S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

http://semr.math.nsc.ru

Том 11, стр. 380–407 (2014) УДК 510.67, 512.53

MSC 03C07, 03G15, 20N02, 08A02, 08A55

ALGEBRAS OF DISTRIBUTIONS FOR ISOLATING FORMULAS OF A COMPLETE THEORY

I.V. SHULEPOV, S.V. SUDOPLATOV

Abstract. We define a class of algebras describing links of binary isolating formulas on a set of realizations for a family of 1-types of a complete theory. We prove that a set of labels for binary isolating formulas on a set of realizations for a 1-type p forms a groupoid of a special form if there is an atomic model over a realization of p. We describe the class of these groupoids and consider features of these groupoids in a general case and for special theories. A description of the class of partial groupoids relative to families of 1-types is given.

Keywords: type, complete theory, groupoid of binary isolating formulas, join of groupoids, deterministic structure.

In [29] (see also [24]–[30]), a series of constructions is introduced admitting to realize key properties of countable theories and to obtain a classification of countable models of small (in particular, of Ehrenfeucht) theories with respect to two basic characteristics: Rudin–Keisler preorders and distribution functions for the numbers of limit models. The construction of these theories is essentially based on the definition of special directed graphs with colored vertices and arcs as well as on the definition of $(n+1)$-ary predicates that turn prime models over realizations of n-types to prime models over realizations of 1-types and reducing links between prime models over finite sets to links between prime models over elements such that these links are defined by principal arcs and edges.

In the paper, we consider a general approach to description of binary links between realizations of 1-types in terms of labels of pairwise non-equivalent isolating

Shulepov I.V., Sudoplatov S.V., Algebras of distributions for isolating formulas of a complete theory.

The authors were supported by RFFI (grant 12-01-00460-a).

Received July, 8, 2013, published May, 29, 2014.

formulas, being represented implicitly or for some special cases in [18]–[20], [24]–[31]. This approach is naturally interpretable in the class of relation partial algebras [10, 14].

In Section 1, we define a class of algebras distributing binary isolating formulas and introduce preliminary definitions, notations, and properties of algebras connected with relations of isolation and semi-isolation. In Sections 2, we describe some basic examples for these algebras and for types basing these algebras. In Section 3, we define a groupoid $\mathfrak{P}_{\nu(p)}$ of principal formulas on the set of all realizations of a 1-type p (assuming that there is an atomic model over a realization of p) with respect to a *regular* labelling function $\nu(p)$ for pairwise non-equivalent principal formulas $\varphi(a,y)$ for which $\varphi(a,x) \vdash p(x)$ holds, $\models p(a)$. In Section 4, we collect the basic properties of groupoids $\mathfrak{P}_{\nu(p)}$ and the significant subgroupoids of $\mathfrak{P}_{\nu(p)}$. In Section 5, using the successively-annihilating sums we construct two kinds of monoids $\mathfrak{P}_{\nu(p)}$ containing an arbitrary group. In Section 6, we produce a list of properties characterizing the class of groupoids $\mathfrak{P}_{\nu(p)}$. Features of these groupoids for the class of special theories are exposed in Section 7. In Section 8, we define the notion of join of groupoids and show the mechanism of extension of basic properties of $\mathfrak{P}_{\nu(p)}$ to the class of partial groupoids being joins of groupoids $\mathfrak{P}_{\nu(p)}$. In final Section 9, we produce a list of properties characterizing the class of partial groupoids corresponding to algebras of distributions for binary isolating formulas on a family of types.

We use the standard relation algebraic, model-theoretical, semigroup, and graph-theoretic terminology [3]–[11], [13, 14, 17] as well as some notions, notations, and constructions in [29].

1. Preliminary notions, notations, and properties

Definition 1.1 [1, 29, 30]. Let T be a complete theory, $\mathcal{M} \models T$. Consider types $p(x), q(y) \in S(\varnothing)$, realized in $\mathcal{M}$, and all (p,q)-*preserving* formulas $\varphi(x,y)$ of T, i. e., formulas for which there is $a \in M$ such that $\models p(a)$ and $\varphi(a,y) \vdash q(y)$. Now, for each such a formula $\varphi(x,y)$, we define a binary relation $R_{p,\varphi,q} \rightleftharpoons \{(a,b) \mid \mathcal{M} \models p(a) \wedge \varphi(a,b)\}$. If $(a,b) \in R_{p,\varphi,q}$, then (a,b) is called a (p,φ,q)-*arc*. If $\varphi(a,y)$ is principal (over a), the (p,φ,q)-arc (a,b) is also *principal*. If, in addition, $\varphi(x,b)$ is principal (over b), the set $[a,b] \rightleftharpoons \{(a,b),(b,a)\}$ is said to be a *principal* (p,φ,q)-*edge*. (p,φ,q)-arcs and (p,φ,q)-edges are called *arcs* and *edges* respectively if we say about fixed or some formula $\varphi(x,y)$. If (a,b) is a principal arc and (b,a) is not a principal arc (on any formula) then (a,b) is called *irreversible*.

For types $p(x), q(y) \in S(\varnothing)$, we denote by $\mathrm{PF}(p,q)$ the set

$$\{\varphi(x,y) \mid \varphi(a,y) \text{ is a principal formula}, \ \varphi(a,y) \vdash q(y), \text{ where } \models p(a)\}.$$

Let $\mathrm{PE}(p,q)$ be the set of all pairs of formulas $(\varphi(x,y), \psi(x,y)) \in \mathrm{PF}(p,q)$ such that for any (some) realization a of p the sets of solutions for $\varphi(a,y)$ and $\psi(a,y)$ coincide.

Clearly, $\mathrm{PE}(p,q)$ is an equivalence relation on the set $\mathrm{PF}(p,q)$. Notice that each $\mathrm{PE}(p,q)$-class E corresponds to either a principal edge or to an irreversible principal arc connecting realizations of p and q by some (any) formula in E. Thus the quotient $\mathrm{PF}(p,q)/\mathrm{PE}(p,q)$ is represented as a disjoint union of sets $\mathrm{PFS}(p,q)$ and $\mathrm{PFN}(p,q)$, where $\mathrm{PFS}(p,q)$ consists of $\mathrm{PE}(p,q)$-classes corresponding to principal edges and $\mathrm{PFN}(p,q)$ consists of $\mathrm{PE}(p,q)$-classes corresponding to irreversible principal arcs.

The sets $\mathrm{PF}(p,p)$, $\mathrm{PE}(p,p)$, $\mathrm{PFS}(p,p)$, and $\mathrm{PFN}(p,p)$ are denoted by $\mathrm{PF}(p)$, $\mathrm{PE}(p)$, $\mathrm{PFS}(p)$, and $\mathrm{PFN}(p)$ respectively.

Let T be a complete theory without finite models, $U = U^- \dot\cup \{0\} \dot\cup U^+$ be an alphabet of cardinality $\geq |S(T)|$, consisting of *negative elements* $u^- \in U^-$, *positive elements* $u^+ \in U^+$, and zero 0. As usual, we write $u < 0$ for any $u \in U^-$ and $u > 0$ for any $u \in U^+$.[1] The set $U^- \cup \{0\}$ is denoted by $U^{\leq 0}$ and $U^+ \cup \{0\}$ is denoted by $U^{\geq 0}$. Elements of U are called *labels*.

Let $\nu(p,q)$: $\mathrm{PF}(p,q)/\mathrm{PE}(p,q) \to U$ be an injective *labelling functions*, $p(x)$, $q(y) \in S(\varnothing)$, for which negative elements correspond to classes in $\mathrm{PFN}(p,q)/\mathrm{PE}(p,q)$ and non-negative elements correspond to classes in $\mathrm{PFS}(p,q)/\mathrm{PE}(p,q)$ such that 0 is defined only for $p = q$ and is represented by the formula $(x \approx y)$, $\nu(p) \rightleftharpoons \nu(p,p)$. We additionally assume that $\rho_{\nu(p)} \cap \rho_{\nu(q)} = \{0\}$ for $p \neq q$ (where, as usual, we denote by ρ_f the image of the function f) and $\rho_{\nu(p,q)} \cap \rho_{\nu(p',q')} = \varnothing$ if $p \neq q$ and $(p,q) \neq (p',q')$. Labelling functions with the properties above as well families of these functions are said to be *regular*. Below we shall consider only regular labelling functions and their regular families.

We denote by $\theta_{p,u,q}(x,y)$ a formula in $\mathrm{PF}(p,q)$ with the label $u \in \rho_{\nu(p,q)}$. If a type p is fixed and $p = q$ then a formula $\theta_{p,u,q}(x,y)$ is denoted by $\theta_u(x,y)$.

Note that if $\theta_{p,u,q}(x,y)$ and $\theta_{q,v,p}(x,y)$ are formulas witnessing that for realizations a and b of p and q respectively the pairs (a,b) and (b,a) are principal arcs then the formula $\theta_{p,u,q}(x,y) \wedge \theta_{q,v,p}(y,x)$ witnesses that $[a,b]$ is a principal edge. Moreover the (non-negative) label v corresponds uniquely to the *invertible* label u and vice versa. The labels u and v are *reciprocally inverse* and are denoted by v^{-1} and u^{-1} respectively.

For types $p_1, p_2, \ldots, p_{k+1} \in S^1(\varnothing)$ and sets $X_1, X_2, \ldots, X_k \subseteq U$ of labels we denote by

$$P(p_1, X_1, p_2, X_2, \ldots, p_k, X_k, p_{k+1})$$

the set of all labels $u \in \rho_{\nu(p_1,p_{k+1})}$ corresponding to formulas $\theta_{p_1,u,p_{k+1}}(x,y)$ satisfying, for realizations a of p_1 and some $u_1 \in X_1 \cap \rho_{\nu(p_1,p_2)}, \ldots, u_k \in X_k \cap \rho_{\nu(p_k,p_{k+1})}$, the following condition:

$$\theta_{p_1,u,p_{k+1}}(a,y) \vdash \theta_{p_1,u_1,p_2,u_2,\ldots,p_k,u_k,p_{k+1}}(a,y),$$

where

$$\begin{gathered}\theta_{p_1,u_1,p_2,u_2,\ldots,p_k,u_k,p_{k+1}}(x,y) \rightleftharpoons \\ \rightleftharpoons \exists x_2, x_3, \ldots, x_k(\theta_{p_1,u_1,p_2}(x,x_2) \wedge \theta_{p_2,u_2,p_3}(x_2,x_3) \wedge \ldots \\ \ldots \wedge \theta_{p_{k-1},u_{k-1},p_k}(x_{k-1},x_k) \wedge \theta_{p_k,u_k,p_{k+1}}(x_k,y)).\end{gathered}$$

Thus the Boolean $\mathcal{P}(U)$ of U is the universe of an *algebra of distributions of binary isolating formulas* with k-ary operations

$$P(p_1, \cdot, p_2, \cdot, \ldots, p_k, \cdot, p_{k+1}),$$

where $p_1, \ldots, p_{k+1} \in S^1(\varnothing)$. This algebra has a natural restriction to any family $R \subseteq S^1(\varnothing)$.

Clearly, replacing the set of labels bijectively we get an isomorphic algebra. In particular, there is a *canonical algebra*, where labels are presented by elements

$$\bigcup_{p,q} \mathrm{PF}(p,q)/\mathrm{PE}(p,q).$$

[1]If U is at most countable, we assume that U is a subset of the set $\mathbb{Z}$ of integers.

Nevertheless, we shall use an abstract set U of labels reflecting their signs and clarifying algebraic properties for operations on $\mathcal{P}(U)$.[2]

Note that if some set X_i is disjoint with $\rho_{\nu(p_i,p_{i+1})}$, in particular, if it is empty then

$$P(p_1, X_1, p_2, X_2, \dots, p_k, X_k, p_{k+1}) = \varnothing.$$

Note also that if $X_i \not\subseteq \rho_{\nu(p_i,p_{i+1})}$ for some i then

$$P(p_1, X_1, p_2, X_2, \dots, p_k, X_k, p_{k+1}) =$$
$$= P(p_1, X_1 \cap \rho_{\nu(p_1,p_2)}, p_2, X_2 \cap \rho_{\nu(p_1,p_2)}, \dots, p_k, X_k \cap \rho_{\nu(p_k,p_{k+1})}, p_{k+1}).$$

In view of the previous equality, it is enough to assume $X_i \subseteq \rho_{\nu(p_i,p_{i+1})}$, $i = 1, \dots, k$, for the values $P(p_1, X_1, p_2, X_2, \dots, p_k, X_k, p_{k+1})$.

If each set X_i is a singleton consisting of an element u_i then we use u_i instead of X_i in $P(p_1, X_1, p_2, X_2, \dots, p_k, X_k, p_{k+1})$ and write

$$P(p_1, u_1, p_2, u_2, \dots, p_k, u_k, p_{k+1}).$$

By the definition the following equality holds:

$$P(p_1, X_1, p_2, X_2, \dots, p_k, X_k, p_{k+1}) =$$
$$= \cup\{P(p_1, u_1, p_2, u_2, \dots, p_k, u_k, p_{k+1}) \mid u_1 \in X_1, \dots, u_k \in X_k\}.$$

Hence the specification of $P(p_1, X_1, p_2, X_2, \dots, p_k, X_k, p_{k+1})$ is reduced to the specifications of $P(p_1, u_1, p_2, u_2, \dots, p_k, u_k, p_{k+1})$. Note also that $P(p, X, q) = X$ for any $X \subseteq \rho_{\nu(p,q)}$.

Clearly, if $u_i = 0$ then $p_i = p_{i+1}$ for nonempty sets

$$P(p_1, u_1, p_2, u_2, \dots, p_i, 0, p_{i+1}, \dots, p_k, u_k, p_{k+1})$$

and the following conditions hold:

$$P(p_1, 0, p_1) = \{0\},$$
$$P(p_1, u_1, p_2, u_2, \dots, p_i, 0, p_{i+1}, u_{i+1}, p_{i+2}, \dots, p_k, u_k, p_{k+1}) =$$
$$= P(p_1, u_1, p_2, u_2, \dots, p_i, u_{i+1}, p_{i+2}, \dots, p_k, u_k, p_{k+1}).$$

If all types p_i equal to a type p then we write $P_p(X_1, X_2, \dots, X_k)$ and $P_p(u_1, u_2, \dots, u_k)$ as well as $\lfloor X_1, X_2, \dots, X_k \rfloor_p$ and $\lfloor u_1, u_2, \dots, u_k \rfloor_p$ instead of

$$P(p_1, X_1, p_2, X_2, \dots, p_k, X_k, p_{k+1})$$

and

$$P(p_1, u_1, p_2, u_2, \dots, p_k, u_k, p_{k+1})$$

respectively. We omit the index $\cdot_p$ if the type p is fixed. In this case, we write $\theta_{u_1,u_2,\dots,u_k}(x, y)$ instead of $\theta_{p,u_1,p,u_2,\dots,p,u_k,p}(x, y)$.

Definition 1.2 [15]. Let $\mathcal{M}$ be a model of a theory T, $\bar{a}$ and $\bar{b}$ be tuples in $\mathcal{M}$, A be a subset of M. The tuple $\bar{a}$ *semi-isolates* the tuple $\bar{b}$ over the set A if there exists a formula $\varphi(\bar{a}, \bar{y}) \in \mathrm{tp}(\bar{b}/A\bar{a})$ for which $\varphi(\bar{a}, \bar{y}) \vdash \mathrm{tp}(\bar{b}/A)$ holds. In this case we say

[2]Considering formulas instead of labels, the value $P(p_1, X_1, p_2, X_2, \dots, p_k, X_k, p_{k+1})$ depends on the choice of formulas θ and same formulas can be used for distinct tuples of types. We can assume that for $(p_1, p_{k+1}) \neq (p'_1, p'_{k+1})$, $\theta_{p_1,u,p_{k+1}}(x, y)$ and $\theta_{p'_1,u',p'_{k+1}}(x, y)$ are separated by some $\psi_1(x) \in p_1(x) \setminus p'_1(x)$ (with $\exists y \theta_{p_1,u,p_{k+1}}(x, y) \vdash \psi_1(x)$ and $\exists y \theta_{p'_1,u',p'_{k+1}}(x, y) \vdash \neg\psi_1(x)$) and $\psi_{k+1}(y) \in p_{k+1}(y) \setminus p'_{k+1}(y)$ (with $\exists x \theta_{p_1,u,p_{k+1}}(x, y) \vdash \psi_{k+1}(y)$ and $\exists x \theta_{p'_1,u',p'_{k+1}}(x, y) \vdash \neg\psi_{k+1}(y)$). At the same time for infinite families of types this procedure may fail. For these algebras, we put the difference into consideration and essentially use labels instead sets of formulas.

that the formula $\varphi(\bar{a},\bar{y})$ (with parameters in A) *witnesses that* $\bar{b}$ *is semi-isolated over* $\bar{a}$ *with respect to* A.

Similarly, a tuple $\bar{a}$ *isolates* a tuple $\bar{b}$ over A if there exists a formula $\varphi(\bar{a},\bar{y}) \in \mathrm{tp}(\bar{b}/A\bar{a})$ for which $\varphi(\bar{a},\bar{y}) \vdash \mathrm{tp}(\bar{b}/A)$ and $\varphi(\bar{a},\bar{y})$ is a principal (i. e., isolating) formula. In this case we say that the formula $\varphi(\bar{a},\bar{y})$ (with parameters in A) *witnesses that* $\bar{b}$ *is isolated over* $\bar{a}$ *with respect to* A.

If $\bar{a}$ (semi-)isolates $\bar{b}$ over $\varnothing$, we simply say that $\bar{a}$ (*semi-*)*isolates* $\bar{b}$; and if a formula $\varphi(\bar{a},\bar{y})$ witnesses that $\bar{a}$ (semi-)isolates $\bar{b}$ over $\varnothing$ then we say that $\varphi(\bar{a},\bar{y})$ *witnesses that* $\bar{a}$ (*semi-*)*isolates* $\bar{b}$.

If $q \in S(T)$ then SI_q (in the model $\mathcal{M}$) denotes the relation of semi-isolation (over $\varnothing$) on the set of all realizations of q:

$$\mathrm{SI}_q \rightleftharpoons \{(\bar{a},\bar{b}) \mid \mathcal{M} \models q(\bar{a}) \wedge q(\bar{b}) \text{ and } \bar{a} \text{ semi-isolates } \bar{b}\}.$$

Similarly, we denote by I_q (in the model $\mathcal{M}$) the relation of isolation (over $\varnothing$) on the set of all realizations of q:

$$I_p \rightleftharpoons \{(\bar{a},\bar{b}) \mid \ \mathcal{M} \models q(\bar{a}) \wedge q(\bar{b}) \text{ and } \bar{a} \text{ isolates } \bar{b}\}.$$

For a family $R \subset S(T)$ of 1-types we denote by I_R (in the model $\mathcal{M}$) the set

$$\{(a,b) \mid \mathrm{tp}(a),\mathrm{tp}(b) \in R \text{ and } a \text{ isolates } b\}$$

and by SI_R (in $\mathcal{M}$) the set

$$\{(a,b) \mid \mathrm{tp}(a),\mathrm{tp}(b) \in R \text{ and } a \text{ semi-isolates } b\}.$$

Clearly, $I_R \subseteq \mathrm{SI}_R$ and, for any set of realizations of types in R, the relations I_R and SI_R are reflexive. As shown in [15], the relation of semi-isolation on the set of tuples in an arbitrary model is transitive and, in particular, any relation SI_R is transitive.

Lemma 1.3 [1, 2, 12, 33, 34]. (1) *If a tuple* $\bar{a}$ *isolates a tuple* $\bar{b}$*, whereas* $\bar{b}$ *does not isolate* $\bar{a}$*, then* $\bar{b}$ *does not semi-isolate* $\bar{a}$.

(2) *If* $(a,b) \in I_R$ *and* $(b,a) \in \mathrm{SI}_R$ *then* $(b,a) \in I_R$.

Proof. (1) Suppose that $\varphi(\bar{a},\bar{y})$ isolates $\mathrm{tp}(\bar{b}/\bar{a})$. Assume the contrary (i. e., $\bar{b}$ semi-isolates $\bar{a}$) and take a formula $\psi(\bar{x},\bar{b})$ witnessing that $\bar{b}$ semi-isolates $\bar{a}$. Now as $\mathrm{tp}(\bar{a}/\bar{b})$ is non-isolated, there exists a formula $\chi(\bar{x},\bar{y})$ such that $\varphi(\bar{x},\bar{b}) \wedge \psi(\bar{x},\bar{b}) \wedge \chi(\bar{x},\bar{b})$ and $\varphi(\bar{x},\bar{b}) \wedge \psi(\bar{x},\bar{b}) \wedge \neg\chi(\bar{x},\bar{b})$ are both consistent. Moreover both formulas imply $\mathrm{tp}(\bar{a})$. Hence $\varphi(\bar{a},\bar{y}) \wedge \chi(\bar{a},\bar{y})$ and $\varphi(\bar{a},\bar{y}) \wedge \neg\chi(\bar{a},\bar{y})$ are both consistent. This contradicts the fact that $\varphi(\bar{a},\bar{y})$ is a principal formula.

(2) follows immediately from (1). □

Proposition 1.4. (1) *If* $p, q \in R$ *are principal types then* $\rho_{\nu(p,q)} \cup \rho_{\nu(q,p)} \subseteq U^{\geq 0}$.

(2) *If* $p, q \in R$*,* p *is a principal type and* q *is a non-principal type then* $\rho_{\nu(p,q)} = \varnothing$ *and* $\rho_{\nu(q,p)} \subseteq U^{-}$.

Proof is obvious. □

Corollary 1.5. *If* $p(x)$ *is a principal type then* $\rho_{\nu(p)} \subseteq U^{\geq 0}$.

Proposition 1.6. *Let* $p_1, p_2, \ldots, p_{k+1}$ *be types in* $S^1(\varnothing)$*. The following assertions hold.*

(1) *If* $u_i \in \rho_{\nu(p_i,p_{i+1})}$, $i = 1, \ldots, k$*, and some* u_i *is negative then*

$$P(p_1, u_1, p_2, u_2, \ldots, p_k, u_k, p_{k+1}) \subseteq U^{-}.$$

(2) *If* $u_i \in \rho_{\nu(p_i,p_{i+1})}$, $i = 1, \ldots, k$, *and all elements* u_i *are not negative then*

$$P(p_1, u_1, p_2, u_2, \ldots, p_k, u_k, p_{k+1}) \subseteq U^{\geq 0}.$$

(3) *If* $u_i \in \rho_{\nu(p_i,p_{i+1})}$, $i = 1, \ldots, k$, *and all elements* u_i *are non-negative, then all elements of the set*

$$X \rightleftharpoons P(p_1, u_1, p_2, u_2, \ldots, p_k, u_k, p_{k+1})$$

are invertible and the set $X^{-1} \rightleftharpoons \{v^{-1} \mid v \in X\}$ *coincides with the set*

$$P(p_{k+1}, u_k^{-1}, p_k, u_{k-1}^{-1}, \ldots, p_2, u_1^{-1}, p_1).$$

Proof. (1) Let v be a label in $P(p_1, u_1, p_2, u_2, \ldots, p_k, u_k, p_{k+1})$. Consider realizations a_i of p_i such that

$$\models \theta_{p_i,u_i,p_{i+1}}(a_i, a_{i+1}), \ \ i = 1, \ldots, k, \ \ \models \theta_{p_1,v,p_{k+1}}(a_1, a_{k+1}).$$

For the family $R = \{p_1, p_2, \ldots, p_{k+1}\}$ we have $(a_1, a_{k+1}) \in I_R$, $(a_i, a_{i+1}) \in I_R$, $i = 1, \ldots, k$, and so $(a_i, a_j) \in \mathrm{SI}_R$ for $i \leq j$. If $u_i < 0$ then $(a_{i+1}, a_i) \notin I_R$ and then, by Lemma 1.3, $(a_{i+1}, a_i) \notin \mathrm{SI}_R$. If $v \geq 0$ then $(a_{k+1}, a_1) \in I_R$ and, by transitivity of SI_R and $(a_{i+1}, a_{k+1}), (a_{k+1}, a_1), (a_1, a_i) \in \mathrm{SI}_R$ we get $(a_{i+1}, a_i) \in \mathrm{SI}_R$ that is impossible. Since the element $v \in P(p_1, u_1, p_2, u_2, \ldots, p_k, u_k, p_{k+1})$ is taken arbitrarily the set $P(p_1, u_1, p_2, u_2, \ldots, p_k, u_k, p_{k+1})$ consists of negative elements.

(2) Take again elements $v, a_1, a_2, \ldots, a_{k+1}$ as for (1). If $u_i \geq 0$ then $(a_{i+1}, a_i) \in I_R$, $i = 1, \ldots, k$. By transitivity of the relation SI_R, the element a_{k+1} semi-isolates the element a_1. In view of $(a_1, a_{k+1}) \in I_R$, by Lemma 1.3, we have $(a_{k+1}, a_1) \in I_R$ and so $v \geq 0$. Since the element

$$v \in P(p_1, u_1, p_2, u_2, \ldots, p_k, u_k, p_{k+1})$$

is taken arbitrarily the set $P(p_1, u_1, p_2, u_2, \ldots, p_k, u_k, p_{k+1})$ consists of non-negative elements.

(3) follows immediately from (2). □

Corollary 1.7. *Restrictions of* U *to the sets* $U^{\leq 0}$ *and* $U^{\geq 0}$ *form subalgebras of the algebra of distributions of binary isolating formulas. Each element of the restriction to* $U^{\geq 0}$ *has a unique inverse element. The operation of inversion is coordinated with the operations of the algebra.*

2. Examples

Consider some examples for distributions of labels of binary isolating formulas on sets of realizations of types $p(x) \in S(\varnothing)$ for countable theories T.

I. If $|\rho_{\nu(p)}| = 1$ then $(x \approx y)$ is the unique principal formula up to equivalence. It is possible only in the following cases:

(1) T is small (i. e., with countable $S(\varnothing)$) and satisfies some of the following condition:

(a) $p(x)$ is a principal type with a unique realization;

(b) $p(x)$ is a non-principal type such that if a set $\{\varphi(a, y) \wedge \neg(a \approx y)\} \cup p(y)$ is consistent, where $\varphi(x, y)$ is a formula of T, $\models p(a)$, then $\varphi(a, y) \not\vdash p(y)$;

(2) T is a theory with continuum many types and for any formula $\varphi(x, y)$ of T and for a realization a of $p(x)$ if the set $\{\varphi(a, y) \wedge \neg(a \approx y)\} \cup p(y)$ is consistent and $\varphi(a, y) \vdash p(y)$ then there are no isolating formulas $\psi(a, y)$ such that $\psi(a, y) \vdash \varphi(a, y) \wedge \neg(a \approx y)$.

The case 1,a is represented by a type being realized by a constant; the cases 1,b and 2 are represented by theories of unary predicates with non-principal types $p(x)$ and having countably many and continuum many types respectively.

II. Let $\rho_{\nu(p)} = \{0, 1\}$. Then $1^{-1} = 1$ and any realization a of p is linked with a unique realization b of p for which $\models \theta_1(a, b)$ and, moreover, $\models \theta_1(b, a)$. Then the set of realizations of p splits on two-element equivalence classes consisting of θ_1-edges. If p is a principal type of a small theory then a θ_1-edge is unique, and if p is non-principal, then the number of these edges can vary from 1 to infinity depending on a model of a theory.

III. Let $\rho_{\nu(p)} = \{-1, 0\}$ be a set for a small theory T. By Corollary 1.5, the type $p(x)$ is non-principal and the formula $\theta_{-1}(x, y)$ witnesses that SI_p is non-symmetric. The formula $\theta_{-1,-1}(x, y) \rightleftharpoons \exists z(\theta_{-1}(x, z) \wedge \theta_{-1}(z, y))$ is also witnessing that SI_p is non-symmetric. By assumption the formula $\theta_{-1,-1}(a, y)$ is equivalent to the formula $\theta_{-1}(a, y)$. It means that, on the set of all realizations of p, the relation described by the formula $\theta_{-1}(x, y) \vee (x \approx y)$ is an infinite partial order. This partial order is dense since if an element a has a covering element then the formula $\theta_{-1}(a, y)$ is equivalent to the disjunction of consistent formulas $\theta_{-1}(a, y) \wedge \theta_{-1,-1}(a, y)$ and $\theta_{-1}(a, y) \wedge \neg\theta_{-1,-1}(a, y)$, but it is impossible for the principal formula $\theta_{-1}(a, y)$.

We consider, as a theory with $\rho_{\nu(p)} = \{-1, 0\}$, the Ehrenfeucht's theory T, i. e. the theory of a structure $\mathcal{M}$, formed from the structure $\langle \mathbb{Q}; < \rangle$ by adding constants c_k, $c_k < c_{k+1}$, $k \in \omega$, such that $\lim\limits_{k\to\infty} c_k = \infty$. The type $p(x)$, isolated by the set of formulas $c_k < x$, $k \in \omega$, has exactly two non-equivalent isolating formulas: $\theta_{-1}(a, y) = (a < y)$ and $\theta_0(a, y) = (a \approx y)$, where $\models p(a)$.

IV. Let $\rho_{\nu(p)} = \{-1, 0, 1\}$. Realizing this equality, we consider the Ehrenfeucht's example, where each element a is replaced by an $<$-antichain consisting of two elements a' and a'' such that $\models \theta_1(a', a'') \wedge \theta_1(a'', a')$. Then we have the following equalities for the type $p(x)$ isolated by the set of formulas $c'_k < x$, $k \in \omega$:

$$P_p(-1, -1) = P_p(-1, 1) = P_p(1, -1) = \{-1\}, \ \ P_p(1, 1) = \{0\}.$$

V. The equality $\rho_{\nu(p)} = \{-2, -1, 0\}$ with $P_p(-2, -2) = \{-2\}$ and

$$P_p(-2, -1) = P_p(-1, -2) = P_p(-1, -1) = \{-1\}$$

can be fulfilled by two dense strict orders $<_1$ and $<_2$ on the set of all realizations of a non-principal type such that $<_1$ immerses $<_2$: $<_1 \circ <_2 \, = \, <_2 \circ <_1 \, = \, <_1$.

VI. Consider a dense linearly ordered set $\mathcal{M} = \langle \mathbb{Q}; < \rangle$, $T = \mathrm{Th}(\mathcal{M})$, and the unique 1-type p of T. Define a labelling function $\nu(p)$, for which 0 corresponds to the formula $(x \approx y)$, 1 to $(x < y)$, and 2 to $(y < x)$. We have $\rho_{\nu(p)} = \{0, 1, 2\}$, $P_p(1, 2) = P_p(2, 1) = \rho_{\nu(p)}$, $P_p(1, 1) = \{1\}$, $P_p(2, 2) = \{2\}$.

VII. Take a group $\langle G; * \rangle$ and define on the set G binary predicates Q_g, $g \in G$, by the following rule:

$$Q_g = \{(a, b) \in G^2 \mid a * g = b\}.$$

If $p(x)$ is a type (of a theory T) realized in any model $\mathcal{M} \models T$ containing G exactly by elements in G connected by definable relations Q_g, then the type p is isolated, the set G is finite, and $\rho_{\nu(p)}$ consists of non-negative elements bijective with elements in G. If $\rho_{\nu(p)}$ consists of non-negative elements, is bijective with G, and the set of realizations of a principal type p is not fixed, then, assuming the smallness of the theory, the set G is infinite and the number of connected components with respect to the relation $Q \rightleftharpoons \bigcup\limits_{g\in G} Q_g$ is not bounded. At last if the type p is not isolated then the number of Q-components on sets of realizations of p is also unbounded although the set G can be finite.

The Cayley table of the group $\langle G; *\rangle$ defines operations $P_p(\cdot,\ldots,\cdot)$ on the set $\rho_{\nu(p)}$ in accordance with links between the relations Q_g.

VIII. Applying to a concrete group we consider the structure $\mathcal{M} \rightleftharpoons \langle \mathbb{Z}; s^{(1)}\rangle$ with the unary *successor function* s: $\mathbb{Z} \leftrightarrow \mathbb{Z}$, where $s(n) = n+1$ for each $n \in \mathbb{Z}$. For the unique 1-type p of the theory $\mathrm{Th}(\mathcal{M})$, the set of pairwise non-equivalent formulas $\theta_u(x,y)$ is exhausted by the list: $y \approx \underbrace{s\ldots s}_{n\text{ times}}(x)$ and $x \approx \underbrace{s\ldots s}_{n\text{ times}}(y)$, $n \in \omega$. The set $\rho_{\nu(p)}$ consists of non-negative elements linked by the additive group of integers.

3. Algebra of distributions of binary isolating formulas on a set of realizations of a type

We consider a complete theory T, a type $p(x) \in S(T)$, a regular labelling function $\nu(p)$: $\mathrm{PF}(p)/\mathrm{PE}(p) \to U$, and a family of sets $P_p(u_1,\ldots,u_k)$, $u_1,\ldots,u_k \in \rho_{\nu(p)}$, $k \in \omega$, of labels for binary isolating formulas.

We denote by $\mathcal{M}_p$ and by $\mathcal{M}(a)$ an atomic model over a realization a of p.

Below we prove some basic properties for sets

$$\lfloor u_1,\ldots,u_k\rfloor \rightleftharpoons P_p(u_1,\ldots,u_k).$$

Proposition 3.1. 1. *For any $u_1,u_2,u_3 \in \rho_{\nu(p)}$ the following inclusions are satisfied:*

$$\lfloor\lfloor u_1,u_2\rfloor,u_3\rfloor \subseteq \lfloor u_1,u_2,u_3\rfloor,$$

$$\lfloor u_1,\lfloor u_2,u_3\rfloor\rfloor \subseteq \lfloor u_1,u_2,u_3\rfloor.$$

2. (Left semi-associativity) *If a model $\mathcal{M}_p$ exists then, for any $u_1,u_2,u_3 \in \rho_{\nu(p)}$,*

$$\lfloor\lfloor u_1,u_2\rfloor,u_3\rfloor = \lfloor u_1,u_2,u_3\rfloor.$$

3. (Criterion for right semi-associativity) *If the model $\mathcal{M}(a)$ exists, where $\models p(a)$, then for any $u_1,u_2,u_3 \in \rho_{\nu(p)}$ the equality*

$$\lfloor u_1,\lfloor u_2,u_3\rfloor\rfloor = \lfloor u_1,u_2,u_3\rfloor$$

holds if and only if for any $v \in \lfloor u_1,u_2,u_3\rfloor$ the formula $\theta_{u_1}(a,y_1)\wedge\theta_{u_2,u_3}(y_1,y)\wedge \theta_v(a,y)$ is realized in $\mathcal{M}(a)$ by a principal arc (b,c).

4. ((≥ 0)-associativity) *If the model $\mathcal{M}(a)$ exists, where $\models p(a)$, then for any $u_1,u_2,u_3 \in \rho_{\nu(p)}$, where $u_1 \geq 0$,*

$$\lfloor\lfloor u_1,u_2\rfloor,u_3\rfloor = \lfloor u_1,u_2,u_3\rfloor = \lfloor u_1,\lfloor u_2,u_3\rfloor\rfloor.$$

Proof. 1. For the proof of $\lfloor\lfloor u_1,u_2\rfloor,u_3\rfloor \subseteq \lfloor u_1,u_2,u_3\rfloor$, we take an arbitrary element $v \in \lfloor\lfloor u_1,u_2\rfloor,u_3\rfloor$. Then $v \in \lfloor v',u_3\rfloor$ for some $v' \in \lfloor u_1,u_2\rfloor$, and for any realization a of p we have

$$\theta_{v'}(a,x_2) \vdash \theta_{u_1,u_2}(a,x_2), \tag{1}$$

$$\theta_v(a,y) \vdash \theta_{v',u_3}(a,y). \tag{2}$$

By (1), we obtain

$$\theta_{v',u_3}(a,y) \vdash \theta_{u_1,u_2,u_3}(a,y). \tag{3}$$

Thus, (2) and (3) imply

$$\theta_v(a,y) \vdash \theta_{u_1,u_2,u_3}(a,y),$$

and, consequently, $v \in \lfloor u_1, u_2, u_3 \rfloor$.

Now we prove the inclusion $\lfloor u_1, \lfloor u_2, u_3 \rfloor\rfloor \subseteq \lfloor u_1, u_2, u_3 \rfloor$. Take an arbitrary element $v \in \lfloor u_1, \lfloor u_2, u_3 \rfloor\rfloor$. Then $v \in \lfloor u_1, v' \rfloor$ for some $v' \in \lfloor u_2, u_3 \rfloor$, and for any realization a of p we have

$$\theta_{v'}(a,y) \vdash \theta_{u_2,u_3}(a,y), \tag{4}$$

$$\theta_v(a,y) \vdash \theta_{u_1,v'}(a,y). \tag{5}$$

By (4), we obtain

$$\theta_{u_1,v'}(a,y) \vdash \theta_{u_1,u_2,u_3}(a,y). \tag{6}$$

Thus, (5) and (6) imply

$$\theta_v(a,y) \vdash \theta_{u_1,u_2,u_3}(a,y),$$

and, consequently, $v \in \lfloor u_1, u_2, u_3 \rfloor$.

2. Take a realization a of p and an element $v \in \lfloor u_1, u_2, u_3 \rfloor$. Then, for the principal formula $\theta_v(a,y)$, we have $\theta_v(a,y) \vdash \theta_{u_1,u_2,u_3}(a,y)$ and so

$$\mathcal{M}(a) \models \theta_{u_1}(a,b_1) \wedge \theta_{u_2}(b_1,b_2) \wedge \theta_{u_3}(b_2,c) \wedge \theta_v(a,c)$$

for some realizations b_1, b_2, and c of p. Since the model $\mathcal{M}(a)$ is atomic over a we have $\theta_{v'}(a,x_2) \vdash \theta_{u_1,u_2}(a,x_2)$ and $\mathcal{M}(a) \models \theta_{v'}(a,b_2)$ for some $v' \in \lfloor u_1, u_2 \rfloor$. Then $\theta_v(a,y) \vdash \theta_{v',u_3}(a,y)$ and hence $v \in \lfloor v', u_3 \rfloor$. Since the element $v \in \lfloor u_1, u_2, u_3 \rfloor$ is chosen arbitrarily, we obtain $\lfloor u_1, u_2, u_3 \rfloor \subseteq \lfloor \lfloor u_1, u_2 \rfloor, u_3 \rfloor$ that implies, by 1, the equality $\lfloor \lfloor u_1, u_2 \rfloor, u_3 \rfloor = \lfloor u_1, u_2, u_3 \rfloor$.

3. It is clear in view of 1.

4. By 1 and 2, it suffices to prove $\lfloor u_1, u_2, u_3 \rfloor \subseteq \lfloor u_1, \lfloor u_2, u_3 \rfloor\rfloor$ for any $u_1, u_2, u_3 \in \rho_{\nu(p)}$, where $u_1 \geq 0$. Let v be an arbitrary element in $\lfloor u_1, u_2, u_3 \rfloor$. Since $u_1 \geq 0$ there is the label u_1^{-1} and, in $\mathcal{M}(a)$, there are realizations b_1, b_2, c of p such that

$$\mathcal{M}(a) \models \theta_{u_1^{-1}}(a,b_1) \wedge \theta_{u_2}(a,b_2) \wedge \theta_{u_2,u_3}(a,c) \wedge \theta_v(b_1,c).$$

Since the type $\mathrm{tp}(c/a)$ is principal, we have $\mathcal{M}(a) \models \theta_{v'}(a,c)$ for some label v'. As $v' \in \lfloor u_2, u_3 \rfloor$ and $v \in \lfloor u_1, v' \rfloor$ we obtain $v \in \lfloor u_1, \lfloor u_2, u_3 \rfloor\rfloor$.

If $u_2 \geq 0$ and $u_3 \geq 0$ we also have the required inclusion by the following arguments. Since $v \geq 0$ by Proposition 1.6 (2), and there is a non-negative element $v^{-1} \in \lfloor u_3^{-1}, u_2^{-1}, u_1^{-1} \rfloor$, then, by 2, we have $v^{-1} \in \lfloor \lfloor u_3^{-1}, u_2^{-1} \rfloor, u_1^{-1} \rfloor$. Applying Proposition 1.6 (3), we obtain $v \in \lfloor u_1, \lfloor u_2, u_3 \rfloor\rfloor$. □

Proposition 3.1 implies

Corollary 3.2. *If there is a model $\mathcal{M}(a)$, where $\models p(a)$, then the following conditions hold:*

1. *For any $u_1, u_2, u_3 \in \rho_{\nu(p)}$, the equalities*

$$\lfloor \lfloor u_1, u_2 \rfloor, u_3 \rfloor = \lfloor u_1, u_2, u_3 \rfloor \supseteq \lfloor u_1, \lfloor u_2, u_3 \rfloor\rfloor$$

are satisfied.

2. (Criterion of associativity) *For any $u_1, u_2, u_3 \in \rho_{\nu(p)}$, the equality*

$$\lfloor \lfloor u_1, u_2 \rfloor, u_3 \rfloor = \lfloor u_1, \lfloor u_2, u_3 \rfloor\rfloor$$

hold if and only if $u_1 \geq 0$ *or, for any* $v \in \lfloor u_1, u_2, u_3 \rfloor$*, the formula* $\theta_{u_1}(a, y_1) \wedge \theta_{u_2,u_3}(y_1, y) \wedge \theta_v(a, y)$ *is realized in* $\mathcal{M}(a)$ *by a principal arc* (b, c)*.*

Note that if $\mathcal{M}_p$ does not exist the associativity (as well as semi-associativities) can be failed. For instance, if $\lfloor u_1, u_2 \rfloor = \varnothing$ then $\lfloor \lfloor u_1, u_2 \rfloor, u_3 \rfloor$ is also empty although $\lfloor u_1, \lfloor u_2, u_3 \rfloor \rfloor \neq \varnothing$ is admissible.

By Proposition 3.1, having $\mathcal{M}_p$ the associativity can be failed only by some labels u_1, u_2, u_3 with $u_1 < 0$. By Proposition 1.6 (1), in this case any label $v \in \lfloor u_1, u_2, u_3 \rfloor$ is also negative. The mechanism presented in the following example shows that the fault of right semi-associativity is admitted for any distribution of signs for nonzero labels u_2, u_3: there are small theories with

$$\lfloor u_1, u_2, u_3 \rfloor \neq \lfloor u_1, \lfloor u_2, u_3 \rfloor \rfloor. \tag{7}$$

Example 3.3. Obtaining (7) with $u_1 < 0$, $u_2, u_3 \neq 0$, and a label $v \in \lfloor u_1, u_2, u_3 \rfloor \setminus \lfloor u_1, \lfloor u_2, u_3 \rfloor \rfloor$ (i. e., by Proposition 3.1 (3), for the non-realizability of the formula $\varphi(a, y_1, y_2) \rightleftharpoons \theta_{u_1}(a, y_1) \wedge \theta_{u_2,u_3}(y_1, y_2) \wedge \theta_v(a, y_2)$ by principal arcs) we consider the schema of the realization of a non-p-principal $(2, p)$-type in a model $\mathcal{M}_p$ of small theory presented in [29, Example 1.3.1] (see also [22]). Defining the type $p(x)$ we introduce a Q_{u_1}- and Q_v-*ordered* (for binary predicates Q_{u_1} and Q_v corresponding to the labels u_1 and v) coloring Col: $M_0 \to \omega \cup \{\infty\}$ of some graph Γ producing unary predicates $\mathrm{Col}_n = \{a \in M_0 \mid \mathrm{Col}(a) = n\}$, $n \in \omega$, such that:

(a) for any $m \leq n < \omega$ and $\alpha = u_1, v$, there are elements $a, b \in M_0$ for which $\models \mathrm{Col}_m(a) \wedge \mathrm{Col}_n(b) \wedge Q_\alpha(a, b)$;

(b) if $m < n < \omega$ then there are no elements $c, d \in M_0$ for which $\models \mathrm{Col}_m(c) \wedge \mathrm{Col}_n(d) \wedge Q_\alpha(d, c)$.

Moreover, using a generic construction [29, Chapter 2] for Γ we obtain the unique non-principal 1-type $p(x)$ and it is isolated by the set $\{\neg\mathrm{Col}_n(x) \mid n \in \omega\}$.

For each label u_i, $i \in \{2, 3\}$, depending on its label, we define a binary predicate Q_{u_i} linking only elements of the same color if u_i is positive, and with the Q_{u_i}-ordering of Col if $u_i < 0$. Now we introduce labels v'_n, $n \in \omega$, being negative if $u_2 < 0$ or $u_3 < 0$ and positive otherwise, such that $\lfloor u_2, u_3 \rfloor = \{v'_n \mid n \in \omega\}$. We define pairwise disjoint predicates $Q_{v'_n}$ linking only elements of the same color if $v'_n > 0$, and linking with the $Q_{v'_n}$-ordering of Col if $v'_n < 0$. Moreover, we require the following condition: for any element a_k of the color k the formula $\varphi(a_k, y_1, y_2)$ is realized by principal $Q_{v'_n}$-arcs exactly with $n \geq k$. It means that, for $\models p(a)$, the formula $\varphi(a, y_1, y_2)$ is not realized by principal arcs, since this formula witnesses that the non-p-principal $(2, p)$-type $q(y_1, y_2) \rightleftharpoons p(y_1) \cup p(y_2) \cup \{\theta_{u_2,u_3}(y_1, y_2)\} \cup \{\neg\theta_{v'_n}(y_1, y_2) \mid n \in \omega\}$ is realized in $\mathcal{M}_p$. □

If the model $\mathcal{M}_p$ exists then, using the left semi-associativity, by induction on the number of brackets one prove that all operations $\lfloor \cdot, \cdot, \ldots, \cdot \rfloor$ acting on sets in $\mathcal{P}(\rho_{\nu(p)}) \setminus \{\varnothing\}$ are generated by the binary operation $\lfloor \cdot, \cdot \rfloor$ on the set $\mathcal{P}(\rho_{\nu(p)}) \setminus \{\varnothing\}$. If we have the right semi-associativity, the values $\lfloor X_1, X_2, \ldots, X_k \rfloor$, $X_1, X_2, \ldots, X_k \subseteq \rho_{\nu(p)}$, do not depend on sequences of placements of brackets for

$$X_{i,i+1,\ldots,i+m+n} \rightleftharpoons \lfloor X_{i,i+1,\ldots,i+m}, X_{i+m+1,i+m+2,\ldots,i+m+n} \rfloor,$$

where $X_{1,2,\ldots,k} = \lfloor X_1, X_2, \ldots, X_k \rfloor$.

Thus, having $\mathcal{M}_p$, the groupoid $\mathfrak{P}_{\nu(p)} \rightleftharpoons \langle \mathcal{P}(\rho_{\nu(p)}) \setminus \{\varnothing\}; \lfloor \cdot, \cdot \rfloor \rangle$, being a (left) *semi-associative* algebra, admits to represent all operations $\lfloor \cdot, \cdot, \ldots, \cdot \rfloor$ by terms of the language $\lfloor \cdot, \cdot \rfloor$. Below the operation $\lfloor \cdot, \cdot \rfloor$ will be also denoted by $\cdot$ and we shall write uv instead of $u \cdot v$. If the right semi-associativity fails we shall assume, for $u_1 u_2 \dots u_k$, the following distribution of parentheses: $(((u_1 \cdot u_2) \cdot \dots) \cdot u_k)$.

Since by the choice of the label 0 for the formula $(x \approx y)$ the equalities $X \cdot \{0\} = X$ and $\{0\} \cdot X = X$ are true for any $X \subseteq \rho_{\nu(p)}$, the groupoid $\mathfrak{P}_{\nu(p)}$ has the unit $\{0\}$, and it is a monoid if the algebra is right semi-associative. We have

$$Y \cdot Z = \bigcup \{yz \mid y \in Y, z \in Z\}$$

for any sets $Y, Z \in \mathcal{P}(\rho_{\nu(p)}) \setminus \{\varnothing\}$ in this structure.

Thus the following proposition holds.

Proposition 3.4. *For any complete theory T, any type $p \in S^1(T)$ having the model $\mathcal{M}_p$, and the regular labelling function $\nu(p)$, any operation $P_p(\cdot, \cdot, \dots, \cdot)$ on the set $\mathcal{P}(\rho_{\nu(p)}) \setminus \{\varnothing\}$ is interpretable by a term of the groupoid $\mathfrak{P}_{\nu(p)}$.*

The groupoid $\mathfrak{P}_{\nu(p)}$ is called the *groupoid of binary isolating formulas over the labelling function $\nu(p)$* or the *$I_{\nu(p)}$-groupoid.*

Propositions 1.6 and 3.1 imply

Proposition 3.5. *For any complete theory T, any type $p \in S^1(T)$ having the model $\mathcal{M}_p$, and the regular labelling function $\nu(p)$, the restriction of the groupoid $\mathfrak{P}_{\nu(p)}$ to the set of non-positive (respectively non-negative) labels is a semi-associative subalgebra of $\mathfrak{P}_{\nu(p)}$ with the unit $\{0\}$ (and, moreover, it is a monoid).*

4. Characterization for transitivity of the relation I_p. Deterministic, almost deterministic $I_{\nu(p)}$-groupoids and elements

The following assertion gives a characterization of transitivity of the relation I_p. For simplicity we formulate and prove it for a 1-type p although the proof implies the validity for any complete type r of a theory with a model $\mathcal{M}_r$.

Proposition 4.1. *Let $p(x)$ be a complete type of a complete theory T having a model $\mathcal{M}_p$, and $\nu(p)$ be a regular labelling function. The following conditions are equivalent:*

(1) *the relation I_p (on a set of realizations of p in a model $\mathcal{M} \models T$) is transitive;*

(2) *for any labels $u_1, u_2 \in \rho_{\nu(p)}$, the set $P_p(u_1, u_2)$ is finite.*

Proof. Let a, b, c be realizations of p such that $(a, b) \in I_p$ and $(b, c) \in I_p$ witnessed by isolating formulas $\theta_{u_1}(a, y)$ and $\theta_{u_2}(b, y)$. If the set $P_p(u_1, u_2)$ is finite and consists of labels $v_1, \dots, v_k$ then, by existence of $\mathcal{M}_p$, the formula $\theta_{u_1,u_2}(a, y)$ is equivalent to the formula $\bigvee\limits_{i=1}^{k} \theta_{v_i}(a, y)$. Since $\models \theta_{u_1,u_2}(a, c)$ we have $\models \bigvee\limits_{i=1}^{k} \theta_{v_i}(a, c)$ and hence $\models \theta_{v_i}(a, c)$ for some i. Thus, $(a, c) \in I_p$ and it is witnessed by the formula $\theta_{v_i}(x, y)$. In view of arbitrary choice of elements a, b, c the implication (2) $\Rightarrow$ (1) is true.

Now, we assume that, for some $u_1, u_2 \in \rho_{\nu(p)}$, the set $P_p(u_1, u_2)$ is infinite. Then by compactness, for a realization a of p, the set

$$q(a, y) \rightleftharpoons \{\theta_{u_1,u_2}(a, y)\} \cup \{\neg\theta_v(a, y) \mid v \in P_p(u_1, u_2)\}$$

is consistent. Consider realizations b and c of p such that $\models \theta_{u_1}(a, b) \wedge \theta_{u_2}(b, c)$ and $\models q(a, c)$. We have $(a, b) \in I_p$, $(b, c) \in I_p$, and $(a, c) \notin I_p$ by the construction of q. Thus the relation I_p is not transitive and we obtain (1) $\Rightarrow$ (2). $\square$

Definition 4.2. A structure $\mathfrak{P}_{\nu(p)}$ is called (*almost*) *deterministic* if the set $\lfloor u_1, u_2 \rfloor$ is a singleton (is nonempty and finite) for any $u_1, u_2 \in \rho_{\nu(p)}$.

Example 4.3. By the definition any polygonometrical theory $\mathrm{Th}(\mathrm{pm}(G_1,\, G_2, \mathcal{P}))$ (see [23, 32]) has a unique 1-type $p(x) \in S(\varnothing)$ and, thus, the structure $\mathfrak{P}_{\nu(p)}$ is a monoid with non-negative labels. The determinacy of $\mathfrak{P}_{\nu(p)}$ means that the group G_1 of sides is unit and there are at most two points in $\mathcal{P}$, or the group G_2 of angles is unit and either $\mathfrak{P}_{\nu(p)}$ contains unique line or G_1 is infinite. The almost determinacy of $\mathfrak{P}_{\nu(p)}$ means that the group G_2 is finite. □

Proposition 4.4. *If there is a model $\mathcal{M}_p$ and the structure $\mathfrak{P}_{\nu(p)}$ is almost deterministic then $\mathfrak{P}_{\nu(p)}$ is a monoid.*

Proof. As noticed in Proposition 3.1, the unique obstacle, for $\mathfrak{P}_{\nu(p)}$ to be a monoid, can be only the existence of labels u_1, u_2, u_3, v, $u_1 < 0$, $v < 0$, for which $v \in \lfloor u_1, u_2, u_3 \rfloor$ and there are no $v' \in \lfloor u_2, u_3 \rfloor$ with $v \in \lfloor u_1, v' \rfloor$. But, by the hypothesis, the set $\lfloor u_2, u_3 \rfloor$ consists of finitely many labels $v_1, \ldots, v_k$. Now we take in $\mathcal{M}(a)$, where $\models p(a)$, elements b, c, d such that

$$\mathcal{M}(a) \models \theta_{u_1}(a, b) \wedge \theta_{u_2}(b, c) \wedge \theta_{u_3}(c, d) \wedge \theta_v(a, d).$$

Since the formula $\theta_{u_2,u_3}(b, y)$ is equivalent to the formula $\bigvee\limits_{i=1}^{k} \theta_{v_i}(b, y)$, there is a required label $v' = v_i$ such that $\mathcal{M}(a) \models \theta_{v'}(b, d)$. □

Any deterministic structure $\mathfrak{P}_{\nu(p)}$ is a monoid (being almost deterministic). It is generated by the monoid $\mathfrak{P}'_{\nu(p)} = \langle \rho_{\nu(p)}; \odot \rangle$, where $\lfloor u, v \rfloor = \{u \odot v\}$ for $u, v \in \rho_{\nu(p)}$.

Thus, the deterministic monoids can be defined by usual Cayley tables for monoids on a set of labels in U while the almost deterministic monoids are represented by one-to-finite functions with two arguments, i. e., by ternary predicates with finitely many third coordinates for fixed first and second coordinates.

Considering deterministic structures $\mathfrak{P}$, being restrictions of the monoid $\mathfrak{P}_{\nu(p)}$ to some subalphabets U_0 of the alphabet U, we denote by $\mathfrak{P}'$ the *generating* monoid $\langle U_0; \odot \rangle$ such that $\lfloor u, v \rfloor \cap U_0 = \{u \odot v\}$ for $u, v \in U_0$.

The following proposition is a reformulation of Proposition 4.1.

Proposition 4.5. *Let $p(x)$ be a complete type of a theory T having a model $\mathcal{M}_p$, $\nu(p)$ be a regular labelling function. The following conditions are equivalent:*

(1) *the relation I_p (on a set of realizations of p in a model $\mathcal{M} \models T$) is transitive;*

(2) *the structure $\mathfrak{P}_{\nu(p)}$ is an almost deterministic monoid.*

Note that there are no principal edges linking distinct realizations of p if and only if the relation I_p is antisymmetric. Since I_p is reflexive, the definition of $\nu(p)$ and Propositions 1.6, 4.5 imply

Corollary 4.6. *Let $p(x)$ be a complete type of a theory T having a model $\mathcal{M}_p$, $\nu(p)$ be a regular labelling function. The following conditions are equivalent:*

(1) *the relation I_p (on the set of realizations of p in any model $\mathcal{M} \models T$) is a partial order;*

(2) *the structure $\mathfrak{P}_{\nu(p)}$ is an almost deterministic monoid and $\rho_{\nu(p)} \subseteq U^{\leq 0}$.*

This partial order I_p is identical if and only if $\rho_{\nu(p)} = \{0\}$. If I_p is not identical, it has infinite chains.

Definition 4.7 [1, 24, 29, 30]. A countable model $\mathcal{M}$ of theory T is *limit* (accordingly *limit over a type $p \in S(T)$*) if $\mathcal{M}$ is not prime over tuples and $\mathcal{M} = \bigcup\limits_{n \in \omega} \mathcal{M}(\bar{a}_n)$, where $(\mathcal{M}(\bar{a}_n))_{n\in\omega}$ is an elementary chain of prime models over tuples $\bar{a}_n$ (and $\mathcal{M} \models p(\bar{a}_n)$), $n \in \omega$.

A characterization for the (non-)symmetry of a relation I_q for the class of small theories is obtained in [1]:

Theorem 4.8. *Let $q(\bar{x})$ be a complete type of a small theory T. The following conditions are equivalent:*

(1) *there exists a limit model over q;*

(2) *the relation I_q of isolation on the set of realizations of q in a* (*any*) *model $\mathcal{M} \models T$ realizing q is non-symmetric;*

(3) *in some* (*any*) *model $\mathcal{M} \models T$ realizing q, there exist realizations $\bar{a}$ and $\bar{b}$ of q such that the type* $\mathrm{tp}(\bar{b}/\bar{a})$ *is principal and $\bar{b}$ does not semi-isolate $\bar{a}$ and, in particular,* SI_q *is non-symmetric on $\mathcal{M}$.*

Proposition 4.5 and Theorem 4.8 imply

Corollary 4.9. *Let $p(x)$ be a complete type of a small theory T, $\nu(p)$ be a regular labelling function. The following conditions are equivalent:*

(1) I_p (*on the set of realizations of p in any model $\mathcal{M} \models T$*) *is an equivalence relation;*

(2) *the structure $\mathfrak{P}_{\nu(p)}$ is an almost deterministic monoid and there are no limit models over p;*

(3) *the structure $\mathfrak{P}_{\nu(p)}$ is an almost deterministic monoid and consists of non-negative labels.*

In Corollary 4.9, the equivalence of (1) and (3) is implied by the existence of $\mathcal{M}_p$ without the assumption of smallness of T.

Definition 4.10. An element $u \in \rho_{\nu(p)}$ is called (*almost*) *deterministic* if for any/some realization a of p the formula $\theta_u(a, y)$ has unique solution (has finitely many solutions).

Note that there is no negative almost deterministic element u for a theory T having an atomic model and finitely many non-principal 1-types in $S(T)$.[3] Indeed, otherwise the presence of a negative element u implies that the type $p(x)$ is non-principal and the relation SI_p is not symmetric, that is witnessed by the formula $\theta_u(x, y)$. Since for $\models p(a)$ the isolating formula $\theta_u(a, y)$ has k solutions for some $k \in \omega \setminus \{0\}$, there exists a formula $\varphi(x) \in p(x)$ such that for any realization b of $\varphi(x)$ there are exactly k solutions of the formula $\theta_u(b, y)$. Moreover, since there are finitely many non-principal types we may assume that $\varphi(x)$ is not consistent with each non-principal type $q(x) \neq p(x)$. Let $\models \theta_u(a, d)$ for some a and d realizing the type p. By non-symmetry of the relation SI_p the formula $\varphi(x) \wedge \theta_u(x, d)$ has a solution c which does not realize p and any other non-principal 1-type. So c realizes some principal type, which is isolated by a formula $\psi(x)$. Since $\theta_u(c, y)$ has finitely many solutions there is a formula $\mu(y)$ such that $\models \theta_u(c, d) \wedge \mu(d)$ and $\theta_u(c, y) \wedge \mu(y) \vdash p(y)$. Then the formula $\exists x(\psi(x) \wedge \mu(y) \wedge \theta_u(x, y))$ isolates the non-principal type p, for a contradiction.

At the same time, Example 1.4.3 in [29] illustrates that there are theories T with even deterministic negative elements u, where there are infinitely many non-principal 1-types in $S(T)$.

[3]The following arguments, in fact, are identical to the remark after the proof of [29, Proposition 1.4.2].

Proposition 4.11. *If elements u and v are* (*almost*) *deterministic then any element v' in $u \cdot v$ is* (*almost*) *deterministic.*

Proof. Consider formulas $\theta_u(a, y)$, $\theta_v(a, y)$, and $\theta_{u,v}(a, y)$, where $\models p(a)$. If u and v are deterministic then all these formulas have unique solutions, so the element $v' \in u \cdot v$ is unique, and the formulas $\theta_{u,v}(a, y)$ and $\theta_{v'}(a, y)$ are equivalent.

If u and v are almost deterministic then the formulas $\theta_u(a, y)$, $\theta_v(a, y)$, and $\theta_{u,v}(a, y)$ have finitely many solutions. It implies that the set $u \cdot v$ is finite and there are finitely many solutions for the formulas $\theta_{v'}(a, y)$, $v' \in u \cdot v$. □

Proposition 4.11 immediately implies

Corollary 4.12. *For any groupoid $\mathfrak{P}_{\nu(p)}$ its restriction $\mathfrak{P}_{\nu(p),d}$* (*respectively $\mathfrak{P}_{\nu(p),\mathrm{ad}}$*) *to the set of* (*almost*) *deterministic elements is a monoid.*

The following proposition presents a characterization for determinacy of non-negative elements in $\mathfrak{P}_{\nu(p)}$ assuming existence of the model $\mathcal{M}_p$.

Proposition 4.13. *If the model $\mathcal{M}_p$ exists then an element $u \geq 0$ in $\mathfrak{P}_{\nu(p)}$ is deterministic if and only if $u^{-1} \cdot u = \{0\}$.*

Proof. Let an element u be deterministic, i. e., $\theta_u(a, \mathcal{M}_p) = \{b\}$ for some realizations a and b of p in $\mathcal{M}_p$. Then $\theta_{u^{-1},u}(b, \mathcal{M}_p) = \{b\}$, i. e., $u^{-1} \cdot u = \{0\}$.

We assume now that $u^{-1} \cdot u = \{0\}$ and prove that the formula $\theta_u(a, y)$, where $\models p(a)$, has the unique solution. Assume on the contrary that there are at least two solutions b_1 and b_2. Then we have $\models \theta_{u^{-1}}(b_1, a) \wedge \theta_u(a, b_2)$. Since $0 \in u^{-1} \cdot u$, $\theta_0(b_1, y) = (b_1 \approx y)$, and $\theta_0(b_1, y) \vdash \theta_{u^{-1},u}(b_1, y)$ then the consistency of the formula $\theta_{u^{-1},u}(b_1, y) \wedge \neg\theta_0(b_1, y)$ and the existence of $\mathcal{M}_p$ imply that there is an isolating formula $\theta_v(b_1, y)$, $v \neq 0$, such that $\theta_v(b_1, y) \vdash \theta_{u^{-1},u}(b_1, y)$. It contradicts the condition $u^{-1} \cdot u = \{0\}$. □

Unlike the determinacy there are no similar characterizations for the almost determinacy.

Example 4.14. If $\Gamma = \langle M; R \rangle$ is an acyclic undirected graph consisting of vertices of fixed positive degree v then for the unique 1-type $p(x) \in S(\mathrm{Th}(\Gamma))$, for the principal formulas $\theta_n(x, y)$, where $\models \theta_n(a, b) \Leftrightarrow \rho(a, b) = n$, $n \in \omega$, and for the monoid $\mathfrak{P}_{\nu(p)}$ over the alphabet ω we have $m \cdot n = \{m + n, |m - n|\}$. In particular, $n = n^{-1}$ and $n \cdot n = \{0, 2n\}$. At the same time, the monoid $\mathfrak{P}_{\nu(p)}$ does not depend on $v \in (\omega \cup \{\infty\}) \setminus \{0\}$. □

Proposition 4.15. *If $\mathfrak{P}_{\nu(p)}$ is a deterministic monoid then the structure $\mathfrak{P}'_{\nu(p)}$ is a group if and only if $\rho_{\nu(p)}$ consists of non-negative elements.*

Proof. At first we observe that, by definition, if $u \in \rho_{\nu(p)}$ is negative then there are no labels v such that $u \odot v = 0$. Hence, if $\mathrm{PFN}(p) \neq \varnothing$ then $\mathfrak{P}'_{\nu(p)}$ is not a group.

Now we assume that $\rho_{\nu(p)} \cap U^- = \varnothing$ and prove that the structure $\mathfrak{P}'_{\nu(p)}$ is a group. Indeed, if $\mathrm{PFN}(p) = \varnothing$ then for any element $u \in \rho_{\nu(p)}$ there is the (unique) inverse element $v = u^{-1}$ such that $0 \in u \cdot v$. As the monoid $\mathfrak{P}_{\nu(p)}$ is deterministic we obtain $u \odot v = 0$. □

Corollary 4.16. *If the model $\mathcal{M}_p$ exists, the monoid $\mathfrak{P}_{\nu(p)}$ is deterministic, and $\mathfrak{P}'_{\nu(p)}$ is a group, then all elements in $\mathfrak{P}'_{\nu(p)}$ are deterministic.*

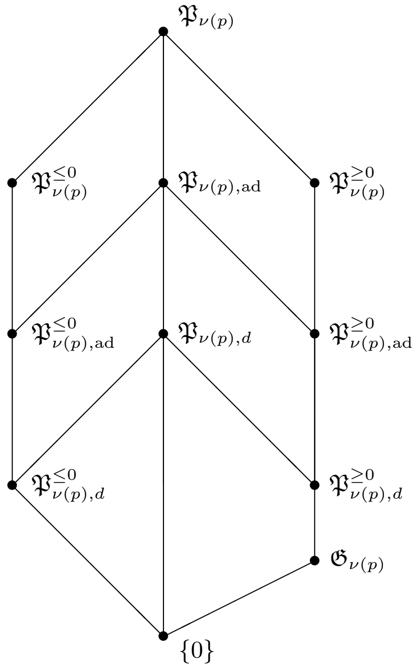

Fig. 1

Proof. Since by Proposition 4.15 the set $\rho_{\nu(p)}$ consists of non-negative elements then, as the monoid $\mathfrak{P}_{\nu(p)}$ is deterministic, by Proposition 4.13 each element in $\mathfrak{P}'_{\nu(p)}$ is deterministic. □

Proposition 4.17. *If the model $\mathcal{M}_p$ exists then the set $\rho^{\geq 0}_{\nu(p),d}$ of all non-negative deterministic elements u in $\rho_{\nu(p)}$, for which elements u^{-1} are also deterministic, forms a deterministic submonoid $\mathfrak{G}_{\nu(p)}$ of the monoid $\mathfrak{P}_{\nu(p),d}$, consisting of deterministic elements of $\mathfrak{P}_{\nu(p)}$, and such that $(\mathfrak{G}_{\nu(p)})'$ is a group.*

Proof. Since for any $u \in \rho^{\geq 0}_{\nu(p),d}$ the element u^{-1} satisfying $u \cdot u^{-1} = u^{-1} \cdot u = \{0\}$ belongs to $\rho^{\geq 0}_{\nu(p),d}$ it suffices to observe that if $u, v \in \rho^{\geq 0}_{\nu(p),d}$ then $u \cdot v$ contains a unique element v' and this element is deterministic by Proposition 4.11. □

A Hasse diagram is presented in Figure 1 illustrating the links of the structure $\mathfrak{P}_{\nu(p)}$ with structures above, being restrictions of $\mathfrak{P}_{\nu(p)}$ to subalphabets of U. Here the superscripts $\cdot^{\leq 0}$ and $\cdot^{\geq 0}$ point out on restrictions of $\mathfrak{P}_{\nu(p)}$ to the sets of non-positive and non-negative elements respectively, the subscripts $\cdot_d$ and $\cdot_{\mathrm{ad}}$ indicate the sets of deterministic and almost deterministic elements. By Propositions 3.1 and 4.4, just $\mathfrak{P}_{\nu(p)}$ and $\mathfrak{P}^{\leq 0}_{\nu(p)}$ may not be monoids.

The following proposition shows that for each label $u \in \rho_{\nu(p)}$, the monoid $\mathfrak{P}_{\nu(p)}$ contains a monoid $\mathfrak{P}_{\nu(p),u}$ with 0 being a restriction of a submonoid of $\mathfrak{P}_{\nu(p)}$ and consisting of all labels $v \in \rho_{\nu(p)}$ for which $u \in u \cdot v$.

Proposition 4.18. *If $\mathfrak{P}_{\nu(p)}$ is a monoid and $u \in (u \cdot v) \cap (u \cdot w)$ for labels u, v, w in $\mathfrak{P}_{\nu(p)}$, then $u \in (u \cdot (v \cdot w))$.*

Proof. Since $u \in (u \cdot v) \cap (u \cdot w)$, for formulas $\theta_u(x, y)$, $\theta_v(x, y)$, $\theta_w(x, y)$ and a realization a of p, there are realizations b, c, d of p, for which

$$\models \theta_u(a, b) \wedge \theta_u(a, c) \wedge \theta_u(a, d) \wedge \theta_v(b, c) \wedge \theta_w(c, d).$$

Then $u \in \lfloor u, v, w \rfloor$. As $\mathfrak{P}_{\nu(p)}$ is a monoid, there is $v' \in (v \cdot w)$ for which $u \in u \cdot v'$. Hence $u \in (u \cdot (v \cdot w))$. □

5. Graph compositions and monoid compositions

Recall [9] that the *composition* $\Gamma_1[\Gamma_2]$ *of graphs* $\Gamma_1 = \langle X_1; R_1 \rangle$ and $\Gamma_2 = \langle X_2; R_2 \rangle$ is the graph $\langle X_1 \times X_2; R \rangle$, where $((a_1, b_1), (a_2, b_2)) \in R$ if and only if some of the following conditions is met:

1) $(a_1, a_2) \in R_1$;

2) $a_1 = a_2$ and $(b_1, b_2) \in R_2$.

Similarly we define the notion of monoid composition.

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be monoids, for which 0 is the unit, $S_1 \subseteq U^{\leq 0}$, and $S_2 \subseteq U^{\geq 0}$. The *composition*, or the *sequentially-annihilating band* (see [8, 13]), $\mathcal{S}_1[\mathcal{S}_2]$ *of monoids* $\mathcal{S}_1$ and $\mathcal{S}_2$ is the algebra $\langle S_1 \cup S_2; \odot \rangle$, where $\langle S_1 \cup S_2; \odot \rangle \upharpoonright S_i = \mathcal{S}_i$ for $i = 1, 2$, and $u \odot v = v \odot u = u$ for $u < 0$ and $v > 0$.

Proposition 5.1 [13]. *Any sequentially-annihilating band $\mathcal{S}_1[\mathcal{S}_2]$ is a monoid.*

Proof is obvious. □

Theorem 5.2. *For any group $\langle G; * \rangle$, where the universe consists of non-negative elements and 0 denotes the group unit, and for the monoid $\langle \{-1, 0\}; + \rangle$ with the zero element 0 and the idempotent element -1, there is a theory T with a type $p \in S(T)$ and a regular labelling function $\nu(p)$ such that the monoid $\mathfrak{P}'_{\nu(p)}$ coincides with the monoid $\langle \{-1, 0\}; + \rangle[\langle G; * \rangle]$.*

Proof. We construct a structure $\mathcal{M}$ such that its theory $T = \mathrm{Th}(\mathcal{M})$ has a type $p(x) \in S(T)$ and a regular labelling function $\nu(p)$ with $\mathfrak{P}'_{\nu(p)} = \langle \{-1, 0\}; + \rangle[\langle G; * \rangle]$. For this aim we consider the Ehrenfeucht's example $\langle \mathbb{Q}; <, c_k \rangle_{k \in \omega}$, $c_k < c_{k+1}$, $k \in \omega$, such that each element a is replaced by a $<$-antichain consisting of $|G|$ elements and forming a free 1-generated polygon over the group $\langle G; * \rangle$ isomorphic to the structure $\mathcal{G} = \langle G; Q_g \rangle_{g \in G}$, where $Q_g = \{(a, b) \in G^2 \mid a * g = b\}$, $g \in G$. Here we replace each constant c_k by a unary predicate R_k consisting of elements of a copy of $\mathcal{G}$. Thus we form the composition $\langle \mathbb{Q}; < \rangle[\mathcal{G}]$ of graphs expanded by relations R_k, $k \in \omega$, $(x < y)$, $\neg(x < y) \wedge \neg(y < x)$, Q_g, $g \in G$. The unique non-principal 1-type $p(x)$ is isolated by set of formulas $\exists y (R_k(y) \wedge (y < x))$, $k \in \omega$. For any realization a of p the list of pairwise non-equivalent isolating formulas $\varphi(a, y)$ with $\varphi(a, y) \vdash p(y)$ is exhausted by the formulas $(a < y)$ and $Q_g(a, y)$, $g \in G$. We define a regular labelling function $\nu(p)$ such that the formula $(a < y)$ has the label -1 and the formulas $Q_g(a, y)$ have non-negative labels g. Since $< \circ < = <$, $< \circ\, Q_g = Q_g \circ < = <$, $g \in G$, and the links between elements of $\rho_{\nu(p)}^{\geq 0}$ are defined by the group $\langle G; * \rangle$, the monoid $\mathfrak{P}'_{\nu(p)}$ coincides with the monoid $\langle \{-1, 0\}; + \rangle[\langle G; * \rangle]$. □

Theorem 5.3. *For any group $\langle G; * \rangle$ consisting of non-negative elements with the unit element 0 and for the monoid $\langle \omega^*; + \rangle$ of non-positive integers, there exists a* *theory T with a type $p \in S(T)$ and a regular labelling function $\nu(p)$ such that the monoid $\mathfrak{P}'_{\nu(p)}$ coincides with the monoid $\langle\omega^*;\, +\rangle[\langle G;\, *\rangle]$.*

Proof. We construct a structure $\mathcal{M}$ such that its theory $T = \mathrm{Th}(\mathcal{M})$ has a type $p(x) \in S(T)$ and a regular labelling function $\nu(p)$ with $\mathfrak{P}'_{\nu(p)} = \langle\omega^*;\, +\rangle[\langle G;\, *\rangle]$.

The language of $\mathcal{M}$ consists of unary predicate symbols Col_n, $n \in \omega$ (forming a coloring of the set M), of binary predicate symbol Q, and of binary predicate symbols Q_g, $g \in G$.

We consider a connected acyclic directed graph $\Gamma = \langle M_0; Q\rangle$, where each element has infinitely many images and infinitely many preimages, i. e., Γ forms a free directed pseudoplane [16, 21, 29].

We define an 1-inessential Q-ordered coloring (see [29]) Col: $M_0 \to \omega \cup \{\infty\}$ of Γ producing unary predicates $\mathrm{Col}_n = \{a \in M_0 \mid \mathrm{Col}(a) = n\}$, $n \in \omega$.

For the graph Γ we define, by induction, relations Q^n, $n \in \mathbf{Z}$: $Q^0 \rightleftharpoons \mathrm{id}_{M_0}$, $Q^{n+1} \rightleftharpoons Q^n \circ Q$, $Q^{-n} \rightleftharpoons (Q^n)^{-1}$, $n \in \omega$.

Note that for the (unique) non-principal type $p(x)$, isolated by the set $\{\neg\mathrm{Col}_m(x) \mid m < \omega\}$ of formulas, and for any realizations a and b of p, the pair (a, b) is a principal arc if and only if $\models Q^n(a, b)$ for some $n \in \omega$.

We assume that the formula $Q^n(x, y)$ has the label $-n \in U^{\leq 0}$, $n \in \omega$. Since for any $m, n \in \omega$ the formula $\exists z(Q^m(x, z) \wedge Q^n(z, y))$ is equivalent to the formula $Q^{m+n}(x, y)$, then for the Q-structure on a set of realizations of p the structure $\mathfrak{P}'_{\nu(p)}$ coincides with $\langle\omega^*;\, +\rangle$.

Now we consider the group $\langle G;\, *\rangle$ and define on the set G binary predicates Q_g, $g \in G$, by the rule: $Q_g = \{(a, b) \in G^2 \mid a * g = b\}$. As in the proof of Theorem 5.2, the structure $\mathcal{G} = \langle G; Q_g\rangle_{g\in G}$ forms a free 1-generated polygon over the group $\langle G;\, *\rangle$.

We define a model of a required theory T as the composition $\Gamma[\mathcal{G}]$ of graphs with colored vertices and arcs such that each vertex a of Γ is replaced by a copy of structure $\mathcal{G}$, for which all elements have the color $\mathrm{Col}(a)$. The relations Q_g, for $\Gamma[\mathcal{G}]$, are composed as the unions of corresponding relations in the copies of $\mathcal{G}$, and the relation Q, in $\Gamma[\mathcal{G}]$, consists of all pairs (a', b'), where $a' \in C_a$, $b' \in C_b$, $(a, b) \in Q$ in Γ, and C_a, C_b are copies of $\mathcal{G}$ replacing vertices $a, b \in M_0$. The composition preserves the uniqueness of the non-principal 1-type $p(x)$.

It remains to note that for any realization a of p the list of pairwise non-equivalent isolating formulas $\varphi(a, y)$ with $\varphi(a, y) \vdash p(y)$ is exhausted by the formulas $Q^n(a, y)$, $n \in \omega$, $Q_g(a, y)$, $g \in G$, and we have $\mathfrak{P}'_{\nu(p)} \upharpoonright \omega^* = \langle\omega^*;\, +\rangle$, $\mathfrak{P}'_{\nu(p)} \upharpoonright G = \langle G;\, *\rangle$ and $Q^n \circ Q_g = Q_g \circ Q^n = Q^n$ for $n > 0$, $g \in G$. □

6. I-GROUPOIDS

In this section, we collect basic structural properties of $\mathfrak{P}_{\nu(p)}$-groupoids and prove that any groupoid $\mathfrak{P}$ satisfying that list of properties coincides with some groupoid $\mathfrak{P}_{\nu(p)}$.

Let $U = U^- \,\dot\cup\, \{0\} \,\dot\cup\, U^+$ be an alphabet consisting of a set U^- of *negative elements*, a set U^+ of *positive elements*, and zero 0. As above we write $u < 0$ for any element $u \in U^-$, $u > 0$ for any element $u \in U^+$, and $u \cdot v$ instead of $\{u\} \cdot \{v\}$ considering an operation $\cdot$ on the set $\mathcal{P}(U) \setminus \{\varnothing\}$.

A groupoid $\mathfrak{P} = \langle\mathcal{P}(U) \setminus \{\varnothing\};\, \cdot\rangle$ is called an *I-groupoid* if it satisfies the following conditions:

• the set $\{0\}$ is the unit of the groupoid $\mathfrak{P}$;

• the operation $\cdot$ of the groupoid $\mathfrak{P}$ is generated by the function $\cdot$ on elements in U such that every elements $u, v \in U$ define a nonempty set $(u \cdot v) \subseteq U$: for any sets $X, Y \in \mathcal{P}(U) \setminus \{\varnothing\}$ the following equality holds:

$$X \cdot Y = \bigcup \{x \cdot y \mid x \in X, y \in Y\};$$

• if $u < 0$ then the sets $u \cdot v$ and $v \cdot u$ consist of negative elements for any $v \in U$;

• if $u > 0$ and $v > 0$ then the set $u \cdot v$ consists of non-negative elements;

• for any $u > 0$ there is a unique *inverse* element $u^{-1} > 0$ such that $0 \in (u \cdot u^{-1}) \cap (u^{-1} \cdot u)$;

• if a positive element u belongs to a set $v_1 \cdot v_2$ then u^{-1} belongs to $v_2^{-1} \cdot v_1^{-1}$;

• for any elements $u_1, u_2, u_3 \in U$ the following inclusion holds:

$$(u_1 \cdot u_2) \cdot u_3 \supseteq u_1 \cdot (u_2 \cdot u_3),$$

and the strict inclusion

$$(u_1 \cdot u_2) \cdot u_3 \supset u_1 \cdot (u_2 \cdot u_3)$$

may be satisfied only for $u_1 < 0$ and $|u_2 \cdot u_3| \geq \omega$;

• the groupoid $\mathfrak{P}$ contains the *deterministic* subgroupoid $\mathfrak{P}_d^{\geq 0}$ (being a monoid) with the universe $\mathcal{P}(U_d^{\geq 0}) \setminus \{\varnothing\}$, where

$$U_d^{\geq 0} = \{u \in U^{\geq 0} \mid u^{-1} \cdot u = \{0\}\};$$

any set $u \cdot v$ is a singleton for $u, v \in U_d^{\geq 0}$.

By the definition each I-groupoid $\mathfrak{P}$ contains I-subgroupoids $\mathfrak{P}^{\leq 0}$ and $\mathfrak{P}^{\geq 0}$ with the universes $\mathcal{P}(U^- \cup \{0\}) \setminus \{\varnothing\}$ and $\mathcal{P}(U^+ \cup \{0\}) \setminus \{\varnothing\}$ respectively. The structure $\mathfrak{P}^{\geq 0}$ is a monoid.

Theorem 6.1. *For any I-groupoid $\mathfrak{P}$ there is a theory T with a type $p(x) \in S^1(T)$ and a regular labelling function $\nu(p)$ such that $\mathfrak{P}_{\nu(p)} = \mathfrak{P}$. If the alphabet is at most countable and the operation of $\mathfrak{P}$ does not force continuum many types then T is small.*

Proof. We fix an I-monoid $\mathfrak{P} = \langle \mathcal{P}(U) \setminus \{\varnothing\}; \cdot \rangle$. The construction of a required theory will be fulfilled in accordance with a construction of a generic structure $\mathcal{M}$ of language $\Sigma = \{\mathrm{Col}_n^{(1)} \mid n \in \omega\} \cup \{Q_u^{(2)} \mid u \in U\}$ [29, Chapter 2] with pairwise disjoint predicates Q_u, with an ordered coloring Col: $M \to \omega \cup \{\infty\}$ with respect to each formula $Q_u(x, y)$, where $u < 0$, and with a unique non-principal 1-type $p(x)$ (isolated by the set $\{\neg\mathrm{Col}_n(x) \mid n \in \omega\}$ of formulas). W.l.o.g. we assume that $|U| \leq \omega$ (for $|U| > \omega$, the construction differs by cardinalities of diagrams describing links for elements of finite sets and by cardinalities of sets of diagrams forming generic structures).

Consider a generic class $(\mathbf{T}_0; \leqslant)$ consisting of all possible diagrams $\Phi(A)$ over finite sets A such that each $\Phi(A)$ contains a maximal consistent set of quantifier-free formulas $\varphi(\bar{a})$, $\bar{a} \in A$, united with a set of formulas $Q_{uv}^{\delta}(a, b)$, $a, b \in A$, $\delta \in \{0, 1\}$, $Q_{uv}(x, y) = \exists z(Q_u(x, z) \wedge Q_v(z, y))$, $u, v \in U$, and $\Phi(A)$ includes formulas with parameters in A, without free variables, and describing the following properties:

(1) for any $u \in U$, any element in A is an image and a preimage of some elements by the relation Q_u;

(2) the relation Q_0 on the set A is identical;

(3) if $a \in A$ then all Q_u-images of a have colors $\geq \mathrm{Col}(a)$ and all Q_u-preimages of a have the colors $\leq \mathrm{Col}(a)$;

(4) if $u > 0$, $a \in A$, and $Q_u(a,b) \in \Phi(A)$ then $Q_{u^{-1}}(b,a) \in \Phi(A)$ and $\mathrm{Col}(b) = \mathrm{Col}(a)$;

(5) if $v \in (u_1 \cdot u_2)$ and $Q_v(a,b) \in \Phi(A)$ then $Q_{u_1u_2}(a,b) \in \Phi(A)$;

(6) for any $u \neq 0$ some diagram $\Psi(B) \supseteq \Phi(A)$ in $\mathbf{T}_0$ defines a graph $\langle B; Q_u\rangle$ with a cycle if and only if $0 \in \underbrace{u \cdot \ldots \cdot u}_{n \text{ times}}$ for some $n > 0$;

(7) if $u \in U_d^{\geq 0}$ then each element $a \in A$ has a unique Q_u-image; the following inductive condition describes the least set $U_{\mathrm{ad}}^{\geq 0} \supseteq U_d^{\geq 0}$ of non-negative elements $u \in U$ for which the sets of Q_u-images and of Q_u-preimages of a are finite: if $(u\cdot u^{-1})\cup(u^{-1}\cdot u)$ consists of finitely many elements belonging to $U_{\mathrm{ad}}^{\geq 0}$ then $u, u^{-1} \in U_{\mathrm{ad}}^{\geq 0}$; if $u^{-1}\cdot u$ consists of finitely many elements belonging to $U_{\mathrm{ad}}^{\geq 0}$ then each element a has finitely many Q_u-images; if $u\cdot u^{-1}$ consists of finitely many elements belonging to $U_{\mathrm{ad}}^{\geq 0}$ then each element a has finitely many Q_u-preimages; for other elements u the numbers of Q_u-images and of Q_u-preimages for elements $a \in A$ is unbounded;

(8) if $u_1, u_2 \in U$ and the set $u_1 \cdot u_2$ is (in)finite then for any element $a \in A$ the set of $Q_{u_1u_2}$-images of a is represented as a union of sets of Q_v-images for all elements $v \in u_1 \cdot u_2$ (and some set of elements that are not Q_u-images of a on any of the relations Q_u);

(9) for any element $v \in ((u_1 \cdot u_2) \cdot u_3) \setminus (u_1 \cdot (u_2 \cdot u_3))$ there is a description forming Example 3.3.

If $\Phi(A), \Psi(B)$ are diagrams in $\mathbf{T}_0$ and $\Phi(A) \subseteq \Psi(B)$, we suppose, by the definition, that $\Phi(A)$ is a *strong subdiagram* of $\Psi(B)$ (i. e., $\Phi(A) \leqslant \Psi(B)$) if $\Phi(A)$, with each element a in A, contains all descriptions (for numbers and Q-links) of its Q_u-images in $\Psi(B)$, where $u^{-1}\cdot u$ consists of finitely many labels belonging to $U_{\mathrm{ad}}^{\geq 0}$.

For the checking that $(\mathbf{T}_0; \leqslant)$ is a self-sufficient generic class, it suffices to observe that for any diagrams $\Phi(A), \Psi(B), \mathrm{X}(C) \in \mathbf{T}_0$ with $\Phi(A) \leqslant \Psi(B)$, $\Phi(A) \leqslant \mathrm{X}(C)$, and $A = B \cap C$ there is a diagram $\Theta(B \cup C) \in \mathbf{T}_0$ such that $\Psi(B) \leqslant \Theta(B \cup C)$ and $\mathrm{X}(C) \leqslant \Theta(B \cup C)$.

For the type $\Theta(B \cup C)$ we choose the set $\Psi(B) \cup \mathrm{X}(C)$ extended by the following formulas for elements $b \in B \setminus A$ and $c \in C \setminus A$:

(a) $\theta_{u,v}(b,c)$, where $Q_u(b,a) \in \Psi(B)$ and $Q_v(a,c) \in \mathrm{X}(C)$ for some $a \in A$;

(b) $\neg\theta_{u,v}(b,c)$, where $\neg Q_u(b,a) \in \Psi(B)$ or $\neg Q_v(a,c) \in \mathrm{X}(C)$ for all $a \in A$;

(c) some formulas $\theta_{v'}(b,c)$, where $Q_u(b,a) \in \Psi(B)$ and $Q_v(a,c) \in \mathrm{X}(C)$ for some $a \in A$, $v' \in u \cdot v$, and the set $u \cdot v$ is finite;

(d) formulas $\neg\theta_{v'}(b,c)$, $v' \in U$, if the previous items do not imply a converse.

We claim that if the operation of $\mathfrak{P}$ does not force continuum many types then, applying the generic construction, one obtains a $(\mathbf{T}_0; \leqslant)$-generic saturated structure $\mathcal{M}$ with the generic theory $T = \mathrm{Th}(\mathcal{M})$, the type $p(x) \in S(T)$, and the regular labelling function $\nu(p)$: $\mathrm{PF}(p)/\mathrm{PE}(p) \to U$ satisfying the condition $\mathfrak{P}_{\nu(p)} = \mathfrak{P}$. By [29, Proposition 1.2.13], each formula $Q_u(x,y)$, $u < 0$, witnesses non-symmetry of the relation SI_p, and each formula $Q_u(x,y)$, $u > 0$, links realizations of p only with realizations of the same type and, being a principal formula of the structure on the set $p(M)$ of realizations of p, has the inverse principal formula $Q_{u^{-1}}(x,y)$ on $p(M)$.

Now we argue to show that $\mathcal{M}$ is saturated. If $U^{\geq 0}_{\rm ad}$ is finite the saturation of $\mathcal{M}$ is implied by [29, Theorem 2.5.1] (see also [28, Theorem 4.1]) in view of the uniform t-amalgamation property that holds by the formula definability of self-sufficient closure of any finite set.

Using the proof of the same theorem, we shall observe that $\mathcal{M}$ is saturated for $|U^{\geq 0}_{\rm ad}| = \omega$. For this aim we enumerate all predicates Q_u, $u \in U$: Q_m, $m \in \omega$.

Let $\mathcal{M}'$ be an ω-saturated model of $\mathrm{Th}(\mathcal{M})$, $\Phi(A)$ and $\Phi(A') = [\Phi(A)]^A_{A'}$ be diagrams in $\mathbf{T}_0$ such that $\mathcal{M} \models \Phi(A)$ and $\mathcal{M}' \models \Phi(A')$. If $\Psi(B') \in \mathbf{T}_0$, $\Phi(A') \leqslant \Psi(B')$, and $\mathcal{M}' \models \Psi(B')$ then the construction of $\mathcal{M}$ implies that there exists a set $B \subset M$ extending A and satisfying $\mathcal{M} \models \Psi(B)$. It means that for a partial isomorphism f: $A \to A'$ between $\mathcal{M}$ and $\mathcal{M}'$ there exists a partial isomorphism g: $B \to B'$ between these structures extending f.

Now, let $\Psi(B) \in \mathbf{T}_0$, $\Phi(A) \leqslant \Psi(B)$, $\mathcal{M} \models \Psi(B)$, and X and Y be disjoint sets of variables, which are in bijective correspondence with sets A and $B \setminus A$. Assume that the formula $\varphi_n(X)$ ($\psi_n(X,Y)$, respectively), $n \in \omega$, describes the following:

(i) finite colors of elements of A (of B);

(ii) negations of colors not exceeding n for elements of A (of B) that are infinite in color;

(iii) the existence, colors of arcs, the existence and colors of some arcs of pathes of length 2 (including all possibilities for colors $\leq n$ of intermediate arcs) connecting elements of A (of B), and the colors $m \leq n$ of arcs outgoing from vertices $a \in A$ ($a \in B$) for which $\exists y Q_m(a,y) \in \Phi(A)$ ($\exists y Q_m(a,y) \in \Psi(B)$), $Q_m = Q_u$, $u \in U^{\geq 0}_{\rm ad}$;

(iv) the non-existence of arcs of colors $\leq n$ and of pathes of length 2 (including all possibilities for colors $\leq n$ of intermediate arcs) connecting elements of A (of B), if these elements are not linked by the pathes, as well as the absence of colors $m \leq n$ for arcs outgoing from vertices $a \in A$ ($a \in B$) for which $\neg\exists y Q_m(a,y) \in \Phi(A)$ ($\neg\exists y Q_m(a,y) \in \Psi(B)$), $Q_m = Q_u$, $u \in U^{\geq 0}_{\rm ad}$.

By the construction of $\mathcal{M}$,

$$\mathcal{M} \models \forall X\, (\varphi_n(X) \to \exists Y\, \psi_n(X,Y)).$$

Hence

$$\mathcal{M}' \models \forall X\, (\varphi_n(X) \to \exists Y\, \psi_n(X,Y)).$$

This implies that the set $\{\psi_n(A',Y) \mid n \in \omega\}$ of formulas is locally realizable in $\mathcal{M}'$; hence, it is realizable in $\mathcal{M}'$ since $\mathcal{M}'$ is ω-saturated. Therefore there exist a set $B' \subset M'$ containing A', and a partial isomorphism g: $B \to B'$ extending the partial isomorphism f.

The possibility for extending any partial isomorphisms f: $A \to A'$ and the known back-and-forth method show that the structure $\mathcal{M}$ with distinguished constants for the elements in $A \subset M$ is isomorphic to a countable elementary substructure of the structure $\mathcal{M}'$ with distinguished constants for the elements in A'. Since the finite sets A and A' connected by a partial isomorphism and preserving a type $\Phi(X)$ are chosen arbitrarily, and $\mathcal{M}'$ is saturated, we conclude that $\mathcal{M}$ realizes any type over a finite set, $\mathcal{M}$ is saturated, and $\mathrm{Th}(\mathcal{M})$ is small.

Note that for the $(\mathbf{T}_0; \leqslant)$-generic structure $\mathcal{M}$, the possibility for extending any finite partial isomorphisms preserving types $\Phi(X)$ in $\mathbf{T}_0$ implies that if $A, B \subset M$, $\mathcal{M} \models \Phi(A)$ and $\mathcal{M} \models \Phi(B)$ then there is an automorphism of $\mathcal{M}$ extending the initial partial isomorphism between A and B. Consequently, $\mathrm{tp}_{\mathcal{M}}(A) = \mathrm{tp}_{\mathcal{M}}(B)$. In particular, for any realization a of p and for any $u \in U$ the formula $Q_u(a,y)$ is isolating and these formulas exhaust the list of all pairwise non-equivalent isolating formulas $\varphi(a,y)$ for which $\varphi(a,y)\vdash p(y)$

Similar arguments are valid for the general case producing a required (not necessary small) theory. $\square$

Remark 6.2. If an I-groupoid $\mathfrak{P}$ is constructed by a set $U^{\geq 0}$ then by the construction above (restricting the construction to a set of realizations of the type infinite in color) there is a transitive theory T with a (unique) type $p(x)\in S(T)$ and a regular labelling function $\nu(p)$ such that $\mathfrak{P}_{\nu(p)}=\mathfrak{P}$.

7. Groupoids of binary isolating formulas on sets of realizations for types of special theories

In this section, we present a specificity of groupoids $\mathfrak{P}_{\nu(p)}$ for types p of *special* theories used for the classifications of countable models of Ehrenfeucht theories [24, 25, 29], of theories with finite Rudin–Keisler preorders [27, 29], of small theories [29, 30], of ω-stable theories with respect to numbers of limit models over types [31], as well as for the investigations of graph links for limit models over types that obtained by quotients of numerical sequences [18, 19, 20]. All these constructions are based on powerful graphs.

Definition 7.1 [26, 29, 32]. Let $\Gamma=\langle X;Q\rangle$ be a graph, and a be a vertex of Γ. The set $\bigtriangledown_Q(a)\rightleftharpoons\bigcup\limits_{n\in\omega}Q^n(a,\Gamma)$ (respectively $\bigtriangleup_Q(a)\rightleftharpoons\bigcup\limits_{n\in\omega}Q^n(\Gamma,a)$) is called an *upper* (*lower*) Q-*cone* of a. We call the Q-cones $\bigtriangledown_Q(a)$ and $\bigtriangleup_Q(a)$ by *cones* and denote by $\bigtriangledown(a)$ and $\bigtriangleup(a)$ respectively if Q is fixed.

A countable acyclic directed graph $\Gamma=\langle X;Q\rangle$ is said to be *powerful* if the following conditions hold:

(a) the automorphism group of Γ is *transitive*, that is any two vertices are connected by an automorphism;

(b) the formula $Q(x,y)$ is equivalent in the theory $\mathrm{Th}(\Gamma)$ to a disjunction of principal formulas;

(c) $\mathrm{acl}(\{a\})\cap\bigtriangleup(a)=\{a\}$ for each vertex $a\in X$;

(d) $\Gamma\models\forall x,y\,\exists z\,(Q(z,x)\wedge Q(z,y))$ (the *pairwise intersection property*).

Below we define the property of powerfulness for the directed graph Γ in terms of the groupoid $\mathfrak{P}_{\nu(p)}$ for the unique 1-type p of the theory $T=\mathrm{Th}(\Gamma)$ assuming that the theory is small.

At first we note that $U^-=\varnothing$ in view of Corollary 1.5 and so $\mathfrak{P}_{\nu(p)}$ is a monoid.

Since the formula $Q(x,y)$ is equivalent to some disjunction $\bigvee\limits_{i=1}^{n}\theta_{u_i}(x,y)$, the acyclicity of Γ means that $0\notin u_{i_1}u_{i_2}\ldots u_{i_k}$ for any $u_{i_1},\ldots,u_{i_k}\in\{u_1,\ldots,u_n\}$. The condition $\mathrm{acl}(\{a\})\cap\bigtriangleup(a)=\{a\}$ is equivalent to that any sets $u_{i_1}^{-1}u_{i_2}^{-1}\ldots u_{i_k}^{-1}$ do not contain almost deterministic elements. The pairwise intersection property means that for any u_i, $i=1,\ldots,n$, and any $v\in U$, the set u_iv contains an element u_j. In particular, if $n=1$ then $u_1\in u_1v$ for any $v\in U$. In this case we say that the element u_1 *induces the pairwise intersection property*, or it is a PIP-*element*.

The characterizations above imply the following

Proposition 7.2. *A small theory T of the language $\{Q^{(2)}\}$ is a theory of a powerful graph $\Gamma=\langle X;Q\rangle$ if and only if T has the unique 1-type p with a regular labelling* *function $\nu(p)$ such that for some elements $u_1, \dots, u_n \in \rho_{\nu(p)}$ the following conditions are satisfied:*

(1) $\vdash Q(x,y) \leftrightarrow \bigvee\limits_{i=1}^{n} \theta_{u_i}(x,y)$;

(2) $0 \notin u_{i_1} u_{i_2} \dots u_{i_k}$ *for any* $u_{i_1}, \dots, u_{i_k} \in \{u_1, \dots, u_n\}$;

(3) *for any* u_i, $i = 1, \dots, n$, *and any* $v \in U$, *the set* $u_i v$ *contains an element* u_j.

Definition 7.3. A monoid $\mathfrak{P}_{\nu(p)}$ is called *special* if $\rho_{\nu(p)} \cap U^- \neq \varnothing$ and for any elements $u_1, u_2, \dots, u_n, v \in \rho_{\nu(p)}$, where $u_1 < 0, \dots, u_n < 0$, $v \geq 0$, and for any element $u' \in u_1 u_2 \dots u_n v$, there is an element $v' \geq 0$ such that $u' \in v' u_1 u_2 \dots u_n$.

A special monoid $\mathfrak{P}_{\nu(p)}$ is called PIP-*special* if each negative $u \in \rho_{\nu(p)}$ is a PIP-element, i. e., $u \in uv$ for any $v \in \rho_{\nu(p)}$.

Having a special monoid (for a special small theory T) the process of construction of a limit model over a type p is reduced to a sequence of θ_{u_n}-extensions, $u_n < 0$, $n \in \omega$, of prime models over realizations of p: for any limit model $\mathcal{M}$ over p there is an elementary chain $(\mathcal{M}(a_n))_{n \in \omega}$, $\models p(a_n)$, such that its union forms $\mathcal{M}$ and $\models \theta_{u_n}(a_{n+1}, a_n)$ is satisfied, $n \in \omega$. In this case the isomorphism type of $\mathcal{M}$ is defined by the sequence $(u_n)_{n \in \omega}$.

As shown in [29], if a PIP-special monoid exists then, by adding of multiplace predicates, each prime model over a tuple of realizations of p is transformed to a model isomorphic to $\mathcal{M}_p$. Thus, the type p is connected with the unique, up to isomorphism, prime model over a realization of p and with some (finite, countable, or continuum) number of limit models over p, which is defined by some quotient for the set of sequences $(u_n)_{n \in \omega}$, $u_n \in U^- \cap \rho_{\nu(p)}$, $n \in \omega$. The action of these quotients is defined by some identifications ($w \approx w'$) of words in the alphabet $U^- \cap \rho_{\nu(p)}$ such that if $w = u_1 \dots u_m$ and $w' = u'_1 \dots u'_n$ then for any $v \in U^{\geq 0} \cap \rho_{\nu(p)}$ and $u_0 \in u_1 \dots u_m v$, there exists $v' \in U^{\geq 0} \cap \rho_{\nu(p)}$ with $u_0 \in v' u'_1 u'_2 \dots u'_n$.

To conclude this section we describe some connections of $I_{\nu(p)}$-monoids with the strict order property.

Definition 7.4. Let T be a theory with a type p having the model $\mathcal{M}_p$, $\mathfrak{P}_{\nu(p)}$ an $I_{\nu(p)}$-groupoid, and X a subset of $\rho_{\nu(p)}$ having a cardinality λ. We say that X is (formula) *definable* if for a realization a of p the set of solutions of $L_{\lambda^+,\omega}$-formula $\varphi(a,y) \rightleftharpoons \bigvee\limits_{u \in X} \theta_u(a,y)$ in $\mathcal{M}_p$ is $L_{\omega,\omega}$-definable in $\mathcal{M}_p$ by a formula $\psi(a,y)$. In this case we say that the formula $\psi(x,y)$ *witnesses* definability of X.

We say that a groupoid $\mathfrak{P}_{\nu(p)}$ *generates the strict order property* if for some definable set $X \subseteq \rho_{\nu(p)}$, for a witnessing formula $\varphi(x,y)$, and for some realizations a and b of p satisfying $\models \theta_v(b,a)$ with a label $v \in \rho_{\nu(p)}$, the inclusion $\varphi(a, \mathcal{M}_p) \subset \varphi(b, \mathcal{M}_p)$ holds.

Proposition 7.5. *If T is a small theory with a type p, and the groupoid $\mathfrak{P}_{\nu(p)}$ has a definable set $X \subseteq \rho_{\nu(p)}$ containing an element $u < 0$ with $u \cdot X \subseteq X$, then $\mathfrak{P}_{\nu(p)}$ generates the strict order property.*

Proof. Take a definable set $Y = X \cup \{0\}$ and consider a witnessing formula $\varphi(x,y)$. Since $u \cdot X \subseteq X$ then $u \cdot Y \subseteq Y$ and, for any realizations a and b of p with $\mathcal{M} \models \theta_u(b,a)$, we have $\varphi(a, \mathcal{M}_p) \subseteq \varphi(b, \mathcal{M}_p)$. At the same time, $0 \in Y$ implies $b \in \varphi(b, \mathcal{M}_p)$, and if $b \in \varphi(a, \mathcal{M}_p)$ then a isolates b that is impossible by $u < 0$. Thus, $\varphi(a, \mathcal{M}_p) \subset \varphi(b, \mathcal{M}_p)$ and $\mathfrak{P}_{\nu(p)}$ generates the strict order property. $\square$

Corollary 7.6. *Let T be a small theory with a type p, and for some nonempty finite set $X \subseteq U^- \cap \rho_{\nu(p)}$ there be a natural number n such that $X^{n+1} \subseteq \bigcup_{i=1}^{n} X^i$, where $X^1 = X$, $X^{i+1} = X^i \cdot X$. Then the groupoid $\mathfrak{P}_{\nu(p)}$ generates the strict order property.*

Proof. Clearly, the finite set X is definable and the sets X^i and $Y \rightleftharpoons \bigcup_{i=1}^{n} X^i$ are also definable. Since $X^{n+1} \subseteq Y$ then for any element $u \in X$ we have $u \cdot Y \subseteq Y$. Since $u < 0$ then by Proposition 7.5 the groupoid $\mathfrak{P}_{\nu(p)}$ generates the strict order property. □

Corollary 7.7. *If T is a small theory with a type p and $U^- \cap \rho_{\nu(p)}$ is a nonempty finite set then the groupoid $\mathfrak{P}_{\nu(p)}$ generates the strict order property.*

Proof. Consider the set $X = U^- \cap \rho_{\nu(p)}$. As X is finite it is definable. Since X contains all negative labels in $\rho_{\nu(p)}$, by Proposition 1.6, we have $u \cdot X \subseteq X$ for any $u < 0$ in $\rho_{\nu(p)}$. Therefore, by Proposition 7.5, the groupoid $\mathfrak{P}_{\nu(p)}$ generates the strict order property. □

8. Partial groupoid of binary isolating formulas on a set of realizations for a family of 1-types

In this section, the results above for a structure of a type are generalized for a structure on a set of realizations for a family of types.

Let R be a nonempty family of types in $S^1(T)$. We denote by $\nu(R)$ a regular family of labelling functions

$$\nu(p,q)\colon \mathrm{PF}(p,q)/\mathrm{PE}(p,q) \to U, \ \ p,q \in R,$$

$$\rho_{\nu(R)} \rightleftharpoons \bigcup_{p,q\in R} \rho_{\nu(p,q)}.$$

Similarly Proposition 3.1, we obtain that, having atomic models $\mathcal{M}_p$ for all types $p \in R$ (for instance, if T is small), the function P, being partial for $|R| > 1$, on the set $R\times(\mathcal{P}(U)\setminus\{\varnothing\})\times R$, which maps each tuple of triples $(p_1, u_1, p_2), \ldots, (p_k, u_k, p_{k+1})$, where $u_1 \in \rho_{\nu(p_1,p_2)}, \ldots, u_k \in \rho_{\nu(p_k,p_{k+1})}$, to the set of triples (p_1, v, p_{k+1}), where $v \in P(p_1, u_1, p_2, u_2, \ldots, p_k, u_k, p_{k+1})$, is *left semi-associative*:

$$\begin{gathered} P(P(p_1,u_1,p_2,u_2,p_3),u_3,p_4) = P(p_1,u_1,p_2,u_2,p_3,u_3,p_4) \supseteq \\ \supseteq P(p_1,u_1,P(p_2,u_2,p_3,u_3,p_4)) \end{gathered} \tag{8}$$

for $u_1 \in \rho_{\nu(p_1,p_2)}$, $u_2 \in \rho_{\nu(p_2,p_3)}$, $u_3 \in \rho_{\nu(p_3,p_4)}$.

Having the models $\mathcal{M}_p$ we consider the semi-associative structure

$$\mathfrak{P}_{\nu(R)} \rightleftharpoons \langle R \times (\mathcal{P}(U) \setminus \{\varnothing\}) \times R; \cdot\rangle$$

with the partial operation $\cdot$ such that

$$(p_1,X_1,p_2)\cdot(p_2,X_2,p_3) = \bigcup\{(p_1,u_1,p_2)\cdot(p_2,u_2,p_3) \mid u_1 \in X_1, u_2 \in X_2\},$$

$$(p_1,u_1,p_2)\cdot(p_2,u_2,p_3) = \{(p_1,v,p_3) \mid v \in P(p_1,u_1,p_2,u_2,p_3)\},$$

$$u_1 \in \rho_{\nu(p_1,p_2)}, u_2 \in \rho_{\nu(p_2,p_3)}.$$

The groupoids $\mathfrak{P}_{\nu(p)}$, $p \in R$, are naturally embeddable into this structure. The structure $\mathfrak{P}_{\nu(R)}$ is called a *join of groupoids* $\mathfrak{P}_{\nu(p)}$, $p \in R$, relative to the family $\nu(R)$ of labelling functions and it is denoted by $\bigoplus_{p\in R} \mathfrak{P}_{\nu(p)}$. If $\rho_{\nu(p,q)} = \varnothing$ for all $p \neq q$

the join $\bigoplus\limits_{p\in R}\mathfrak{P}_{\nu(p)}$ is *free*, it is isomorphically represented as the disjoint union of the groupoids $\mathfrak{P}_{\nu(p)}$, and it is denoted by $\bigsqcup\limits_{p\in R}\mathfrak{P}_{\nu(p)}$.

By (8), we obtain

Proposition 8.1. *For any complete theory T, for any nonempty family $R\subset S(T)$ of 1-types having models $\mathcal{M}_p$ for each $p\in P$, and for any regular family $\nu(R)$ of labelling functions, each n-ary partial operation*

$$P(p_1,\cdot,p_2,\cdot,p_3\dots,p_n,\cdot,p_{n+1})$$

on the set $\mathcal{P}(U)\setminus\{\varnothing\}$ is interpretable by a term of the structure $\bigoplus\limits_{p\in R}\mathfrak{P}_{\nu(p)}$ with fixed types $p_1,\dots,p_{n+1}\in R$.

By Proposition 1.6 we obtain the following analogue of Proposition 3.4.

Proposition 8.2. *For any complete theory T, for any nonempty family $R\subset S(T)$ of 1-types, and for any regular family $\nu(R)$ of labelling functions, the restriction of the structure $\mathfrak{P}_{\nu(R)}$ to the set of negative* (*respectively non-positive, non-negative*) *labels is closed under the partial operation $\cdot$.*

In view of Proposition 8.2, the structure $\mathfrak{P}_{\nu(R)}$ has substructures $\mathfrak{P}^{\leq 0}_{\nu(R)}$ and $\mathfrak{P}^{\geq 0}_{\nu(R)}$, generated by triples (p,u,q) with $u\leq 0$ and $u\geq 0$ respectively, $p,q\in R$. Here, for any triple (p,u,q) in $\mathfrak{P}^{\geq 0}_{\nu(R)}$ the triple (q,u^{-1},p) is also attributed to $\mathfrak{P}^{\geq 0}_{\nu(R)}$.

A structure $\mathfrak{P}_{\nu(R)}$ is called (*almost*) *deterministic* if the set $(p,u,q)\cdot(q,v,r)$ is a singleton (finite) for any triples (p,u,q) and (q,v,r) in $\mathfrak{P}_{\nu(R)}$ with $u\in\rho_{\nu(p,q)}$ and $v\in\rho_{\nu(q,r)}$.

The deterministic structure $\mathfrak{P}_{\nu(R)}$ is generated by the structure $\mathfrak{P}'_{\nu(R)}=\langle R\times U\times R;\odot\rangle$, where $(p,u,q)\cdot(q,v,r)=\{(p,u,q)\odot(q,v,r)\}$ for $p,q,r\in R$, $u,v\in U$.

Adapting the proof of Proposition 4.1 to a family R of 1-types we obtain

Proposition 8.3. *For any complete theory T, for any nonempty family $R\subset S(T)$ of 1-types having models $\mathcal{M}_p$ for each $p\in P$, and for any regular family $\nu(R)$ of labelling functions, the following conditions are equivalent:*

(1) *the relation I_R is transitive for any model $\mathcal{M}\models T$;*

(2) *the structure $\mathfrak{P}_{\nu(R)}$ is almost deterministic.*

Note that the absence of principal edges linking distinct realizations of types in R is equivalent to the antisymmetry of the relation I_R. Since I_R is reflexive (by the formula $(x\approx y)$), the definition of the family $\nu(R)$ and Propositions 1.6, 8.3 imply

Corollary 8.4. *For any complete theory T, for any nonempty family $R\subset S(T)$ of 1-types having models $\mathcal{M}_p$ for each $p\in P$, and for any regular family $\nu(R)$ of labelling functions, the following conditions are equivalent:*

(1) *the relation I_R is a partial order on the set of realizations of types of R in any model $\mathcal{M}\models T$;*

(2) *the structure $\mathfrak{P}_{\nu(R)}$ is almost deterministic and $\rho_{\nu(R)}\subseteq U^{\leq 0}$.*

The partial order I_R is identical if and only if $\rho_{\nu(R)}=\{0\}$. The non-identical partial order I_R has infinite chains if and only if $|\rho_{\nu(p)}|>1$ for some $p\in R$ or there is a sequence p_n, $n\in\omega$, of pairwise distinct types in R such that $|\rho_{\nu(p_n,p_{n+1})}|\geq 1$, $n\in\omega$, or $|\rho_{\nu(p_{n+1},p_n)}|\geq 1$, $n\in\omega$.

Lemma 1.3 and Proposition 8.3 imply

Corollary 8.5. *For any complete theory T, for any nonempty family $R \subset S(T)$ of 1-types having models $\mathcal{M}_p$ for each $p \in P$, and for any regular family $\nu(R)$ of labelling functions, the following conditions are equivalent:*

(1) *I_R is an equivalence relation on the set of realizations of types of R in any model $\mathcal{M} \models T$;*

(2) *the structure $\mathfrak{P}_{\nu(R)}$ is almost deterministic and $\rho_{\nu(R)} \subseteq U^{\geq 0}$.*

An element $u \in U$ is called (*almost*) *deterministic* with respect to the regular family $\nu(R)$ of labelling functions if for some realization a of a type in R and for some type $q \in R$, the formula $\theta_{\mathrm{tp}(a),u,q}(a, y)$ is consistent and has a unique solution (has finitely many solutions).

Repeating the proof of Proposition 4.11 we obtain

Proposition 8.6. *For any structure $\mathfrak{P}_{\nu(R)}$ its restriction $\mathfrak{P}_{\nu(R),d}$ (respectively $\mathfrak{P}_{\nu(R),\mathrm{ad}}$) to the set of (almost) deterministic elements is closed under the partial operation of the structure $\mathfrak{P}_{\nu(R)}$.*

Using the proof of Proposition 4.13 the following proposition holds.

Proposition 8.7. *If for the types $p, q \in S^1(T)$ the models $\mathcal{M}_p$ and $\mathcal{M}_q$ exist then an element $u \geq 0$ in $\rho_{\nu(p,q)}$ is deterministic if and only if $(q, u^{-1}, p) \cdot (p, u, q) = \{(q, 0, q)\}$.*

Proposition 8.8. *If the structure $\mathfrak{P}_{\nu(R)}$ is deterministic then the structure $\mathfrak{P}'_{\nu(R)}$ is a join of groups if and only if each set $\rho_{\nu(p)}$, $p \in R$, consists of non-negative elements.*

Proof is identical to the proof of Proposition 4.15 for each set $\rho_{\nu(p)}$. □

Corollary 8.9. *If R is a nonempty family of 1-types in $S^1(T)$, there are models $\mathcal{M}_p$ for $p \in R$, $\mathfrak{P}_{\nu(R)}$ is a deterministic structure, and $\mathfrak{P}'_{\nu(R)}$ is a join of groups, then all elements in $\mathfrak{P}'_{\nu(p)}$, $p \in R$, are deterministic.*

Proof. Since, by Proposition 8.8, the sets $\rho_{\nu(p)}$ consist of non-negative elements, the determinacy of the structure $\mathfrak{P}_{\nu(R)}$ and Proposition 8.7 imply that each element in $\mathfrak{P}'_{\nu(p)}$, $p \in R$, is deterministic. □

Repeating the proof of Proposition 4.17 we obtain

Proposition 8.10. *If R is a nonempty family of 1-types in $S^1(T)$, there exists models $\mathcal{M}_p$ for $p \in R$, and $\nu(R)$ is a regular family of labelling functions, then for the structure $\mathfrak{P}_{\nu(R)}$ the set $\rho^{\geq 0}_{\nu(R),d}$ of all non-negative deterministic elements u in $\rho_{\nu(R)}$, for which the elements u^{-1} are also deterministic, forms the deterministic substructure $\mathfrak{G}^{\geq 0}_{\nu(R),d}$ of $\mathfrak{P}_{\nu(R)}$ such that $(\mathfrak{G}^{\geq 0}_{\nu(R),d})'$ is a join of groups.*

The results above substantiate the transformation of the diagram in Figure 1 replacing the type p by a nonempty family $R \subseteq S^1(\varnothing)$.

9. $I_{\mathcal{R}}$-structures

Definition 9.1. Let $\mathcal{R}$ be a nonempty set, $U = U^- \dot\cup \{0\} \dot\cup U^+$ be an alphabet consisting of a set U^- of *negative elements*, of a set U^+ of *positive elements* and a zero 0. If p and q are elements in $\mathcal{R}$, we write $u < 0$ and $(p, u, q) < 0$ for any $u \in U^-$, $u > 0$ and $(p, u, q) > 0$ for any $u \in U^+$. For the set $\mathcal{R}^2$ of all pairs (p, q), $p, q \in \mathcal{R}$, we consider a *regular* family $\mu(\mathcal{R})$ of sets $\mu(p, q) \subseteq U$ such that

• $0 \in \mu(p,q)$ if and only if $p = q$;

• $\mu(p,p) \cap \mu(q,q) = \{0\}$ for $p \neq q$;

• $\mu(p,q) \cap \mu(p',q') = \varnothing$ for $p \neq q$ and $(p,q) \neq (p',q')$;

• $\bigcup\limits_{p,q \in \mathcal{R}} \mu(p,q) = U$.

Below we write $\mu(p)$ instead of $\mu(p,p)$, and considering a partial operation $\cdot$ on the set $\mathcal{R} \times (\mathcal{P}(U) \setminus \{\varnothing\}) \times \mathcal{R}$ we shall write, as above, $(p,u,q) \cdot (q,v,r)$ instead of $(p,\{u\},q) \cdot (q,\{v\},r)$.

A left semi-associative structure $\mathfrak{P} = \langle \mathcal{R} \times (\mathcal{P}(U) \setminus \{\varnothing\}) \times \mathcal{R}; \cdot \rangle$ with a regular family $\mu(\mathcal{R})$ of sets is called an *$I_{\mathcal{R}}$-structure* if the partial operation $\cdot$ of $\mathfrak{P}$ has values $(p,X,q) \cdot (p',Y,q')$ only for $p' = q$, $\varnothing \neq X \subseteq \mu(p,q)$, $\varnothing \neq Y \subseteq \mu(p',q')$, and is generated by the partial function $\cdot$ for elements in U where $(p,x,q) \cdot (q,y,r)$ forms a nonempty set of triples (p,z,r), $z \in \mu(p,r)$, if $x \in \mu(p,q)$ and $y \in \mu(q,r)$: for any sets $X, Y \in \mathcal{P}(U) \setminus \{\varnothing\}$, $\varnothing \neq X \subseteq \mu(p,q)$, $\varnothing \neq Y \subseteq \mu(q,r)$,

$$(p,X,q) \cdot (q,Y,r) = \bigcup \{(p,x,q) \cdot (q,y,r) \mid x \in X, y \in Y\},$$

as well as the following conditions hold:

• each restriction $\mathfrak{P}_{\mu(p)}$ of $\mathfrak{P}$ to the set $\{p\} \times (\mathcal{P}(\mu(p)) \setminus \{\varnothing\}) \times \{p\}$ is isomorphic to an I-groupoid with the universe $\mathcal{P}(\mu(p)) \setminus \{\varnothing\}$, $p \in \mathcal{R}$;

• if $u \in \mu(p,q)$ and $u < 0$ then the sets $(p,u,q \cdot (q,v,r)$ and $(r,v',p) \cdot (p,u,q)$ consist of negative elements for any $v \in \mu(q,r)$ and $v' \in (r,p)$;

• if $u \in \mu(p,q)$, $v \in \mu(q,r)$, $u > 0$, and $v > 0$, then the set $(p,u,q) \cdot (q,v,r)$ consists of non-negative elements;

• for any element $u \in \mu(p,q)$ with $u > 0$ there is a unique *inverse* element $u^{-1} \in \mu(q,p)$, $u^{-1} > 0$, such that $(p,0,p) \in (p,u,q) \cdot (q,u^{-1},p)$ and $(q,0,q) \in (q,u^{-1},p) \cdot (p,u,q)$;

• if an element (p,u,r) is positive and belongs to the set $(p,v_1,q) \cdot (q,v_2,r)$ then the element (r,u^{-1},p) belongs to the set $(r,v_2^{-1},q) \cdot (q,v_1^{-1},p)$;

• for any elements $(p,u_1,q), (q,u_2,r), (r,u_3,t)$ the following inclusion holds:

$$((p,u_1,q) \cdot (q,u_2,r)) \cdot (r,u_3,t) \supseteq (p,u_1,q) \cdot ((q,u_2,r) \cdot (r,u_3,t)),$$

and the strict inclusion

$$((p,u_1,q) \cdot (q,u_2,r)) \cdot (r,u_3,t) \supset p,u_1,q) \cdot ((q,u_2,r) \cdot (r,u_3,t))$$

may be satisfied only for $u_1 < 0$ and $|(q,u_2,r) \cdot (r,u_3,t)| \geq \omega$;

• the structure $\mathfrak{P}$ contains the *deterministic* substructure $\mathfrak{P}_d^{\geq 0}$, being the restriction to the set

$$U_d^{\geq 0} = \{u \in U^{\geq 0} \mid (q,u^{-1},p) \cdot (p,u,q) = \{(q,0,q)\} \text{ for some } p,q \in \mathcal{R}\};$$

every set $(p,u,q) \cdot (q,v,r)$ is a singleton for $u \in U_d^{\geq 0} \cap \mu(p,q)$ and $v \in U_d^{\geq 0} \cap \mu(q,r)$.

By the definition, any $I_{\mathcal{R}}$-structure $\mathfrak{P}$ contains I-subgroupoids $\mathfrak{P}_{\mu(p)}$, $p \in \mathcal{R}$, and $I_{\mathcal{R}}$-substructures $\mathfrak{P}^{\leq 0}$ and $\mathfrak{P}^{\geq 0}$ being restrictions of $\mathfrak{P}$ to the sets $U^{\leq 0}$ and $U^{\geq 0}$ respectively.

Theorem 9.2. *For any $I_{\mathcal{R}}$-structure $\mathfrak{P}$ there exists a theory T with a family $R \subset S(T)$ of 1-types and a regular family $\nu(R)$ of labelling functions such* *that $\mathfrak{P}_{\nu(R)} = \mathfrak{P}$. If the alphabet and the family $\mathcal{R}$ are at most countable, and the operation of $\mathfrak{P}$ does not force continuum many types, then T is small.*

Proof follows the schema for the proof of Theorem 6.1 extended by the schema for the proof of Theorem 3.4.1 in [29]. In view of bulkiness of this proof we only point out the distinctive features leading to the proof of this theorem.

1. For each symbol $p \in \mathcal{R}$ we introduce a unary predicate R_p, which intersects all predicates Col_n, $n \in \omega$, and forms, on the set of realizations of complete 1-type $p'(x)$, being isolated by the set $\{R_p(x)\} \cup \{\neg\mathrm{Col}_n(x) \mid n \in \omega\}$, a structure of isolating formulas corresponding to the I-groupoid $\mathfrak{P}_{\mu(p)}$. Moreover, we suppose that predicates R_p are disjoint.

2. For the elements $u \in \mu(p,q)$ the predicates Q_u link only elements a in R_p with elements b in R_q. Moreover, if $u > 0$ then $\mathrm{Col}(a) = \mathrm{Col}(b)$, and if $u < 0$ then $\mathrm{Col}(a) \leq \mathrm{Col}(b)$ and the coloring Col is Q_u-ordered.

3. The relation $Q^{\geq 0} = \bigcup\limits_{u \geq 0} Q_u$ is an equivalence relation such that its classes are ordered by the relation $Q^{<0} = \bigcup\limits_{u<0} Q_u$. □

In conclusion, we note that, using the operation $\cdot^{\mathrm{eq}}$, the constructions above can be transformed for an arbitrary family of types in $S(T)$.

Acknowledgment. The authors are grateful to the anonymous referee for the detailed reading of the text and comments that helped to improve the exposition.

References

[1] B.S. Baizhanov, S.V. Sudoplatov, and V.V. Verbovskiy, *Conditions for non-symmetric relations of semi-isolation*, Siberian Electronic Math. Reports, **9** (2012), 161–184. MR2954703
[2] E. Casanovas, *The number of countable models*, 2012, preprint.
[3] C.C. Chang and H.J. Keisler, *Model theory*, Elsevier, Amsterdam, 1990. MR1059055
[4] A.H. Clifford and G.B. Preston, *The Algebraic Theory of Semigroups. Vol.* 1, American Mathematical Society, Providence, 1961. MR0132791
[5] A.H. Clifford and G.B. Preston, *The Algebraic Theory of Semigroups. Vol.* 2, American Mathematical Society, Providence, 1967. MR0218472
[6] Yu.L. Ershov and E.A. Palyutin, *Mathematical logic*, Mir, Moscow, 1990. MR1079063
[7] *General algebra. Vol.* 1, O.V. Mel'nikov and others; ed. L.A. Skornyakov, Nauka, Moscow, 1990 (in Russian). MR1137273
[8] *General algebra. Vol.* 2, V.A. Artamonov and others; ed. L.A. Skornyakov, Nauka, Moscow, 1991 (in Russian). MR1146434
[9] F. Harary, *Graph theory*, Addison-Wesley, Reading, Massachusetts, 1969. MR0256911
[10] R. Hirsch and I. Hodkinson, *Relation algebras by games*, Elsevier, Amsterdam, 2002. MR1935083
[11] W. Hodges, *Model theory*, Cambridge University Press, 1993. MR1221741
[12] B. Kim, *On the number of countable models of a countable supersimple theory*, J. London Math. Soc., **60**:3 (1999), 641–645. MR1753804
[13] E.S. Lyapin, Semigroups, American Mathematical Society, Providence, 1974. Zbl 0303.20039
[14] R.D. Maddux, *Relation algebras*, Elsevier, Amsterdam, 2006. MR2269199
[15] A. Pillay, *Countable models of stable theories*, Proc. Amer. Math. Soc., **89**:4 (1983), 666–672. MR0718994
[16] A. Pillay, *A note on one-based theories*, 1989, preprint.
[17] S. Shelah, *Classification theory and the number of non-isomorphic models*, North-Holland, Amsterdam, 1990. MR1083551
[18] I.V. Shulepov, *Quotients of sequences by sets of word identities and limit models*, Algebra and Model Theory 7. Collection of papers, eds.: A.G. Pinus, K.N. Ponomaryov, S.V. Sudoplatov, NSTU, Novosibirsk (2009), 120–130 (in Russian). Zbl 1206.03032

[19] I.V. Shulepov, *On predicate similarity of limit models*, Proceedings of International Conference "Computability and Models", Collection of papers, Edition of D. Serikbayev East Kazakhstan State Technical University, Ust'-Kamenogorsk (2010), 136–138 (in Russian).

[20] I.V. Shulepov, *Predicate similarity of limit models with identifications of signature symbols*, Algebra and Model Theory 8. Collection of papers, eds.: A.G. Pinus, K.N. Ponomaryov, S.V. Sudoplatov, E.I. Timoshenko, NSTU, Novosibirsk (2011), 132–137 (in Russian).

[21] S.V. Sudoplatov, *On powerful types in small theories*, Siberian Math. J., **31**:4 (1990), 629–638. MR1083739

[22] S.V. Sudoplatov, *Type reduction and powerful types*, Siberian Math. J., **33**:1 (1992), 125–133. MR1165687

[23] S.V. Sudoplatov, *Group polygonometries and related algebraic systems*, Contributions to General Algebra 11, Proc. of the Olomouc Workshop '98 on General Algebra, Verlag Johannes Heyn, Klagenfurt (1999), 191–210. MR1696672

[24] S.V. Sudoplatov, *Complete theories with finitely many countable models.* I, Algebra and Logic, **43** (2004), 62–69. MR2073447

[25] S.V. Sudoplatov, *Complete theories with finitely many countable models.* II, Algebra and Logic, **45**:3 (2006), 180–200. MR2289092

[26] S.V. Sudoplatov, *Powerful digraphs*, Siberian Math. J., **48**:1 (2007), 165–171. MR2304889

[27] S.V. Sudoplatov, *On the number of countable models of complete theories with finite Rudin–Keisler preorders*, Siberian Math. J., **48**:2 (2007), 334–338. MR2330071

[28] S.V. Sudoplatov, *Syntactic approach to constructions of generic models*, Algebra and Logic, **46**:2 (2007), 134–146. MR2356525

[29] S.V. Sudoplatov, *The Lachlan problem*, NSTU, Novosibirsk, 2009 (in Russian, English version is available in: http://www.math.nsc.ru/∼sudoplatov/lachlan_eng_03_09_2008.pdf).

[30] S.V. Sudoplatov, *Hypergraphs of prime models and distributions of countable models of small theories*, J. Math. Sciences, **169** (2010), 680–695.

[31] S.V. Sudoplatov, *On limit models over types in the class of ω-stable theories*, Reports of Irkutsk State University. Series "Mathematics", **3** (2010), 114–120 (in Russian). Zbl 1271.03053

[32] S.V. Sudoplatov, *Group polygonometries*, NSTU, Novosibirsk, 2011, 2013 (in Russian).

[33] P. Tanović, *Theories with constants and three countable models*, Archive for Math. Logic, **46** (2007), 517–527. MR2321591

[34] P. Tanović, *Asymmetric* RK*-minimal types*, Archive for Math. Logic, **49** (2010), 367–377. MR2609988

Ilya Vladimirovich Shulepov
Novosibirsk State University,
ul. Pirogova, 2,
630090, Novosibirsk, Russia
E-mail address: salvodore@mail.ru

Sergey Vladimirovich Sudoplatov
Sobolev Institute of Mathematics,
pr. Koptyuga, 4,
630090, Novosibirsk, Russia;
Novosibirsk State Technical University,
pr. K.Marx, 20,
630073, Novosibirsk, Russia;
Novosibirsk State University,
ul. Pirogova, 2,
630090, Novosibirsk, Russia
E-mail address: sudoplat@math.nsc.ru