{div}_y \mathbf{Z}_2^{ij}(\mathbf{y}, 0) \\ & \quad + \left(m_1 \left(\nu_1 - \frac{2}{3} \mu_1 \right) + m_2 \left(\nu_2 - \frac{2}{3} \mu_2 \right) \right) \frac{m_2 \rho_{2f}^0}{\rho_f^0} \operatorname{div}_y \mathbf{Z}_4^{ij}(\mathbf{y}, 0) \\ & \quad + \frac{m_1 m_2}{\rho_f^0} \alpha_c^0 (\rho_{1f}^0 - \rho_{2f}^0) \operatorname{div}_y \mathbf{Z}_3^{ij}(\mathbf{y}) - \frac{m_1 m_2}{\rho_f^0} \alpha_c^0 (\rho_{1f}^0 - \rho_{2f}^0) \operatorname{div}_y \mathbf{Z}_1^{ij}(\mathbf{y}) \\ & \quad + \frac{m_1 m_2}{\rho_f^0} \left(\rho_{1f}^0 \left(\nu_2 - \frac{2}{3} \mu_2 \right) - \rho_{2f}^0 \left(\nu_1 - \frac{2}{3} \mu_1 \right) \right) \operatorname{div}_y \mathbf{Z}_4^{ij}(\mathbf{y}, 0) \\ & \quad \left. \left. \left. - \frac{m_1 m_2}{\rho_f^0} \left(\rho_{1f}^0 \left(\nu_2 - \frac{2}{3} \mu_2 \right) - \rho_{2f}^0 \left(\nu_1 - \frac{2}{3} \mu_1 \right) \right) \operatorname{div}_y \mathbf{Z}_2^{ij}(\mathbf{y}, 0) \right) \operatorname{div}_y \boldsymbol{\varphi}_2(\mathbf{x}, t, \mathbf{y}) \right. \right. \\ & \quad + \left(2(m_1 \mu_1 + m_2 \mu_2) \frac{m_1 \rho_{1f}^0}{\rho_f^0} \mathbb{D}(\mathbf{y}, \mathbf{Z}_2^{ij}(\mathbf{y}, 0)) + 2(m_1 \mu_1 + m_2 \mu_2) \frac{m_2 \rho_{2f}^0}{\rho_f^0} \mathbb{D}(\mathbf{y}, \mathbf{Z}_4^{ij}(\mathbf{y}, 0)) \right. \\ & \quad \left. \left. + 2(\rho_{1f}^0 \mu_2 - \rho_{2f}^0 \mu_1) \frac{m_1 m_2}{\rho_f^0} \mathbb{D}(\mathbf{y}, \mathbf{Z}_4^{ij}(\mathbf{y}, 0)) \right. \right. \\ & \quad \left. \left. - 2(\rho_{1f}^0 \mu_2 - \rho_{2f}^0 \mu_1) \frac{m_1 m_2}{\rho_f^0} \mathbb{D}(\mathbf{y}, \mathbf{Z}_2^{ij}(\mathbf{y}, 0)) \right) : \mathbb{D}(\mathbf{y}, \boldsymbol{\varphi}_2(\mathbf{x}, t, \mathbf{y})) \right] \\ & \quad + (1 - \bar{\chi}(\mathbf{y})) \left[\left(\left(\eta - \frac{2}{3} \lambda \right) \delta_{ij} + \left(\eta - \frac{2}{3} \lambda \right) \operatorname{div}_y \mathbf{Z}_5^{ij}(\mathbf{y}) \right) \operatorname{div}_y \boldsymbol{\varphi}_2(\mathbf{x}, t, \mathbf{y}) \right. \\ & \quad \left. + 2\lambda \sum_{k,l=1}^3 (1/2) (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \mathbb{D}_{kl}(\mathbf{y}, \boldsymbol{\varphi}_2(\mathbf{x}, t, \mathbf{y})) + 2\lambda \mathbb{D}(\mathbf{y}, \mathbf{Z}_5^{ij}(\mathbf{y})) : \mathbb{D}(\mathbf{y}, \boldsymbol{\varphi}_2(\mathbf{x}, t, \mathbf{y})) \right] \left. \right\} \\ & \quad + \sum_{i,j=1}^3 \int_0^t \frac{\partial u_{ci}^*}{\partial x_j}(\mathbf{x}, s) \left\{ \bar{\chi}(\mathbf{y}) \left[\left(\alpha_c^0 \frac{m_1 \rho_{1f}^0}{\rho_f^0} \operatorname{div}_y \mathbf{Z}_2^{ij}(\mathbf{y}, t-s) + \alpha_c^0 \frac{m_2 \rho_{2f}^0}{\rho_f^0} \operatorname{div}_y \mathbf{Z}_4^{ij}(\mathbf{y}, t-s) \right. \right. \right. \\ & \quad + \left(m_1 \left(\nu_1 - \frac{2}{3} \mu_1 \right) + m_2 \left(\nu_2 - \frac{2}{3} \mu_2 \right) \right) \frac{m_1 \rho_{1f}^0}{\rho_f^0} \operatorname{div}_y \frac{\partial \mathbf{Z}_2^{ij}}{\partial t}(\mathbf{y}, t-s) \\ & \quad + \left(m_1 \left(\nu_1 - \frac{2}{3} \mu_1 \right) + m_2 \left(\nu_2 - \frac{2}{3} \mu_2 \right) \right) \frac{m_2 \rho_{2f}^0}{\rho_f^0} \operatorname{div}_y \frac{\partial \mathbf{Z}_4^{ij}}{\partial t}(\mathbf{y}, t-s) \\ & \quad \left. \left. + \frac{m_1 m_2}{\rho_f^0} \alpha_c^0 (\rho_{1f}^0 - \rho_{2f}^0) \operatorname{div}_y \mathbf{Z}_4^{ij}(\mathbf{y}, t-s) - \frac{m_1 m_2}{\rho_f^0} \alpha_c^0 (\rho_{1f}^0 - \rho_{2f}^0) \operatorname{div}_y \mathbf{Z}_2^{ij}(\mathbf{y}, t-s) \right. \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{m_1 m_2}{\rho_f^0} \left(\rho_{1f}^0 \left(\nu_2 - \frac{2}{3} \mu_2 \right) - \rho_{2f}^0 \left(\nu_1 - \frac{2}{3} \mu_1 \right) \right) \operatorname{div}_y \frac{\partial \mathbf{Z}_4^{ij}}{\partial t}(\mathbf{y}, t-s) \\
& - \frac{m_1 m_2}{\rho_f^0} \left(\rho_{1f}^0 \left(\nu_2 - \frac{2}{3} \mu_2 \right) - \rho_{2f}^0 \left(\nu_1 - \frac{2}{3} \mu_1 \right) \right) \operatorname{div}_y \frac{\partial \mathbf{Z}_2^{ij}}{\partial t}(\mathbf{y}, t-s) \operatorname{div}_y \varphi_2(\mathbf{x}, t, \mathbf{y}) \\
& \quad + \left(2(m_1 \mu_1 + m_2 \mu_2) \frac{m_1 \rho_{1f}^0}{\rho_f^0} \mathbb{D} \left(y, \frac{\partial \mathbf{Z}_2^{ij}}{\partial t}(\mathbf{y}, t-s) \right) \right. \\
& \quad \left. + 2(m_1 \mu_1 + m_2 \mu_2) \frac{m_2 \rho_{2f}^0}{\rho_f^0} \mathbb{D} \left(y, \frac{\partial \mathbf{Z}_4^{ij}}{\partial t}(\mathbf{y}, t-s) \right) \right. \\
& \left. + 2(\rho_{1f}^0 \mu_2 - \rho_{2f}^0 \mu_1) \frac{m_1 m_2}{\rho_f^0} \mathbb{D} \left(y, \frac{\partial \mathbf{Z}_4^{ij}}{\partial t}(\mathbf{y}, t-s) - \frac{\partial \mathbf{Z}_2^{ij}}{\partial t}(\mathbf{y}, t-s) \right) : \mathbb{D}(y, \varphi_2(\mathbf{x}, t, \mathbf{y})) \right] \} ds \\
& \quad + \sum_{i,j=1}^3 \frac{\partial^2 u_{ci}^*}{\partial t \partial x_j}(\mathbf{x}, t) \left\{ \bar{\chi}(\mathbf{y}) \left[\left((m_1 \left(\nu_1 - \frac{2}{3} \mu_1 \right) + m_2 \left(\nu_2 - \frac{2}{3} \mu_2 \right)) \right) \delta_{ij} \right. \right. \\
& \quad + \left(m_1 \left(\nu_1 - \frac{2}{3} \mu_1 \right) + m_2 \left(\nu_2 - \frac{2}{3} \mu_2 \right) \right) \frac{m_1 \rho_{1f}^0}{\rho_f^0} \operatorname{div}_y \mathbf{Z}_1^{ij}(\mathbf{y}) \\
& \quad + \left(m_1 \left(\nu_1 - \frac{2}{3} \mu_1 \right) + m_2 \left(\nu_2 - \frac{2}{3} \mu_2 \right) \right) \frac{m_2 \rho_{2f}^0}{\rho_f^0} \operatorname{div}_y \mathbf{Z}_3^{ij}(\mathbf{y}) \\
& \quad + \frac{m_1 m_2}{\rho_f^0} \left(\rho_{1f}^0 \left(\nu_2 - \frac{2}{3} \mu_2 \right) - \rho_{2f}^0 \left(\nu_1 - \frac{2}{3} \mu_1 \right) \right) \operatorname{div}_y \mathbf{Z}_3^{ij}(\mathbf{y}) \\
& \quad \left. - \frac{m_1 m_2}{\rho_f^0} \left(\rho_{1f}^0 \left(\nu_2 - \frac{2}{3} \mu_2 \right) - \rho_{2f}^0 \left(\nu_1 - \frac{2}{3} \mu_1 \right) \right) \operatorname{div}_y \mathbf{Z}_1^{ij}(\mathbf{y}) \operatorname{div}_y \varphi_2(\mathbf{x}, t, \mathbf{y}) \right. \\
& \quad \left. + 2(m_1 \mu_1 + m_2 \mu_2) \sum_{k,l=1}^3 (1/2) (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \mathbb{D}_{kl}(y, \varphi_2(\mathbf{x}, t, \mathbf{y})) \right. \\
& \quad \left. + \left(2(m_1 \mu_1 + m_2 \mu_2) \frac{m_1 \rho_{1f}^0}{\rho_f^0} \mathbb{D}(y, \mathbf{Z}_1^{ij}(\mathbf{y})) + 2(m_1 \mu_1 + m_2 \mu_2) \frac{m_2 \rho_{2f}^0}{\rho_f^0} \mathbb{D}(y, \mathbf{Z}_3^{ij}(\mathbf{y})) \right. \right. \\
& \left. \left. + 2(\rho_{1f}^0 \mu_2 - \rho_{2f}^0 \mu_1) \frac{m_1 m_2}{\rho_f^0} \mathbb{D}(y, \mathbf{Z}_3^{ij}(\mathbf{y}) - \mathbf{Z}_1^{ij}(\mathbf{y})) : \mathbb{D}(y, \varphi_2(\mathbf{x}, t, \mathbf{y})) \right] \right\} \} \Big|_y dx dt = 0,
\end{aligned}$$

(5.4)

$$\begin{aligned}
& \int_0^\tau \int_\Omega \psi_1(\mathbf{x}, t) \left\langle \sum_{i,j=1}^3 \frac{\partial u_{ci}^*}{\partial x_j}(\mathbf{x}, t) \left\{ \left[\frac{m_1 m_2}{\rho_f^0} \alpha_r^0 (\rho_{1f}^0 - \rho_{2f}^0) \operatorname{div}_y (\mathbf{Z}_3^{ij}(\mathbf{y}) - \mathbf{Z}_1^{ij}(\mathbf{y})) \right. \right. \right. \\
& \quad + \frac{m_1 m_2}{\rho_f^0} \left(\rho_{2f}^0 \left(\nu_1 - \frac{2}{3} \mu_1 \right) + \rho_{1f}^0 \left(\nu_2 - \frac{2}{3} \mu_2 \right) \right) \operatorname{div}_y (\mathbf{Z}_4^{ij}(\mathbf{y}, 0) - \mathbf{Z}_2^{ij}(\mathbf{y}, 0)) \\
& \quad + \alpha_r^0 \delta_{ij} + \alpha_r^0 \frac{m_1 \rho_{1f}^0}{\rho_f^0} \operatorname{div}_y \mathbf{Z}_1^{ij}(\mathbf{y}) + \alpha_r^0 \frac{m_2 \rho_{2f}^0}{\rho_f^0} \operatorname{div}_y \mathbf{Z}_3^{ij}(\mathbf{y}) \\
& \quad + \left(m_2 \left(\nu_2 - \frac{2}{3} \mu_2 \right) - m_1 \left(\nu_1 - \frac{2}{3} \mu_1 \right) \right) \frac{m_1 \rho_{1f}^0}{\rho_f^0} \operatorname{div}_y \mathbf{Z}_2^{ij}(\mathbf{y}, 0) \\
& \quad \left. \left. + \left(m_2 \left(\nu_2 - \frac{2}{3} \mu_2 \right) - m_1 \left(\nu_1 - \frac{2}{3} \mu_1 \right) \right) \frac{m_2 \rho_{2f}^0}{\rho_f^0} \operatorname{div}_y \mathbf{Z}_4^{ij}(\mathbf{y}, 0) \right] \operatorname{div}_y \psi_2(\mathbf{y}) \right. \\
& \quad \left. \left. \left. \right\} \right\rangle
\end{aligned}$$

$$\begin{aligned}
& + 2 \frac{m_1 m_2}{\rho_f^0} (\rho_{2f}^0 \mu_1 + \rho_{1f}^0 \mu_2) \mathbb{D}(y, \mathbf{Z}_4^{ij}(\mathbf{y}, 0) - \mathbf{Z}_2^{ij}(\mathbf{y}, 0)) \\
& + 2(m_2 \mu_2 - m_1 \mu_1) \left(\frac{m_1 \rho_{1f}^0}{\rho_f^0} \mathbb{D}(y, \mathbf{Z}_2^{ij}(\mathbf{y}, 0)) + \frac{m_2 \rho_{2f}^0}{\rho_f^0} \mathbb{D}(y, \mathbf{Z}_4^{ij}(\mathbf{y}, 0)) : \mathbb{D}(y, \boldsymbol{\psi}_2(\mathbf{y})) \right) \Big\} \\
& + \sum_{i,j=1}^3 \int_0^t \frac{\partial u_{ci}^*}{\partial x_j}(\mathbf{x}, s) \left\{ \left[\alpha_r^0 (\rho_{1f}^0 - \rho_{2f}^0) \frac{m_1 m_2}{\rho_f^0} \operatorname{div}_y (\mathbf{Z}_4^{ij}(\mathbf{y}, t-s) - \mathbf{Z}_2^{ij}(\mathbf{y}, t-s)) \right. \right. \\
& + \frac{m_1 m_2}{\rho_f^0} \left(\rho_{2f}^0 \left(\nu_1 - \frac{2}{3} \mu_1 \right) + \rho_{1f}^0 \left(\nu_2 - \frac{2}{3} \mu_2 \right) \right) \operatorname{div}_y \frac{\partial (\mathbf{Z}_4^{ij}(\mathbf{y}, t-s) - \mathbf{Z}_2^{ij}(\mathbf{y}, t-s))}{\partial t} \\
& \quad + \alpha_r^0 \frac{m_1 \rho_{1f}^0}{\rho_f^0} \operatorname{div}_y \mathbf{Z}_2^{ij}(\mathbf{y}, t-s) + \alpha_r^0 \frac{m_2 \rho_{2f}^0}{\rho_f^0} \operatorname{div}_y \mathbf{Z}_4^{ij}(\mathbf{y}, t-s) \\
& \quad + \left(m_2 \left(\nu_2 - \frac{2}{3} \mu_2 \right) - m_1 \left(\nu_1 - \frac{2}{3} \mu_1 \right) \right) \frac{m_1 \rho_{1f}^0}{\rho_f^0} \operatorname{div}_y \frac{\partial \mathbf{Z}_2^{ij}}{\partial t}(\mathbf{y}, t-s) \\
& \quad + \left. \left(m_2 \left(\nu_2 - \frac{2}{3} \mu_2 \right) - m_1 \left(\nu_1 - \frac{2}{3} \mu_1 \right) \right) \frac{m_2 \rho_{2f}^0}{\rho_f^0} \operatorname{div}_y \frac{\partial \mathbf{Z}_4^{ij}}{\partial t}(\mathbf{y}, t-s) \right] \operatorname{div}_y \boldsymbol{\psi}_2(\mathbf{y}) \\
& \quad + \left[2 \frac{m_1 m_2}{\rho_f^0} (\rho_{2f}^0 \mu_1 + \rho_{1f}^0 \mu_2) \mathbb{D}\left(y, \frac{\partial (\mathbf{Z}_4^{ij}(\mathbf{y}, t-s) - \mathbf{Z}_2^{ij}(\mathbf{y}, t-s))}{\partial t}\right) \right. \\
& \quad \quad + 2(m_2 \mu_2 - m_1 \mu_1) \frac{m_1 \rho_{1f}^0}{\rho_f^0} \mathbb{D}\left(y, \frac{\partial \mathbf{Z}_2^{ij}}{\partial t}(\mathbf{y}, t-s)\right) \\
& \quad \quad \left. + 2(m_2 \mu_2 - m_1 \mu_1) \frac{m_2 \rho_{2f}^0}{\rho_f^0} \mathbb{D}\left(y, \frac{\partial \mathbf{Z}_4^{ij}}{\partial t}(\mathbf{y}, t-s)\right) \right] : \mathbb{D}(y, \boldsymbol{\psi}_2(\mathbf{y})) \Big\} ds \\
& + \sum_{i,j=1}^3 \frac{\partial^2 u_{ci}^*}{\partial t \partial x_j}(\mathbf{x}, t) \left\{ \left[\frac{m_1 m_2}{\rho_f^0} \left(\rho_{2f}^0 \left(\nu_1 - \frac{2}{3} \mu_1 \right) + \rho_{1f}^0 \left(\nu_2 - \frac{2}{3} \mu_2 \right) \right) \operatorname{div}_y (\mathbf{Z}_3^{ij}(\mathbf{y}) - \mathbf{Z}_1^{ij}(\mathbf{y})) \right. \right. \\
& \quad + \left(m_2 \left(\nu_2 - \frac{2}{3} \mu_2 \right) - m_1 \left(\nu_1 - \frac{2}{3} \mu_1 \right) \right) \left(\delta_{ij} + \frac{m_1 \rho_{1f}^0}{\rho_f^0} \operatorname{div}_y \mathbf{Z}_1^{ij}(\mathbf{y}) \right) \\
& \quad + \left(m_2 \left(\nu_2 - \frac{2}{3} \mu_2 \right) - m_1 \left(\nu_1 - \frac{2}{3} \mu_1 \right) \right) \frac{m_2 \rho_{2f}^0}{\rho_f^0} \operatorname{div}_y \mathbf{Z}_3^{ij}(\mathbf{y}) \Big] \operatorname{div}_y \boldsymbol{\psi}_2(\mathbf{y}) \\
& \quad + \left[2 \frac{m_1 m_2}{\rho_f^0} (\rho_{2f}^0 \mu_1 + \rho_{1f}^0 \mu_2) \mathbb{D}(y, \mathbf{Z}_3^{ij}(\mathbf{y}) - \mathbf{Z}_1^{ij}(\mathbf{y})) \right. \\
& \quad \quad + 2(m_2 \mu_2 - m_1 \mu_1) \sum_{k,l=1}^3 (1/2) (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \mathbb{D}_{kl}(y, \boldsymbol{\psi}_2(\mathbf{y})) \\
& \quad \quad + 2(m_2 \mu_2 - m_1 \mu_1) \frac{m_1 \rho_{1f}^0}{\rho_f^0} \mathbb{D}(y, \mathbf{Z}_1^{ij}(\mathbf{y})) \\
& \quad \quad \left. + 2(m_2 \mu_2 - m_1 \mu_1) \frac{m_2 \rho_{2f}^0}{\rho_f^0} \mathbb{D}(y, \mathbf{Z}_3^{ij}(\mathbf{y})) \right] : \mathbb{D}(y, \boldsymbol{\psi}_2(\mathbf{y})) \Big\} \Big\}_{\mathcal{V}_f} d\mathbf{x} dt = 0.
\end{aligned}$$

Notice that we have aggregated the terms in the integral equalities (5.3) and (5.4) so that we can clearly see that these equalities are valid for all possible functions \mathbf{u}_c^* independently of a choice of test functions $\boldsymbol{\psi}_1$, $\boldsymbol{\varphi}_2$ and $\boldsymbol{\psi}_2$, if we succeed that

the functions \mathbf{Z}_1^{ij} , \mathbf{Z}_2^{ij} , \mathbf{Z}_3^{ij} , \mathbf{Z}_4^{ij} , and \mathbf{Z}_5^{ij} are the solutions of the problems defined on the pattern cell \mathcal{Y} . These problems are formulated below in accordance with the method outlined in [11]. The constants A_1, \dots, A_{16} in these problems are explicitly written out below in the formulas (5.9a)–(5.9p).

Problem C 1. *It is necessary to find the vector-function $\mathbf{Z}_5^{ij} \in W_2^1(\mathcal{Y}_s)$ ($i, j = 1, 2, 3$) satisfying the equation*

$$(5.5a) \quad \operatorname{div}_y \left((\eta - (2/3)\lambda)(\delta_{ij}\mathbb{I} + \operatorname{div}_y \mathbf{Z}_5^{ij}(\mathbf{y})\mathbb{I}) + 2\lambda(\mathbb{J}^{ij} + \mathbb{D}(y, \mathbf{Z}_5^{ij}(\mathbf{y}))) \right) = 0 \quad \text{on } \mathcal{Y}_s,$$

the boundary condition

$$(5.5b) \quad \left((\eta - (2/3)\lambda)(\delta_{ij}\mathbb{I} + \operatorname{div}_y \mathbf{Z}_5^{ij}(\mathbf{y})\mathbb{I}) + 2\lambda(\mathbb{J}^{ij} + \mathbb{D}(y, \mathbf{Z}_5^{ij}(\mathbf{y}))) \right) \mathbf{n} = 0 \\ \text{on } \partial\mathcal{Y}_s \setminus \partial\mathcal{Y},$$

and the conditions of periodicity and normalization

$$(5.5c) \quad \mathbf{Z}_5^{ij}(\mathbf{y}) \text{ is 1-periodic and } \langle \mathbf{Z}_5^{ij} \rangle_{\mathcal{Y}_s} = 0.$$

Problem C 2. *It is necessary to find the vector-functions $\mathbf{Z}_1^{ij}, \mathbf{Z}_3^{ij} \in W_2^1(\mathcal{Y}_f)$ ($i, j = 1, 2, 3$) satisfying the system of equations*

$$(5.6a) \quad \operatorname{div}_y \left(A_1 \delta_{ij} \mathbb{I} + A_2 \mathbb{J}^{ij} + A_3 \operatorname{div}_y \mathbf{Z}_1^{ij}(\mathbf{y})\mathbb{I} + A_4 \operatorname{div}_y \mathbf{Z}_3^{ij}(\mathbf{y})\mathbb{I} + A_5 \mathbb{D}(y, \mathbf{Z}_1^{ij}(\mathbf{y})) \right. \\ \left. + A_6 \mathbb{D}(y, \mathbf{Z}_3^{ij}(\mathbf{y})) \right) = 0 \quad \text{on } \mathcal{Y}_f,$$

(5.6b)

$$\operatorname{div}_y \left(A_7 \delta_{ij} \mathbb{I} + A_8 \mathbb{J}^{ij} + A_9 \operatorname{div}_y \mathbf{Z}_1^{ij}(\mathbf{y})\mathbb{I} + A_{10} \operatorname{div}_y \mathbf{Z}_3^{ij}(\mathbf{y})\mathbb{I} + A_{11} \mathbb{D}(y, \mathbf{Z}_1^{ij}(\mathbf{y})) \right. \\ \left. + A_{12} \mathbb{D}(y, \mathbf{Z}_3^{ij}(\mathbf{y})) \right) = 0 \quad \text{on } \mathcal{Y}_f,$$

the boundary conditions

$$(5.6c) \quad \left(A_1 \delta_{ij} \mathbb{I} + A_2 \mathbb{J}^{ij} + A_3 \operatorname{div}_y \mathbf{Z}_1^{ij}(\mathbf{y})\mathbb{I} + A_4 \operatorname{div}_y \mathbf{Z}_3^{ij}(\mathbf{y})\mathbb{I} + A_5 \mathbb{D}(y, \mathbf{Z}_1^{ij}(\mathbf{y})) \right. \\ \left. + A_6 \mathbb{D}(y, \mathbf{Z}_3^{ij}(\mathbf{y})) \right) \mathbf{n} = 0 \quad \text{on } \partial\mathcal{Y}_f \setminus \partial\mathcal{Y},$$

(5.6d)

$$\left(A_7 \delta_{ij} \mathbb{I} + A_8 \mathbb{J}^{ij} + A_9 \operatorname{div}_y \mathbf{Z}_1^{ij}(\mathbf{y})\mathbb{I} + A_{10} \operatorname{div}_y \mathbf{Z}_3^{ij}(\mathbf{y})\mathbb{I} + A_{11} \mathbb{D}(y, \mathbf{Z}_1^{ij}(\mathbf{y})) \right. \\ \left. + A_{12} \mathbb{D}(y, \mathbf{Z}_3^{ij}(\mathbf{y})) \right) \mathbf{n} = 0 \quad \text{on } \partial\mathcal{Y}_f \setminus \partial\mathcal{Y},$$

and the conditions of periodicity and normalization

$$(5.6e) \quad \mathbf{Z}_1^{ij}(\mathbf{y}), \mathbf{Z}_3^{ij}(\mathbf{y}) \text{ are 1-periodic, } \langle \mathbf{Z}_1^{ij} \rangle_{\mathcal{Y}_f} = 0, \text{ and } \langle \mathbf{Z}_3^{ij} \rangle_{\mathcal{Y}_f} = 0.$$

Problem C 3. *It is necessary to find the initial data $\mathbf{Z}_2^{ij}(\cdot, 0), \mathbf{Z}_4^{ij}(\cdot, 0) \in W_2^1(\mathcal{Y}_f)$ ($i, j = 1, 2, 3$) satisfying the system of equations*

$$(5.7a) \quad \operatorname{div}_y \left(\alpha_c^0 \delta_{ij} \mathbb{I} + A_{13} \operatorname{div}_y \mathbf{Z}_1^{ij}(\mathbf{y})\mathbb{I} + A_{14} \operatorname{div}_y \mathbf{Z}_3^{ij}(\mathbf{y})\mathbb{I} + A_3 \operatorname{div}_y \mathbf{Z}_2^{ij}(\mathbf{y}, 0)\mathbb{I} \right. \\ \left. + A_4 \operatorname{div}_y \mathbf{Z}_4^{ij}(\mathbf{y}, 0)\mathbb{I} + A_5 \mathbb{D}(y, \mathbf{Z}_2^{ij}(\mathbf{y}, 0)) + A_6 \mathbb{D}(y, \mathbf{Z}_4^{ij}(\mathbf{y}, 0)) \right) = 0 \quad \text{on } \mathcal{Y}_f,$$

$$(5.7b) \quad \operatorname{div}_y \left(\alpha_r^0 \delta_{ij} \mathbb{I} + A_{15} \operatorname{div}_y \mathbf{Z}_1^{ij}(\mathbf{y}) \mathbb{I} + A_{16} \operatorname{div}_y \mathbf{Z}_3^{ij}(\mathbf{y}) \mathbb{I} + A_9 \operatorname{div}_y \mathbf{Z}_2^{ij}(\mathbf{y}, 0) \mathbb{I} \right. \\ \left. + A_{10} \operatorname{div}_y \mathbf{Z}_4^{ij}(\mathbf{y}, 0) \mathbb{I} + A_{11} \mathbb{D}(y, \mathbf{Z}_2^{ij}(\mathbf{y}, 0)) + A_{12} \mathbb{D}(y, \mathbf{Z}_4^{ij}(\mathbf{y}, 0)) \right) = 0 \quad \text{on } \mathcal{Y}_f,$$

the boundary conditions

$$(5.7c) \quad \left(\alpha_c^0 \delta_{ij} \mathbb{I} + A_{13} \operatorname{div}_y \mathbf{Z}_1^{ij}(\mathbf{y}) \mathbb{I} + A_{14} \operatorname{div}_y \mathbf{Z}_3^{ij}(\mathbf{y}) \mathbb{I} + A_3 \operatorname{div}_y \mathbf{Z}_2^{ij}(\mathbf{y}, 0) \mathbb{I} \right. \\ \left. + A_4 \operatorname{div}_y \mathbf{Z}_4^{ij}(\mathbf{y}, 0) \mathbb{I} + A_5 \mathbb{D}(y, \mathbf{Z}_2^{ij}(\mathbf{y}, 0)) + A_6 \mathbb{D}(y, \mathbf{Z}_4^{ij}(\mathbf{y}, 0)) \right) \mathbf{n} = 0 \\ \text{on } \partial \mathcal{Y}_f \setminus \partial \mathcal{Y},$$

$$(5.7d) \quad \left(\alpha_r^0 \delta_{ij} \mathbb{I} + A_{15} \operatorname{div}_y \mathbf{Z}_1^{ij}(\mathbf{y}) \mathbb{I} + A_{16} \operatorname{div}_y \mathbf{Z}_3^{ij}(\mathbf{y}) \mathbb{I} + A_9 \operatorname{div}_y \mathbf{Z}_2^{ij}(\mathbf{y}, 0) \mathbb{I} \right. \\ \left. + A_{10} \operatorname{div}_y \mathbf{Z}_4^{ij}(\mathbf{y}, 0) \mathbb{I} + A_{11} \mathbb{D}(y, \mathbf{Z}_2^{ij}(\mathbf{y}, 0)) + A_{12} \mathbb{D}(y, \mathbf{Z}_4^{ij}(\mathbf{y}, 0)) \right) \mathbf{n} = 0 \\ \text{on } \partial \mathcal{Y}_f \setminus \partial \mathcal{Y},$$

and the periodicity and normalization conditions

$$(5.7e) \quad \mathbf{Z}_2^{ij}(\mathbf{y}, 0), \mathbf{Z}_4^{ij}(\mathbf{y}, 0) \text{ are 1-periodic, } \langle \mathbf{Z}_2^{ij}(\cdot, 0) \rangle_{\mathcal{Y}_f} = 0, \text{ and } \langle \mathbf{Z}_3^{ij}(\cdot, 0) \rangle_{\mathcal{Y}_f} = 0.$$

Problem C 4. It is necessary to find the functions $\mathbf{Z}_2^{ij}, \mathbf{Z}_4^{ij} \in L^1(0, T; W_2^1(\mathcal{Y}_f))$ ($i, j = 1, 2, 3$) satisfying the system of equations

$$(5.8a) \quad \operatorname{div}_y \left(A_{13} \operatorname{div}_y \mathbf{Z}_2^{ij}(\mathbf{y}, t) \mathbb{I} + A_{14} \operatorname{div}_y \mathbf{Z}_4^{ij}(\mathbf{y}, t) \mathbb{I} + A_3 \operatorname{div}_y \frac{\partial \mathbf{Z}_2^{ij}}{\partial t}(\mathbf{y}, t) \mathbb{I} \right. \\ \left. + A_4 \operatorname{div}_y \frac{\partial \mathbf{Z}_4^{ij}}{\partial t}(\mathbf{y}, t) \mathbb{I} + A_5 \mathbb{D}(y, \frac{\partial \mathbf{Z}_2^{ij}}{\partial t}(\mathbf{y}, t)) + A_6 \mathbb{D}(y, \frac{\partial \mathbf{Z}_4^{ij}}{\partial t}(\mathbf{y}, t)) \right) = 0 \\ \text{on } \mathcal{Y}_f,$$

$$(5.8b) \quad \operatorname{div}_y \left[\chi(\mathbf{y}) \left(A_{15} \operatorname{div}_y \mathbf{Z}_2^{ij}(\mathbf{y}, t) \mathbb{I} + A_{16} \operatorname{div}_y \mathbf{Z}_4^{ij}(\mathbf{y}, t) \mathbb{I} + A_9 \operatorname{div}_y \frac{\partial \mathbf{Z}_2^{ij}}{\partial t}(\mathbf{y}, t) \mathbb{I} \right. \right. \\ \left. \left. + A_{10} \operatorname{div}_y \frac{\partial \mathbf{Z}_4^{ij}}{\partial t}(\mathbf{y}, t) \mathbb{I} + A_{11} \mathbb{D}(y, \frac{\partial \mathbf{Z}_2^{ij}}{\partial t}(\mathbf{y}, t)) + A_{12} \mathbb{D}(y, \frac{\partial \mathbf{Z}_4^{ij}}{\partial t}(\mathbf{y}, t)) \right) \right] = 0 \\ \text{on } \mathcal{Y}_f,$$

the boundary conditions

$$(5.8c) \quad \left(A_{13} \operatorname{div}_y \mathbf{Z}_2^{ij}(\mathbf{y}, t) \mathbb{I} + A_{14} \operatorname{div}_y \mathbf{Z}_4^{ij}(\mathbf{y}, t) \mathbb{I} + A_3 \operatorname{div}_y \frac{\partial \mathbf{Z}_2^{ij}}{\partial t}(\mathbf{y}, t) \mathbb{I} \right. \\ \left. + A_4 \operatorname{div}_y \frac{\partial \mathbf{Z}_4^{ij}}{\partial t}(\mathbf{y}, t) \mathbb{I} + A_5 \mathbb{D}(y, \frac{\partial \mathbf{Z}_2^{ij}}{\partial t}(\mathbf{y}, t)) + A_6 \mathbb{D}(y, \frac{\partial \mathbf{Z}_4^{ij}}{\partial t}(\mathbf{y}, t)) \right) \mathbf{n} = 0 \\ \text{on } \partial \mathcal{Y}_f \setminus \partial \mathcal{Y},$$

$$(5.8d) \quad \left(A_{15} \operatorname{div}_y \mathbf{Z}_2^{ij}(\mathbf{y}, t) \mathbb{I} + A_{16} \operatorname{div}_y \mathbf{Z}_4^{ij}(\mathbf{y}, t) \mathbb{I} + A_9 \operatorname{div}_y \frac{\partial \mathbf{Z}_2^{ij}}{\partial t}(\mathbf{y}, t) \mathbb{I} \right. \\ \left. + A_{10} \operatorname{div}_y \frac{\partial \mathbf{Z}_4^{ij}}{\partial t}(\mathbf{y}, t) \mathbb{I} + A_{11} \mathbb{D}(y, \frac{\partial \mathbf{Z}_2^{ij}}{\partial t}(\mathbf{y}, t)) + A_{12} \mathbb{D}(y, \frac{\partial \mathbf{Z}_4^{ij}}{\partial t}(\mathbf{y}, t)) \right) \mathbf{n} = 0 \\ \text{on } \partial \mathcal{Y}_f \setminus \partial \mathcal{Y},$$

the initial conditions

$$(5.8e) \quad \mathbf{Z}_2^{ij}|_{t=0} = \mathbf{Z}_2^{ij}(\mathbf{y}, 0), \quad \mathbf{Z}_4^{ij}|_{t=0} = \mathbf{Z}_4^{ij}(\mathbf{y}, 0), \quad \mathbf{y} \in \mathcal{Y}_f,$$

where

$$(5.8f) \quad \mathbf{Z}_2^{ij}(\mathbf{y}, 0) \text{ and } \mathbf{Z}_4^{ij}(\mathbf{y}, 0) \text{ are the solutions of Problem C3,}$$

and the conditions of periodicity and normalization

$$(5.8g) \quad \begin{aligned} &\mathbf{Z}_2^{ij}(\mathbf{y}, t), \mathbf{Z}_4^{ij}(\mathbf{y}, t) \text{ are 1-periodic in } \mathbf{y}, \\ &\langle \mathbf{Z}_2^{ij}(\cdot, t) \rangle_{\mathcal{Y}_f} = 0, \text{ and } \langle \mathbf{Z}_3^{ij}(\cdot, t) \rangle_{\mathcal{Y}_f} = 0, \quad t \in (0, T]. \end{aligned}$$

In (5.5a)–(5.6d) and further in the article by \mathbf{e}^j we denote the vectors of the standard Descartes basis in \mathbb{R}^3 and by \mathbb{J}^{ij} we denote the 3×3 -matrix $\mathbb{J}^{ij} \stackrel{def}{=} (1/2)(\mathbf{e}^i \otimes \mathbf{e}^j + \mathbf{e}^j \otimes \mathbf{e}^i)$, where $\mathbf{e}^k \otimes \mathbf{e}^l$ is a diad of the two basis vectors: for any $\mathbf{a} \in \mathbb{R}^3$ the diad is defined by the formula $(\mathbf{e}^k \otimes \mathbf{e}^l)\mathbf{a} = a_l \mathbf{e}^k$. Thus, \mathbb{J}^{ij} is the matrix with the components as follows: (i, j) -th and (j, i) -th components are equal to 1 and all other components vanish. In Problems C1–C4 the constants A_i are:

$$(5.9a) \quad A_1 := m_1 \left(\nu_1 - \frac{2}{3} \mu_1 \right) + m_2 \left(\nu_2 - \frac{2}{3} \mu_2 \right),$$

$$(5.9b) \quad A_2 := 2(m_1 \mu_1 + m_2 \mu_2),$$

$$(5.9c) \quad \begin{aligned} A_3 := &\frac{m_1 \rho_{1f}^0}{\rho_f^0} \left(m_1 \left(\nu_1 - \frac{2}{3} \mu_1 \right) + m_2 \left(\nu_2 - \frac{2}{3} \mu_2 \right) \right) \\ &- \frac{m_1 m_2}{\rho_f^0} \left(\rho_{1f}^0 \left(\nu_2 - \frac{2}{3} \mu_2 \right) - \rho_{2f}^0 \left(\nu_1 - \frac{2}{3} \mu_1 \right) \right), \end{aligned}$$

$$(5.9d) \quad \begin{aligned} A_4 := &\frac{m_2 \rho_{2f}^0}{\rho_f^0} \left(m_1 \left(\nu_1 - \frac{2}{3} \mu_1 \right) + m_2 \left(\nu_2 - \frac{2}{3} \mu_2 \right) \right) \\ &+ \frac{m_1 m_2}{\rho_f^0} \left(\rho_{1f}^0 \left(\nu_2 - \frac{2}{3} \mu_2 \right) - \rho_{2f}^0 \left(\nu_1 - \frac{2}{3} \mu_1 \right) \right), \end{aligned}$$

$$(5.9e) \quad A_5 := 2 \frac{m_1 \rho_{1f}^0}{\rho_f^0} (m_1 \mu_1 + m_2 \mu_2) - 2 \frac{m_1 m_2}{\rho_f^0} (\rho_{1f}^0 \mu_2 - \rho_{2f}^0 \mu_1),$$

$$(5.9f) \quad A_6 := 2 \frac{m_2 \rho_{2f}^0}{\rho_f^0} (m_1 \mu_1 + m_2 \mu_2) + 2 \frac{m_1 m_2}{\rho_f^0} (\rho_{1f}^0 \mu_2 - \rho_{2f}^0 \mu_1),$$

$$(5.9g) \quad A_7 := m_2 \left(\nu_2 - \frac{2}{3} \mu_2 \right) - m_1 \left(\nu_1 - \frac{2}{3} \mu_1 \right),$$

$$(5.9h) \quad A_8 := 2(m_2 \mu_2 - m_1 \mu_1)$$

$$(5.9i) \quad \begin{aligned} A_9 := &\frac{m_1 \rho_{1f}^0}{\rho_f^0} \left(m_2 \left(\nu_2 - \frac{2}{3} \mu_2 \right) - m_1 \left(\nu_1 - \frac{2}{3} \mu_1 \right) \right) \\ &- \frac{m_1 m_2}{\rho_f^0} \left(\rho_{2f}^0 \left(\nu_1 - \frac{2}{3} \mu_1 \right) + \rho_{1f}^0 \left(\nu_2 - \frac{2}{3} \mu_2 \right) \right), \end{aligned}$$

$$(5.9j) \quad A_{10} := \frac{m_2 \rho_{2f}^0}{\rho_f^0} \left(m_2 \left(\nu_2 - \frac{2}{3} \mu_2 \right) - m_1 \left(\nu_1 - \frac{2}{3} \mu_1 \right) \right) \\ + \frac{m_1 m_2}{\rho_f^0} \left(\rho_{2f}^0 \left(\nu_1 - \frac{2}{3} \mu_1 \right) + \rho_{1f}^0 \left(\nu_2 - \frac{2}{3} \mu_2 \right) \right),$$

$$(5.9k) \quad A_{11} := 2 \frac{m_1 \rho_{1f}^0}{\rho_f^0} (m_2 \mu_2 - m_1 \mu_1) - 2 \frac{m_1 m_2}{\rho_f^0} (\rho_{2f}^0 \mu_1 + \rho_{1f}^0 \mu_2),$$

$$(5.9l) \quad A_{12} := 2 \frac{m_2 \rho_{2f}^0}{\rho_f^0} (m_2 \mu_2 - m_1 \mu_1) + 2 \frac{m_1 m_2}{\rho_f^0} (\rho_{2f}^0 \mu_1 + \rho_{1f}^0 \mu_2),$$

$$(5.9m) \quad A_{13} := \alpha_c^0 \left(\frac{m_1 \rho_{1f}^0}{\rho_f^0} - \frac{m_1 m_2}{\rho_f^0} (\rho_{1f}^0 - \rho_{2f}^0) \right),$$

$$(5.9n) \quad A_{14} := \alpha_c^0 \left(\frac{m_2 \rho_{2f}^0}{\rho_f^0} + \frac{m_1 m_2}{\rho_f^0} (\rho_{1f}^0 - \rho_{2f}^0) \right),$$

$$(5.9o) \quad A_{15} := \alpha_r^0 \left(\frac{m_1 \rho_{1f}^0}{\rho_f^0} - \frac{m_1 m_2}{\rho_f^0} (\rho_{1f}^0 - \rho_{2f}^0) \right),$$

$$(5.9p) \quad A_{16} := \alpha_r^0 \left(\frac{m_2 \rho_{2f}^0}{\rho_f^0} + \frac{m_1 m_2}{\rho_f^0} (\rho_{1f}^0 - \rho_{2f}^0) \right).$$

The following assertion states that Problems C1–C4 for the vector-functions \mathbf{Z}_1^{ij} , \mathbf{Z}_2^{ij} , \mathbf{Z}_3^{ij} , \mathbf{Z}_4^{ij} , and \mathbf{Z}_5^{ij} are well-posed. Hence the two-scale limiting functions \mathbf{U}_c and \mathbf{U}_r admit the representations (5.1) and (5.2) indeed. Moreover, these representations are unique.

Proposition 3. *Let geometry of the sets \mathcal{Y}_f and \mathcal{Y}_s and all coefficients in Problems C1–C4 be given. Then each of these problems has a unique solution.*

PROOF. Problems C1–C4 are already well-known and exhaustively studied. Proposition 3 is valid due to [16, Chapter VII, Paragraph 5], [17, Chapter 1, page 18], and [12]. \square

Inserting (5.1) and (5.2) into (4.12a) we obtain the integral equality (3.6) in Definition 2 so that the components of tensor and matrices in (3.6) have the

following forms: The components of the effective elasticity tensor are

$$\begin{aligned}
(5.10a) \quad \mathbb{E}^{ijkl} &= \lambda |\mathcal{Y}_s| (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + \alpha_c^0 |\mathcal{Y}_f| \delta_{ij} \delta_{kl} + \left(\eta - \frac{2}{3} \lambda \right) |\mathcal{Y}_s| \delta_{ij} \delta_{kl} \\
&+ \alpha_c^0 \left(\frac{m_1 \rho_{1f}^0}{\rho_f^0} - \frac{m_1 m_2}{\rho_f^0} (\rho_{1f}^0 - \rho_{2f}^0) \right) \delta_{ij} \langle \operatorname{div}_y \mathbf{Z}_1^{kl}(\mathbf{y}) \rangle_{\mathcal{Y}_f} \\
&+ \left(\frac{m_1 \rho_{1f}^0}{\rho_f^0} \left(m_1 \left(\nu_1 - \frac{2}{3} \mu_1 \right) + m_2 \left(\nu_2 - \frac{2}{3} \mu_2 \right) \right) \right. \\
&\quad \left. - \frac{m_1 m_2}{\rho_f^0} \left(\rho_{1f}^0 \left(\nu_2 - \frac{2}{3} \mu_2 \right) - \rho_{2f}^0 \left(\nu_1 - \frac{2}{3} \mu_1 \right) \right) \right) \delta_{ij} \langle \operatorname{div}_y \mathbf{Z}_2^{kl}(\mathbf{y}, 0) \rangle_{\mathcal{Y}_f} \\
&+ \left(2(\mu_1 m_1 + \mu_2 m_2) \frac{m_1 \rho_{1f}^0}{\rho_f^0} - 2(\rho_{1f}^0 \mu_2 - \rho_{2f}^0 \mu_1) \frac{m_1 m_2}{\rho_f^0} \right) \langle \mathbb{D}_{ij}(y, \mathbf{Z}_2^{kl}(\mathbf{y}, 0)) \rangle_{\mathcal{Y}_f} \\
&+ \alpha_c^0 \left(\frac{m_2 \rho_{2f}^0}{\rho_f^0} + \frac{m_1 m_2}{\rho_f^0} (\rho_{1f}^0 - \rho_{2f}^0) \right) \delta_{ij} \langle \operatorname{div}_y \mathbf{Z}_3^{kl}(\mathbf{y}) \rangle_{\mathcal{Y}_f} \\
&+ \left(\left(m_1 \left(\nu_1 - \frac{2}{3} \mu_1 \right) + m_2 \left(\nu_2 - \frac{2}{3} \mu_2 \right) \right) \frac{m_2 \rho_{2f}^0}{\rho_f^0} \right. \\
&\quad \left. + \frac{m_1 m_2}{\rho_f^0} \left(\rho_{1f}^0 \left(\nu_2 - \frac{2}{3} \mu_2 \right) - \rho_{2f}^0 \left(\nu_1 - \frac{2}{3} \mu_1 \right) \right) \right) \delta_{ij} \langle \operatorname{div}_y \mathbf{Z}_4^{kl}(\mathbf{y}, 0) \rangle_{\mathcal{Y}_f} \\
&+ \left(2(\mu_1 m_1 + \mu_2 m_2) \frac{m_2 \rho_{2f}^0}{\rho_f^0} + 2(\rho_{1f}^0 \mu_2 - \rho_{2f}^0 \mu_1) \frac{m_1 m_2}{\rho_f^0} \right) \langle \mathbb{D}_{ij}(y, \mathbf{Z}_4^{kl}(\mathbf{y}, 0)) \rangle_{\mathcal{Y}_f} \\
&+ \left(\eta - \frac{2}{3} \lambda \right) \delta_{ij} \langle \operatorname{div}_y \mathbf{Z}_5^{kl}(\mathbf{y}) \rangle_{\mathcal{Y}_s} + 2\lambda \langle \mathbb{D}_{ij}(y, \mathbf{Z}_5^{kl}(\mathbf{y})) \rangle_{\mathcal{Y}_s}, \quad i, j, k, l = 1, 2, 3.
\end{aligned}$$

The components of the effective viscosity tensor are

$$\begin{aligned}
(5.10b) \quad \mathbb{M}^{ijkl} &= 2(m_1 \mu_1 + m_2 \mu_2) |\mathcal{Y}_f| (1/2) (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \\
&+ \left(m_1 \left(\nu_1 - \frac{2}{3} \mu_1 \right) + m_2 \left(\nu_2 - \frac{2}{3} \mu_2 \right) \right) \left(|\mathcal{Y}_f| \delta_{ij} \delta_{kl} + \frac{m_1 \rho_{1f}^0}{\rho_f^0} \delta_{ij} \langle \operatorname{div}_y \mathbf{Z}_1^{kl}(\mathbf{y}) \rangle_{\mathcal{Y}_f} \right) \\
&- \frac{m_1 m_2}{\rho_f^0} \left(\rho_{1f}^0 \left(\nu_2 - \frac{2}{3} \mu_2 \right) - \rho_{2f}^0 \left(\nu_1 - \frac{2}{3} \mu_1 \right) \right) \delta_{ij} \langle \operatorname{div}_y \mathbf{Z}_1^{kl}(\mathbf{y}) \rangle_{\mathcal{Y}_f} \\
&+ \left(m_1 \left(\nu_1 - \frac{2}{3} \mu_1 \right) + m_2 \left(\nu_2 - \frac{2}{3} \mu_2 \right) \right) \frac{m_2 \rho_{2f}^0}{\rho_f^0} \delta_{ij} \langle \operatorname{div}_y \mathbf{Z}_3^{kl}(\mathbf{y}) \rangle_{\mathcal{Y}_f} \\
&+ \frac{m_1 m_2}{\rho_f^0} \left(\rho_{1f}^0 \left(\nu_2 - \frac{2}{3} \mu_2 \right) - \rho_{2f}^0 \left(\nu_1 - \frac{2}{3} \mu_1 \right) \right) \delta_{ij} \langle \operatorname{div}_y \mathbf{Z}_3^{kl}(\mathbf{y}) \rangle_{\mathcal{Y}_f} \\
&+ 2(m_1 \mu_1 + m_2 \mu_2) \frac{m_1 \rho_{1f}^0}{\rho_f^0} \langle \mathbb{D}_{ij}(y, \mathbf{Z}_1^{kl}(\mathbf{y})) \rangle_{\mathcal{Y}_f} \\
&- 2 \frac{m_1 m_2}{\rho_f^0} (\rho_{1f}^0 \mu_2 - \rho_{2f}^0 \mu_1) \langle \mathbb{D}_{ij}(y, \mathbf{Z}_1^{kl}(\mathbf{y})) \rangle_{\mathcal{Y}_f} \\
&+ 2(m_1 \mu_1 + m_2 \mu_2) \frac{m_2 \rho_{2f}^0}{\rho_f^0} \langle \mathbb{D}_{ij}(y, \mathbf{Z}_3^{kl}(\mathbf{y})) \rangle_{\mathcal{Y}_f} \\
&+ 2 \frac{m_1 m_2}{\rho_f^0} (\rho_{1f}^0 \mu_2 - \rho_{2f}^0 \mu_1) \langle \mathbb{D}_{ij}(y, \mathbf{Z}_3^{kl}(\mathbf{y})) \rangle_{\mathcal{Y}_f}, \quad i, j, k, l = 1, 2, 3.
\end{aligned}$$

The components of the relaxation tensor are

$$\begin{aligned}
(5.10c) \quad \mathbb{G}^{ijkl}(t) &= \alpha_c^0 \left[\frac{m_1 \rho_{1f}^0}{\rho_f^0} - \frac{m_1 m_2}{\rho_f^0} (\rho_{1f}^0 - \rho_{2f}^0) \right] \delta_{ij} \left\langle \operatorname{div}_y \mathbf{Z}_2^{kl}(\mathbf{y}, t) \right\rangle_{\mathcal{Y}_f} \\
&\quad + \left[\left(m_1 \left(\nu_1 - \frac{2}{3} \mu_1 \right) + m_2 \left(\nu_2 - \frac{2}{3} \mu_2 \right) \right) \frac{m_1 \rho_{1f}^0}{\rho_f^0} \right. \\
&\quad \left. - \frac{m_1 m_2}{\rho_f^0} \left(\rho_{1f}^0 \left(\nu_2 - \frac{2}{3} \mu_2 \right) - \rho_{2f}^0 \left(\nu_1 - \frac{2}{3} \mu_1 \right) \right) \right] \delta_{ij} \left\langle \operatorname{div}_y \frac{\partial \mathbf{Z}_2^{kl}(\mathbf{y}, t)}{\partial t} \right\rangle_{\mathcal{Y}_f} \\
&\quad + 2 \left[(m_1 \mu_1 + m_2 \mu_2) \frac{m_1 \rho_{1f}^0}{\rho_f^0} - (\rho_{1f}^0 \mu_2 - \rho_{2f}^0 \mu_1) \frac{m_1 m_2}{\rho_f^0} \right] \left\langle \mathbb{D}_{ij} \left(y, \frac{\partial \mathbf{Z}_2^{kl}(\mathbf{y}, t)}{\partial t} \right) \right\rangle_{\mathcal{Y}_f} \\
&\quad + \alpha_c^0 \left[\frac{m_2 \rho_{2f}^0}{\rho_f^0} + \frac{m_1 m_2}{\rho_f^0} (\rho_{1f}^0 - \rho_{2f}^0) \right] \delta_{ij} \left\langle \operatorname{div}_y \mathbf{Z}_4^{kl}(\mathbf{y}, t) \right\rangle_{\mathcal{Y}_f} \\
&\quad + \left[\left(m_1 \left(\nu_1 - \frac{2}{3} \mu_1 \right) + m_2 \left(\nu_2 - \frac{2}{3} \mu_2 \right) \right) \frac{m_2 \rho_{2f}^0}{\rho_f^0} \right. \\
&\quad \left. + \frac{m_1 m_2}{\rho_f^0} \left(\rho_{1f}^0 \left(\nu_2 - \frac{2}{3} \mu_2 \right) - \rho_{2f}^0 \left(\nu_1 - \frac{2}{3} \mu_1 \right) \right) \right] \delta_{ij} \left\langle \operatorname{div}_y \frac{\partial \mathbf{Z}_4^{kl}(\mathbf{y}, t)}{\partial t} \right\rangle_{\mathcal{Y}_f} \\
&\quad + 2 \left[(m_1 \mu_1 + m_2 \mu_2) \frac{m_2 \rho_{2f}^0}{\rho_f^0} + (\rho_{1f}^0 \mu_2 - \rho_{2f}^0 \mu_1) \frac{m_1 m_2}{\rho_f^0} \right] \left\langle \mathbb{D}_{ij} \left(y, \frac{\partial \mathbf{Z}_4^{kl}(\mathbf{y}, t)}{\partial t} \right) \right\rangle_{\mathcal{Y}_f}, \\
&\hspace{15em} i, j, k, l = 1, 2, 3.
\end{aligned}$$

Coefficients in 3.8–3.11 are as follows:

$$(5.11a) \quad a_1 = \frac{m_1 m_2}{c_{\rho 1} \rho_{1f}^0 m_2 + c_{\rho 2} \rho_{2f}^0 m_1} \left(c_{\rho 1} \rho_{1f}^0 - c_{\rho 2} \rho_{2f}^0 + \frac{c_{\rho 2} \rho_{2f}^0}{m_2} \right),$$

$$(5.11b) \quad a_2 = \frac{m_1 m_2}{c_{\rho 1} \rho_{1f}^0 m_2 + c_{\rho 2} \rho_{2f}^0 m_1} \left(c_{\rho 2} \rho_{2f}^0 - c_{\rho 1} \rho_{1f}^0 + \frac{c_{\rho 1} \rho_{1f}^0}{m_1} \right),$$

$$\begin{aligned}
(5.11c) \quad \mathbb{K}_1^{ij} &= (c_{\rho 1} \rho_{1f}^0 - c_{\rho 2} \rho_{2f}^0) \left[\delta_{ij} \mathbb{I} + \left\langle \frac{m_1 \rho_{1f}^0}{\rho_f^0} \operatorname{div}_y \mathbf{Z}_1^{ij} + \frac{m_2 \rho_{2f}^0}{\rho_f^0} \operatorname{div}_y \mathbf{Z}_3^{ij} \right\rangle_{\mathcal{Y}_f} \right. \\
&\quad \left. + \left\langle \operatorname{div}_y \mathbf{Z}_5^{ij} \right\rangle_{\mathcal{Y}_s} \right] + \frac{\rho_{1f}^0 \rho_{2f}^0}{\rho_f^0} (c_{\rho 1} m_2 + c_{\rho 2} m_1) \left\langle \operatorname{div}_y \mathbf{Z}_3^{ij} - \operatorname{div}_y \mathbf{Z}_1^{ij} \right\rangle_{\mathcal{Y}_f},
\end{aligned}$$

$$\begin{aligned}
(5.11d) \quad \mathbb{K}_2^{ij} &= (c_{\rho 2} \rho_{2f}^0 - c_{\rho 1} \rho_{1f}^0) \left[\delta_{ij} \mathbb{I} + \left\langle \frac{m_1 \rho_{1f}^0}{\rho_f^0} \operatorname{div}_y \mathbf{Z}_1^{ij} + \frac{m_2 \rho_{2f}^0}{\rho_f^0} \operatorname{div}_y \mathbf{Z}_3^{ij} \right\rangle_{\mathcal{Y}_f} \right. \\
&\quad \left. + \left\langle \operatorname{div}_y \mathbf{Z}_5^{ij} \right\rangle_{\mathcal{Y}_s} \right] + (1/\rho_f^0) (c_{\rho 2} (\rho_{2f}^0)^2 m_2 + c_{\rho 1} (\rho_{1f}^0)^2 m_1) \left\langle \operatorname{div}_y \mathbf{Z}_3^{ij} - \operatorname{div}_y \mathbf{Z}_1^{ij} \right\rangle_{\mathcal{Y}_f},
\end{aligned}$$

$$\begin{aligned}
(5.11e) \quad \mathbb{K}_3^{ij} &= (m_1 + m_2) \left[\delta_{ij} \mathbb{I} + \left\langle \frac{m_1 \rho_{1f}^0}{\rho_f^0} \operatorname{div}_y \mathbf{Z}_1^{ij} + \frac{m_2 \rho_{2f}^0}{\rho_f^0} \operatorname{div}_y \mathbf{Z}_3^{ij} \right\rangle_{\mathcal{Y}_f} \right. \\
&\quad \left. + \left\langle \operatorname{div}_y \mathbf{Z}_5^{ij} \right\rangle_{\mathcal{Y}_s} \right] - \frac{m_1 m_2}{\rho_f^0} (\rho_{2f}^0 - \rho_{1f}^0) \left\langle \operatorname{div}_y \mathbf{Z}_3^{ij} - \operatorname{div}_y \mathbf{Z}_1^{ij} \right\rangle_{\mathcal{Y}_f},
\end{aligned}$$

$$(5.11f) \quad \mathbb{H}_1^{ij}(t) = (c_{\rho 1} \rho_{1f}^0 - c_{\rho 2} \rho_{2f}^0) \left\langle \frac{m_1 \rho_{1f}^0}{\rho_f^0} \operatorname{div}_y \mathbf{Z}_2^{ij}(\cdot, t) + \frac{m_2 \rho_{2f}^0}{\rho_f^0} \operatorname{div}_y \mathbf{Z}_4^{ij}(\cdot, t) \right\rangle_{\mathcal{Y}_f} \\ + \frac{\rho_{1f}^0 \rho_{2f}^0}{\rho_f^0} (c_{\rho 1} m_2 + c_{\rho 2} m_1) \left\langle \operatorname{div}_y \mathbf{Z}_4^{ij}(\cdot, t) - \operatorname{div}_y \mathbf{Z}_2^{ij}(\cdot, t) \right\rangle_{\mathcal{Y}_f},$$

(5.11g)

$$\mathbb{H}_2^{ij}(t) = (c_{\rho 2} \rho_{2f}^0 - c_{\rho 1} \rho_{1f}^0) \left\langle \frac{m_1 \rho_{1f}^0}{\rho_f^0} \operatorname{div}_y \mathbf{Z}_2^{ij}(\cdot, t) + \frac{m_2 \rho_{2f}^0}{\rho_f^0} \operatorname{div}_y \mathbf{Z}_4^{ij}(\cdot, t) \right\rangle_{\mathcal{Y}_f} \\ + (1/\rho_f^0) (c_{\rho 2} (\rho_{2f}^0)^2 m_2 + c_{\rho 1} (\rho_{1f}^0)^2 m_1) \left\langle \operatorname{div}_y \mathbf{Z}_4^{ij}(\cdot, t) - \operatorname{div}_y \mathbf{Z}_2^{ij}(\cdot, t) \right\rangle_{\mathcal{Y}_f},$$

$$(5.11h) \quad \mathbb{H}_3^{ij}(t) = (m_1 + m_2) \left\langle \frac{m_1 \rho_{1f}^0}{\rho_f^0} \operatorname{div}_y \mathbf{Z}_2^{ij}(\cdot, t) + \frac{m_2 \rho_{2f}^0}{\rho_f^0} \operatorname{div}_y \mathbf{Z}_4^{ij}(\cdot, t) \right\rangle_{\mathcal{Y}_f} \\ - \frac{m_1 m_2}{\rho_f^0} (\rho_{2f}^0 - \rho_{1f}^0) \left\langle \operatorname{div}_y \mathbf{Z}_4^{ij}(\cdot, t) - \operatorname{div}_y \mathbf{Z}_2^{ij}(\cdot, t) \right\rangle_{\mathcal{Y}_f}, \\ i, j = 1, 2, 3.$$

Thus Theorem 1 is proved. All effective coefficients are fully derived from microstructure.

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