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ON ISOMORPHISM BETWEEN DISTANCE-REGULAR GRAPHS

S.V. GORYAINOV

Abstract. In [1], two new constructions of antipodal distance-regular graphs related to the group $PSL_2(q)$ have been proposed. The author of [1] remained the question whether these graphs were isomorphic to some known ones unsolved. In this work, we show that distance-regular graphs mentioned above are isomorphic to the Mathon graphs with appropriate values of parameters. .

Keywords: distance-regular graph, graph isomorphism, group action on a graph.

1. Introduction

In [1], two new constructions of antipodal distance-regular graphs related to the group $PSL_2(q)$ have been proposed. The author of [1] remained the question whether these graphs were isomorphic to some known ones unsolved. In this work, we show that distance-regular graphs mentioned above are isomorphic to the Mathon graphs with appropriate values of parameters.

We recall that the group $PSL_2(q)$, $q = p^n$, contains the two conjugacy classes of elements of order p, see Lemma 1 in Section 2. Let Γ_B be a graph with vertex set $B = g^G \cup (g^{-1})^G$, where g^G is a conjugacy class of elements of order p of the group $G = PSL_2(p^n)$, and with edge set $\{\{x, y\} | xy^{-1} \in B\}$, where p is an odd prime number, $q = p^n \geq 5$. Denote by $\widehat{\Gamma_B}$ a graph obtained from the graph Γ_B by removing edges connecting commuting elements of B.

Theorem 1 ([1])**.** *If $q \equiv 1 \pmod 4$, then the graph $\widehat{\Gamma_B} := \widehat{\Gamma_B}(q)$ is distance-regular with intersection array $\{q, q-3, 1; 1, 2, q\}$.*

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Let Γ_J be a graph whose vertex set is the set of all elements of order p of the group G and whose edge set is $\{\{x,y\}|xy^{-1} \in J\}$, where J is a class of conjugate involutions of G.

Theorem 2 ([1])**.** *If $q \equiv 1,3 \pmod 8$, then the graph $\Gamma_J := \Gamma_J(q)$ is disconnected and its connected components are two isomorphic distance-regular graphs, say Γ'_J, Γ''_J, with intersection array $\{q, q-3, 1; 1, 2, q\}$.*

The following theorem is Proposition 12.5.3 from [2].

Theorem 3 ([2])**.** *Let $q = rm + 1$ be a prime power, where $r > 1$ and either m is even or q is a power of 2. Let V be a vector space of dimension 2 over $F = \mathbb{F}_q$ provided with a nondegenerate symplectic form f. Let K be the subgroup of the multiplicative group F^* of index r, and let $b \in F^*$. Then the graph $M(m,q)$ with vertex set $\{Kv \mid v \in V \setminus \{\overline{0}\}\}$ where $Ku \sim Kv$ if and only if $f(u,v) \in bK$ is distance-regular of diameter 3 with $r(q+1)$ vertices and intersection array $\{q, q-m-1, 1; 1, m, q\}$.*

In particular, if $m = 2$ the graph $M(2,q)$ has intersection array $\{q, q-3, 1; 1, 2, q\}$.

Note that $M(m,q)$ does not depend (up to isomorhpism) on the choice of subgroup K, form B and b. For our purposes, we may assume that $b := 1$, where 1 is an identity of $\mathbb{F}_q^*$, the subgroup $K := \{\pm 1\} \leq \mathbb{F}_q^*$ of order 2, and the symplectic form $f := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then, for $u, v \in V \backslash \{\overline{0}\}$, $f(u,v) = \begin{pmatrix} u_1 & u_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ $= u_1v_2 - u_2v_1$, and vertices $Ku, Kv \in V(\Gamma)$ are adjacent if and only if $f(u,v) = u_1v_2 - u_2v_1 \in K = \{\pm 1\}$.

The result of this work is expressed in the following theorem.

Theorem 4. *Graphs Γ'_J, Γ''_J and graph $\Gamma(q)$ are isomorphic to $M(2,q)$.*

In our proof we will make use the fact that $M(m,q)$, $\widehat{\Gamma_B}$, and Γ'_J, Γ''_J are vertex-transitive graphs.

Definition 1. *Let G be a finite group, $H \leq G$ and $S \subseteq G$. Define a graph $\Gamma(G,H,S)$ whose vertices are all cosets gH $(g \in G)$, and $g_1H \sim g_2H \Leftrightarrow Hg_2^{-1}g_1H \subseteq HSH$.*

Clearly, the group G acts transitively on the vertex set of $\Gamma(G,H,S)$ by left multiplication.

It is well-known fact that any vertex-transitive graph can be considered as $\Gamma(G,H,S)$ for appropriate G,H,S. In order to prove Theorem 4, we show that all the graphs from its statement are of type $\Gamma(G,H,S)$ with $G = PSL_2(q)$ and isomorphic subgroups H and sets S equivalent under the action of appropriate automorphisms of G.

2. Preliminaries

For the background on distance-regular graphs, we refer to [2].

In the following lemma we describe the two conjugacy classes of p-elements of $PSL_2(q)$, $q = p^n$ (see also [3, p. 261]).

In what follows, $\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}$ denotes a coset of $SL_2(q)/Z(SL_2(q)) = PSL_2(q)$ with representative $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Lemma 1 ([1])**.** *The conjugacy classes C_1 and C_2 of elements of order p of $PSL_2(q)$ can be represented as*

$$C_j = \left\{ \overline{\begin{pmatrix} 1 & 0 \\ -\gamma & 1 \end{pmatrix}} | \gamma \in R_j \right\} \cup \left\{ \overline{\begin{pmatrix} 1+\gamma\delta_i & \gamma \\ -\gamma\delta_i^2 & 1-\gamma\delta_i \end{pmatrix}} | \gamma \in R_j, \delta_i \in \mathbb{F}_q \right\}, \ j = 1, 2,$$

where R_1 is the set of all nonzero squares of $\mathbb{F}_q$, and R_2 — the set of all non-squares of $\mathbb{F}_q$.

For the convenience of the reader, below we recall some basic facts about vertex-transitive graphs $\Gamma(G,H,S)$. For a graph Γ and its vertex x, denote by $Stab(x)$ the stabilizer of x, $Stab(x) := \{g \in Aut(\Gamma) \mid gx = x\}$.

Lemma 2. *Let Γ be a graph. For a group $G \le Aut(\Gamma)$, suppose that G acts transitively on the vertex set of Γ. Then graphs Γ and $\Gamma(G,H,S)$ are isomorphic, where $H = G \cap Stab(x)$ for some vertex $x \in V(\Gamma)$ and $S = \{g \in G \mid gx \sim x\}$.*

Proof. Fix an arbitrary vertex $x \in \Gamma$. We first show that $HSH = S$. In fact, $g \in S \Leftrightarrow gx \sim x \Leftrightarrow \forall h_1, h_2 \in H \ \ gh_1x \sim h_2x \Leftrightarrow \forall h_1, h_2 \in H \ \ h_2^{-1}gh_1x \sim x \Leftrightarrow \forall h_1, h_2 \in H \ \ h_2^{-1}gh_1 \in S$, i.e. $HSH \subseteq S$. As $S \subseteq HSH$, we have $S = HSH$.

We now define an isomorphism $\phi: \ V(\Gamma) \to G/H$ by the following rule. For a vertex $y \in V(\Gamma)$, let g be an automorphism of Γ such that $gx = y$. Put $\phi(y) = gH$.

Let us show that ϕ is well defined, i.e. it does not depend on the choice of element from gH. If g_1, g_2 are two automorphisms such that $g_1x = g_2x = y$, then $g_2^{-1}g_1x = x$, i.e. $g_2^{-1}g_1 \in Stab(x) \cap G = H$. Hence g_1 and g_2 belong to the same coset of G/H, and the map ϕ is well defined. Clearly, ϕ is a bijection.

Let us show that ϕ is isomorphism, i.e., it preserves the adjacency relation. Choose arbitrary automorphisms $g_1, g_2 \in Aut(\Gamma)$ and consider vertices $y_1 = g_1x$, and $y_2 = g_2x$. Then we have $y_1 \sim y_2 \Leftrightarrow g_1x \sim g_2x \Leftrightarrow \forall h_1, h_2 \in H \ \ g_1h_1x \sim g_2h_2x \Leftrightarrow \forall h_1, h_2 \in H \ \ h_2^{-1}g_2^{-1}g_1h_1x \sim x \Leftrightarrow \forall h_1, h_2 \in H \ \ h_2^{-1}g_2^{-1}g_1h_1 \in S = HSH \Leftrightarrow Hg_2^{-1}g_1H \subseteq HSH \Leftrightarrow g_1H \sim g_2H$. ■

To the rest of this paper, let G denote $PSL_2(q)$, $q = p^n$ and $q \ge 5$.

Lemma 3. *The following holds.*

(i) *The group G acts transitively by left multiplication on the vertex set of the graph $M(2,q)$.*

(ii) *$M(2,q) \simeq \Gamma(G, H_M, S_M)$, where*

$$H_M := G \cap Stab(K\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = \left\{ \overline{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}} \mid b \in \mathbb{F}_q \right\},$$

$$S_M := \{g \in G \mid gK\begin{pmatrix} 1 \\ 0 \end{pmatrix} \sim K\begin{pmatrix} 1 \\ 0 \end{pmatrix}\} = \left\{ \overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \mid ad - bc = 1, c^2 = 1 \right\}.$$

Proof. 1) Consider arbitrary nonzero vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} x' \\ y' \end{pmatrix}$ from V. Let us show that there exists a matrix of $SL_2(q)$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

This gives the following system of equations:

$$\begin{cases} ax+by=x' \\ cx+dy=y' \\ ad-bc=1. \end{cases}$$

Since $\begin{pmatrix} x \\ y \end{pmatrix}$ is nonzero, we may assume that $x \neq 0$. Then we have

$$\begin{cases} a=(x'-by)x^{-1} \\ c=(y'-dy)x^{-1} \\ ad-bc=1 \end{cases},$$

and $dx'-by'=x$.

Since $\begin{pmatrix} x' \\ y' \end{pmatrix}$ is nonzero, we may assume that $x' \neq 0$. Then

$$\begin{cases} a=(x'-by)x^{-1} \\ c=(y'-dy)x^{-1} \\ d=(x+by')(x')^{-1} \end{cases}$$

Substituting an arbitrary $b \in \mathbb{F}_q$, we obtain the required matrix. This shows (i).

2) Let us determine all elements of $PSL_2(q)$ fixing $K\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ by left multiplication. Suppose that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(q)$. If we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} K\begin{pmatrix} 1 \\ 0 \end{pmatrix} = K\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$K\begin{pmatrix} a \\ c \end{pmatrix} = K\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

then

$$\begin{cases} a=\pm 1 \\ c=0 \\ ad-bc=1 \end{cases}.$$

Thus, $H_M = \left\{ \overline{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} | b \in \mathbb{F}_q} \right\}$.

3) Let us determine all elements of $PSL_2(q)$ that map the vertex $K\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to an adjacent vertex. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(q)$. We have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} K\begin{pmatrix} 1 \\ 0 \end{pmatrix} = K\begin{pmatrix} a \\ c \end{pmatrix},$$

and

$$K\begin{pmatrix} a \\ c \end{pmatrix} \sim K\begin{pmatrix} 1 \\ 0 \end{pmatrix} \Leftrightarrow c \in K = \{\pm 1\} \Leftrightarrow c^2 = 1.$$

Thus, $S_M = \{g \in G \mid gK\begin{pmatrix} 1 \\ 0 \end{pmatrix} \sim K\begin{pmatrix} 1 \\ 0 \end{pmatrix}\} = \left\{\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} | ad - bc = 1, c^2 = 1\right\}$.

4) Now Lemma 2 shows (ii). The lemma is proved. ■

3. $\widehat{\Gamma_B} \simeq M(2, q)$

In this section we prove that $\widehat{\Gamma_B} \simeq M(2, q)$.

Lemma 4. *The following holds.*

(i) *Group G acts transitively by conjugation on the vertex set of the graph $\widehat{\Gamma}_B$.*

(ii) *$\widehat{\Gamma}_B \simeq \Gamma(G, H_B, S_B)$, where*

$$H_B := G \cap Stab(\overline{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}) = \left\{\overline{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}} | b \in \mathbb{F}_q\right\},$$

$$S_B := \{g \in G \mid g\overline{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}} \sim \overline{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}\} = \left\{\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} | ad - bc = 1, c^2 = -4\right\}.$$

Proof. 1) The graph $\widehat{\Gamma_B}$ has as vertices all elements of $g^G \cup (g^{-1})^G$, where g is an element of order p. By Theorem 1, we assume that $q \equiv 1 \pmod 4$. Hence -1 is a square of $\mathbb{F}_q$, and $g^{-1} \in g^G$ due to Lemma 1. Therefore the vertex set of $\widehat{\Gamma_B}$ is a conjugacy class (C_1 or C_2) of elements of order p of G. This proves (i).

2) Assume that $V(\widehat{\Gamma_B}) = C_1$. Put $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ so that $\overline{T} \in V(\widehat{\Gamma_B})$.

Note that if $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A \in SL_2(q)$, then $A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ and

$$(1) \quad M := ATA^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 1 - ac & a^2 \\ -c^2 & 1 + ac \end{pmatrix}.$$

Let us determine all elements of $PSL_2(q)$ fixing the vertex $\overline{T}$. We have the equation $\overline{M} = \overline{T}$, i.e.,

$$\overline{\begin{pmatrix} 1 - ac & a^2 \\ -c^2 & 1 + ac \end{pmatrix}} = \overline{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}},$$

which implies that at least one of the following systems of equations:

$$\begin{cases} ad - bc = 1 \\ 1 - ac = 1 \\ a^2 = 1 \\ -c^2 = 0 \\ 1 + ac = 1 \end{cases} \quad \text{and} \quad \begin{cases} ad - bc = 1 \\ 1 - ac = -1 \\ a^2 = -1 \\ -c^2 = 0 \\ 1 + ac = -1 \end{cases}$$

is compatible.

It is easily seen that the second system has no solutions.

The first system is equivalent to the following one:

$$\begin{cases} ad - bc = 1 \\ c = 0 \\ a^2 = 1 \\ a = d \end{cases}.$$

Thus, $H_B = \left\{ \overline{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}} \mid b \in \mathbb{F}_q \right\}$.

3) Find all elements of $PSL_2(q)$ that map the vertex $\overline{T}$ to an adjacent vertex. By the definition of $\widehat{\Gamma_B}$, the vertices $\overline{T}$ and $\overline{M}$, $M = ATA^{-1}$ (see Equation (1)), are adjacent if and only if $\overline{TM^{-1}}$ belongs to the same conjugacy class as of $\overline{T}$ and $\overline{M}$. We have

$$TM^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1+ac & -a^2 \\ c^2 & 1-ac \end{pmatrix} = \begin{pmatrix} 1+ac+c^2 & 1-ac-a^2 \\ c^2 & 1-ac \end{pmatrix}.$$

By Lemma 1, we should consider the following two cases when $\overline{T}$ and $\overline{M}$ can be adjacent:

$$\overline{\begin{pmatrix} 1+ac+c^2 & 1-ac-a^2 \\ c^2 & 1-ac \end{pmatrix}} = \overline{\begin{pmatrix} 1 & 0 \\ -\gamma & 1 \end{pmatrix}}, \tag{2}$$

$$\overline{\begin{pmatrix} 1+ac+c^2 & 1-ac-a^2 \\ c^2 & 1-ac \end{pmatrix}} = \overline{\begin{pmatrix} 1+\gamma\delta & \gamma \\ -\gamma\delta^2 & 1-\gamma\delta \end{pmatrix}}. \tag{3}$$

Equation (2) implies that at least one of the following two systems of equations:

$$\begin{cases} ad-bc = 1 \\ 1+ac+c^2 = 1 \\ 1-ac-a^2 = 0 \\ c^2 = -\gamma \\ 1-ac = 1 \end{cases} \quad \text{or} \quad \begin{cases} ad-bc = 1 \\ 1+ac+c^2 = -1 \\ 1-ac-a^2 = 0 \\ c^2 = \gamma \\ 1-ac = -1 \end{cases}$$

is compatible.

In case of the first system of equations, the sum of the second and the fifth equations gives $c^2 = 0$. This contradicts the fourth equation, as γ is nonzero.

The second system of equations is equivalent to the following one:

$$\begin{cases} ad-bc = 1 \\ c^2 = -4 \\ a^2 = -1 \\ c^2 = \gamma \\ ac = 2 \end{cases},$$

from which it is easily seen that $\overline{A} \in S_B$, where S_B is defined in the statement of this lemma.

Equation (3) implies that at least one of the following two systems of equations:

$$\begin{cases} ad-bc = 1 \\ 1+ac+c^2 = 1+\gamma\delta \\ 1-ac-a^2 = \gamma \\ c^2 = -\gamma\delta^2 \\ 1-ac = 1-\gamma\delta \end{cases} \quad \text{or} \quad \begin{cases} ad-bc = 1 \\ 1+ac+c^2 = -1-\gamma\delta \\ 1-ac-a^2 = -\gamma \\ c^2 = \gamma\delta^2 \\ 1-ac = -1+\gamma\delta \end{cases}$$

is compatible.

From the second and the fifth equations of the first system of equations, we obtain $c^2 = 0$, i.e. $c = 0$. Further, $M = \begin{pmatrix} 1 & a^2 \\ 0 & 1 \end{pmatrix}$, and hence the vertices $\overline{T}$ and $\overline{M}$ commute, therefore they are not adjacent in $\widehat{\Gamma_B}$, a contradiction.

The second system of equations is equivalent to the following one:

$$\begin{cases} ad - bc = 1 \\ c^2 = -4 \\ (a + 2^{-1}c)^2 = \gamma \\ c^2 = \gamma\delta^2 \\ ac = 2 - \gamma\delta \end{cases}.$$

The second and the fourth equations give $\gamma\delta^2 = -4$, and taking into account the fifth equation, we see that $(2 - ac)\delta = -4$. As $\gamma\delta \neq 0$, it follows that $2 - ac \neq 0$, hence $\delta = -4/(2-ac)$. Therefore, for any a, b, c, d such that $ad - bc = 1$, $ac - 2 \neq 0$ and $c^2 = -4$, the system of equations has a solution. In particular, $\overline{A} \in S_B$.

4) Now (ii) directly follows from Lemma 2. ■

Lemma 5. *Let* $t \in \mathbb{F}_q^*$. *Then the map* $\psi_t : \overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \to \overline{\begin{pmatrix} a & t^{-1}b \\ tc & d \end{pmatrix}}$ *is an automorphism of* $PSL_2(q)$.

Proof. Straightforward. ■

Theorem 5. $\widehat{\Gamma_B} \simeq M(2, q)$.

Proof. It follows from Lemmas 3 and 4 that $H_M = H_B$. We have $M(2, q) \simeq \Gamma(G, H_M, S_M)$ by Lemma 3, and $\Gamma(G, H_B, S_B) \simeq \widehat{\Gamma_B}$ by Lemma 4. Let $t \in \mathbb{F}_q^*$ be such that $t^2 = -4$. Then $\psi_t(S_M) = S_B$. Thus, $\Gamma(G, H_M, S_M) \simeq \Gamma(G, H_B, S_B)$ by Lemma 5 and $M(2, q) \simeq \Gamma(G, H_M, S_M) \simeq \Gamma(G, H_B, S_B) \simeq \widehat{\Gamma_B}$. ■

4. $\Gamma'_J \simeq M(2, q)$

Denote by J the conjugacy class of involutions of G (we recall that all involutions of $PSL_2(q)$ are pairwise conjugate). We recall that the graph Γ_J is defined on the set of all elements of order p of the group $G = PSL_2(q)$ with two vertices x, y being adjacent if and only if $xy^{-1} \in J$. It follows from Lemma 1 that $V(\Gamma_J) = C_1 \cup C_2$. By Theorem 2, if $q \equiv 1, 3 \pmod 8$, then the graph Γ_J has exactly two connected components, say Γ'_J, Γ''_J, which are isomorphic distance-regular graphs.

Lemma 6 ([1])**.** *If* $q \equiv 3 \pmod 4$, *then* J *can be represented as follows:*

$$J = \left\{ \overline{\begin{pmatrix} -\alpha\gamma & \beta - \alpha^2\gamma \\ \gamma & \alpha\gamma \end{pmatrix}} | \alpha, \beta, \gamma \in \mathbb{F}_q, \beta\gamma = -1 \right\},$$

and if $q \equiv 1 \pmod 4$, *then* $J =$

$$\left\{ \overline{\begin{pmatrix} -\alpha\gamma & \beta - \alpha^2\gamma \\ \gamma & \alpha\gamma \end{pmatrix}} | \alpha, \beta, \gamma \in \mathbb{F}_q, \beta\gamma = -1 \right\} \cup \left\{ \overline{\begin{pmatrix} \delta & 2\alpha\gamma \\ 0 & -\delta \end{pmatrix}} | \alpha \in \mathbb{F}_q, \delta^2 = -1 \right\}.$$

Lemma 7. *Two vertices of* Γ_J *belong to the same connected component if and only if they belong to the same conjugacy class of elements of order* p.

Proof. We first note that -2 is a square in $\mathbb{F}_q$ if an only if $q \equiv 1(8)$ or $q \equiv 3(8)$.

Further, by Lemma 1, $V(\Gamma_J)$ contains vertices of two types:

$$\overline{\begin{pmatrix} 1 & 0 \\ -\gamma & 1 \end{pmatrix}} \text{ and } \overline{\begin{pmatrix} 1+\gamma\delta & \gamma \\ -\gamma\delta^2 & 1-\gamma\delta \end{pmatrix}}.$$

We claim that the vertices of the first type are pairwise non-adjacent. Indeed,

$$\begin{pmatrix} 1 & 0 \\ -\gamma_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\gamma_2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -\gamma_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \gamma_2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \gamma_2-\gamma_1 & 1 \end{pmatrix},$$

and $\overline{\begin{pmatrix} 1 & 0 \\ \gamma_2-\gamma_1 & 1 \end{pmatrix}} \notin J$ if $\gamma_1 \neq \gamma_2$.

Consider now two adjacent vertices of different types, say $x = \overline{\begin{pmatrix} 1 & 0 \\ -\gamma_1 & 1 \end{pmatrix}}$, and $y = \overline{\begin{pmatrix} 1+\gamma_2\delta & \gamma_2 \\ -\gamma_2\delta^2 & 1-\gamma_2\delta \end{pmatrix}}$. Then

$$xy^{-1} = \overline{\begin{pmatrix} 1 & 0 \\ -\gamma_1 & 1 \end{pmatrix}\begin{pmatrix} 1-\gamma_2\delta & -\gamma_2 \\ \gamma_2\delta^2 & 1+\gamma_2\delta \end{pmatrix}} = \overline{\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}},$$

where $z_{11} = 1-\gamma_2\delta, z_{12} = -\gamma_2, z_{21} = -\gamma_1+\gamma_1\gamma_2\delta+\gamma_2\delta^2, z_{22} = \gamma_1\gamma_2+1+\gamma_2\delta$.

Since $xy^{-1} \in J$, it follows from Lemma 6 that $z_{11} = -z_{22}$, which gives $\gamma_1\gamma_2 = -2$. Hence γ_1 and γ_2 are both squares or non-squares simultaneously.

Finally, consider two adjacent vertices of the second type, say $x = \overline{\begin{pmatrix} 1+\gamma_1\delta_1 & \gamma_1 \\ -\gamma_1\delta_1^2 & 1-\gamma_1\delta_1 \end{pmatrix}}$ and $y = \overline{\begin{pmatrix} 1+\gamma_2\delta_2 & \gamma_2 \\ -\gamma_2\delta_2^2 & 1-\gamma_2\delta_2 \end{pmatrix}}$. Then

$$xy^{-1} = \overline{\begin{pmatrix} 1+\gamma_1\delta_1 & \gamma_1 \\ -\gamma_1\delta_1^2 & 1-\gamma_1\delta_1 \end{pmatrix}\begin{pmatrix} 1-\gamma_2\delta_2 & -\gamma_2 \\ \gamma_2\delta_2^2 & 1+\gamma_2\delta_2 \end{pmatrix}} = \overline{\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}},$$

where

$$z_{11} = 1+\gamma_1\delta_1-\gamma_2\delta_2-\gamma_1\gamma_2\delta_2(\delta_1-\delta_2),$$
$$z_{12} = \gamma_1-\gamma_2-\gamma_1\gamma_2(\delta_1-\delta_2),$$
$$z_{21} = -\gamma_1\delta_1^2+\gamma_2\delta_2^2+\gamma_1\gamma_2\delta_1\delta_2(\delta_1-\delta_2),$$
$$z_{22} = 1-\gamma_1\delta_1+\gamma_2\delta_2+\gamma_1\gamma_2\delta_1(\delta_1-\delta_2).$$

From $z_{11} = -z_{22}$ we see that $\gamma_1\gamma_2(\delta_1-\delta_2)^2 = -2$, hence γ_1 and γ_2 are both squares or non-squares simultaneously. The lemma is proved. ■

By Lemma 7, we may restrict ourselves to considering a connected component Γ_J' of the graph Γ_J, which corresponds, for instance, to the conjugacy class C_1.

Lemma 8. *The following holds.*

(i) *Group G acts transitively by conjugation on the vertex set of Γ_J''.*

(ii) *$\Gamma_J' \simeq \Gamma(G, H_J, S_J)$, where*

$$H_J = H_B,$$

$$S_J := \{g \in G \mid g\overline{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}} \sim \overline{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}\} = \left\{\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \mid ad-bc = 1, c^2 = -2\right\}.$$

Proof. By Lemma 7, the vertex set of Γ'_J is a conjugacy class. This shows (i).

2) It follows from Lemma 4, that H_B is the stabilizer of $\overline{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}$, which may be considered as a vertex of Γ'_J. Put $H_J = H_B$.

3) Put $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Let us determine all elements of G that map the vertex $\overline{T}$ to an adjacent vertex. We will make use the notation from Equation (1). Then

$$TM^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1+ac & -a^2 \\ c^2 & 1-ac \end{pmatrix} = \begin{pmatrix} 1+ac+c^2 & 1-ac-a^2 \\ c^2 & 1-ac \end{pmatrix}.$$

If $\overline{T}$ and $\overline{M}$ are adjacent then $\overline{TM^{-1}} \in J$. By Lemma 6, we should consider two cases:

$$\begin{aligned} &\overline{TM^{-1}} = \overline{\begin{pmatrix} -\alpha\gamma & \beta-\alpha^2\gamma \\ \gamma & \alpha\gamma \end{pmatrix}} \text{ with } \beta\gamma = -1, \\ &\overline{TM^{-1}} = \overline{\begin{pmatrix} \delta & 2\alpha\gamma \\ 0 & -\delta \end{pmatrix}} \text{ with } \delta^2 = -1. \end{aligned} \tag{4}$$

The first case implies that at least one of the following two system of equations:

$$\begin{cases} ad-bc = 1 \\ 1+ac+c^2 = -\alpha\gamma \\ 1-ac-a^2 = \beta-\alpha^2\gamma \\ c^2 = \gamma \\ 1-ac = \alpha\gamma \end{cases} \quad \text{or} \quad \begin{cases} ad-bc = 1 \\ 1+ac+c^2 = \alpha\gamma \\ 1-ac-a^2 = -\beta+\alpha^2\gamma \\ c^2 = -\gamma \\ 1-ac = -\alpha\gamma \end{cases}$$

is compatible.

Consider the first system of equations. Summing the second and the fifth equations, we obtain $c^2 = -2$, which together with $\beta\gamma = -1$ gives $\gamma = -2$ and $\beta = 2^{-1}$. Hence the system of equations takes the following form:

$$\begin{cases} ad-bc = 1 \\ c^2 = -2 \\ 1-ac-a^2 = 2^{-1}+2\alpha^2 \\ 1-ac = -2\alpha \end{cases}.$$

Substituting $1-ac = -2\alpha$ into the third equation and simplifying, we get $-2a^2 = (1+2\alpha)^2$, which, in fact, follows from $1-ac = -2\alpha$ and $c^2 = -2$. Therefore, for any a, b, c, d such that $c^2 = -2$ and $ad-bc = 1$, the system has a solution in α, β, γ with $\beta\gamma = -1$, and $\overline{A} \in S_J$.

In the same manner, one can show that the second system of equations is also compatible.

Finally, the second case in (4) implies that $c = 0$ and $\delta = 1 = -\delta$, a contradiction.

4) Now (ii) directly follows from Lemma 2. ■

Theorem 6. $\Gamma'_J \simeq M(2, q)$.

Proof. It follows from Lemmas 3 and 8 that $H_M = H_J$. We have $\Gamma(G, H_J, S_J) \simeq \Gamma'_J$ by Lemma 8. Let $t \in \mathbb{F}_q^*$ be such that $t^2 = -2$. Then $\psi_t(S_M) = S_J$. Thus,

$\Gamma(G, H_M, S_M) \simeq \Gamma(G, H_J, S_J)$ by Lemma 5 and $M(2, q) \simeq \Gamma(G, H_M, S_M) \simeq \Gamma(G, H_J, S_J) \simeq \Gamma'_J$. ■

Theorem 4 now follows from Theorems 5 and 6.

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Sergey Viktorovich Goryainov
N.N. Krasovsky Institute of Mathematics and Mechanics, UB RAS,
S.Kovalevskaja street 16,
620990, Ekaterinburg, Russia
E-mail address: 44g@mail.ru