S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

http://semr.math.nsc.ru

Том 11, стр. 408–433 (2014) УДК 510.67, 512.53

MSC 03C07, 03G15, 20N02, 08A02, 08A55

ALGEBRAS OF DISTRIBUTIONS FOR SEMI-ISOLATING FORMULAS OF A COMPLETE THEORY

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Abstract. We define a class of algebras describing links of binary semi-isolating formulas on the set of all realizations for a family of 1-types of a complete theory. These algebras include algebras of isolating formulas considered before. We prove that a set of labels for binary semi-isolating formulas on the set of all realizations for a 1-type p forms a monoid of a special form with a partial order inducing ranks for labels, with set-theoretic operations, and with a composition. We describe the class of these structures. A description of the class of structures relative to families of 1-types is given.

Keywords: type, complete theory, algebra of binary semi-isolating formulas, join of monoids, deterministic structure.

In [18], a series of constructions is introduced admitting to realize key properties of countable theories and to obtain a classification of countable models of small (in particular, of Ehrenfeucht) theories with respect to two basic characteristics: Rudin–Keisler preorders and distribution functions for numbers of limit models. The construction of these theories is essentially based on the definition of special directed graphs with colored vertices and arcs as well as on the definition of $(n+1)$-ary predicates that turn prime models over realizations of n-types to prime models over realizations of 1-types and reducing links between prime models over finite sets to links between prime models over elements such that these links are given by definable sets of arcs and edges.

Sudoplatov S.V., Algebras of distributions for semi-isolating formulas of a complete theory.

The author was supported by RFFI (grant 12-01-00460-a).

Received July, 8, 2013, published June, 1, 2014.

In the paper, we develop a general approach to the description of binary links between realizations of 1-types in terms of labels of pairwise non-equivalent isolating formulas [17] to sets of labels of semi-isolating formulas.

We use the standard relation algebraic, model-theoretical, semigroup, and graph-theoretic terminology [2, 3, 4, 6, 7, 8, 10, 12, 13, 16] as well as some notions, notations, and constructions in [17, 18].

1. Preliminary notions, notations, and properties

Definition 1.1 [1, 17, 18, 20]. Let T be a complete theory, $\mathcal{M} \models T$. Consider types $p(x), q(y) \in S(\varnothing)$, realized in $\mathcal{M}$, and all (p,q)-*preserving* (p,q)-*semi-isolating*, $(p \to q)$-, or $(q \leftarrow p)$-*formulas* $\varphi(x,y)$ of T, i. e., formulas for which there is $a \in M$ such that $\models p(a)$ and $\varphi(a,y) \vdash q(y)$. Now, for each such a formula $\varphi(x,y)$, we define a binary relation $R_{p,\varphi,q} \rightleftharpoons \{(a,b) \mid \mathcal{M} \models p(a) \wedge \varphi(a,b)\}$. If $(a,b) \in R_{p,\varphi,q}$, then (a,b) is called a (p,φ,q)-*arc*. If $\varphi(a,y)$ is principal (over a), the (p,φ,q)-arc (a,b) is also *principal*.

If, in addition, $\varphi(x,y)$ is a $(p \leftrightarrow q)$-*formula*, i. e., it is both a $(p \to q)$- and a $(q \to p)$-formula then the set $[a,b] \rightleftharpoons \{(a,b),(b,a)\}$ is said to be a (p,φ,q)-*edge*. If the (p,φ,q)-edge $[a,b]$ consists of principal (p,φ,q)- and (q,φ^{-1},p)-arcs, where $\varphi^{-1}(x,y)$ denotes $\varphi(y,x)$, then $[a,b]$ is a *principal* (p,φ,q)-edge.

(p,φ,q)-arcs and (p,φ,q)-edges are called *arcs* and *edges* respectively if we speak of fixed or some $(p \to q)$-formula $\varphi(x,y)$. If (a,b) is a (p,φ,q)-arc such that the pair (b,a) is not an arc for any (q,p)-formula (that is $(b,a) \notin \mathrm{SI}_{\{p,q\}} \rightleftharpoons \{(c,d) \mid \mathrm{tp}(c), \mathrm{tp}(d) \in \{p,q\}$ and c semi-isolates $d\}$), then (a,b) is called *irreversible*.

For types $p(x), q(y) \in S(\varnothing)$, we denote by $\mathrm{SICF}(p,q)$ the set of all $(p \to q)$-formulas $\varphi(x,y)$ such that $\{\varphi(a,y)\}$ is consistent for some (all) a with $\models p(a)$.

Let $\mathrm{SICE}(p,q)$ be the set of all pairs of formulas $(\varphi(x,y),\psi(x,y)) \in \mathrm{SICF}(p,q)$ such that for any (some) realization a of p the sets of solutions for $\varphi(a,y)$ and $\psi(a,y)$ coincide.

Clearly, $\mathrm{SICE}(p,q)$ is an equivalence relation on the set $\mathrm{SICF}(p,q)$. Notice that each $\mathrm{SICE}(p,q)$-class E corresponds to either a set of (p,φ,q)-edges, or a set of irreversible (p,φ,q)-arcs, or simultaneously a set of (p,φ,q)-edges and of irreversible (p,φ,q)-arcs linking realizations of p and q by any (some) formula φ in E. Thus the quotient $\mathrm{SICF}(p,q)/\mathrm{SICE}(p,q)$ is represented as a disjoint union of sets $\mathrm{SICFE}(p,q)$, $\mathrm{SICFA}(p,q)$, and $\mathrm{SICFM}(p,q)$, where $\mathrm{SICFE}(p,q)$ consists of $\mathrm{SICE}(p,q)$-classes corresponding to sets of edges, $\mathrm{SICFA}(p,q)$ consists of $\mathrm{SICE}(p,q)$-classes corresponding to sets of irreversible arcs, and $\mathrm{SICFM}(p,q)$ consists of $\mathrm{SICE}(p,q)$-classes corresponding to sets containing edges and irreversible arcs.

The sets $\mathrm{SICF}(p,p)$, $\mathrm{SICE}(p,p)$, $\mathrm{SICFE}(p,p)$, $\mathrm{SICFA}(p,p)$, and $\mathrm{SICFM}(p,p)$ are denoted by $\mathrm{SICF}(p)$, $\mathrm{SICE}(p)$, $\mathrm{SICFE}(p)$, $\mathrm{SICFA}(p)$, and $\mathrm{SICFM}(p)$ respectively.

Let T be a complete theory without finite models, $U = U^- \dot\cup \{0\} \dot\cup U^+ \dot\cup U'$ be an alphabet of cardinality $\geq |S(T)|$ and consisting of *negative elements* $u^- \in U^-$, *positive elements* $u^+ \in U^+$, *neutral elements* $u' \in U'$, and zero 0. As usual, we write $u < 0$ for any $u \in U^-$ and $u > 0$ for any $u \in U^+$. The set $U^- \cup \{0\}$ is denoted by $U^{\leq 0}$ and $U^+ \cup \{0\}$ is denoted by $U^{\geq 0}$. Elements of U are called *labels*.

Let $\nu(p,q)\colon \mathrm{SICF}(p,q)/\mathrm{SICE}(p,q) \to U$ be an injective *labelling functions*, $p(x), q(y) \in S(\varnothing)$, for which negative elements correspond to classes in $\mathrm{SICFA}(p,q)/\mathrm{SICE}(p,q)$, positive elements and 0 correspond to classes in

SICFE(p,q)/SICE(p,q) such that 0 is defined only for $p=q$ and is represented by the formula $(x\approx y)$, and neutral elements code classes in SICFM(p,q)/SICE(p,q), $\nu(p)\rightleftharpoons\nu(p,p)$. We additionally assume that $\rho_{\nu(p)}\cap\rho_{\nu(q)}=\{0\}$ for $p\neq q$ (where, as usual, we denote by ρ_f the image of the function f) and $\rho_{\nu(p,q)}\cap\rho_{\nu(p',q')}=\varnothing$ if $p\neq q$ and $(p,q)\neq(p',q')$. Labelling functions with the properties above as well families of these functions are said to be *regular*. Below we shall consider only regular labelling functions and their regular families.

The labels, corresponding to isolating formulas, are said to be *isolating* whereas each label in $\bigcup\limits_{p,q\in S^1(\varnothing)}\rho_{\nu(p,q)}$ is *semi-isolating*. By the definition, each isolating label belongs to $U^-\dot\cup\{0\}\dot\cup U^+$, i. e., it is not neutral.

We denote by $\theta_{p,u,q}(x,y)$ a formula in SICF(p,q) with the label $u\in\rho_{\nu(p,q)}$. If a type p is fixed and $p=q$ then a formula $\theta_{p,u,q}(x,y)$ is denoted by $\theta_u(x,y)$.

Note that if $\theta_{p,u,q}(x,y)$ and $\theta_{q,v,p}(x,y)$ are formulas witnessing that for realizations a and b of p and q respectively the pairs (a,b) and (b,a) belong to SI$_{\{p,q\}}$, then the formula $\varphi(x,y)=\theta_{p,u,q}(x,y)\wedge\theta_{q,v,p}(y,x)$ witnesses that $[a,b]$ is a (p,φ,q)-edge. If the edge $[a,b]$ is principal and $\theta_{p,u,q}(a,y)$ is an isolating formula such that $\models\theta_{p,u,q}(a,b)$, $\models p(a)$, then the label u is *invertible* and the label $v\in U^{\geq 0}$ corresponds uniquely to u such that $\theta_{q,v,p}(b,y)$ is an isolating formula with $\models\theta_{q,v,p}(b,a)$, and vice versa. The labels u and v are *reciprocally inverse* and are denoted by v^{-1} and u^{-1} respectively. In general case, each label $u\in U^{\geq 0}$ has a (nonempty) set of *inverse* labels in $U^{\geq 0}$, denoted also by u^{-1}. Note that independently on a label $u\in U^{\geq 0}$, for which a formula $\theta_{p,u,q}(x,y)$ witnesses that $[a,b]$ is a (p,φ,q)-edge, the set u^{-1} includes all labels $v\in U^{\geq 0}$ such that $[b,a]$ is a $(q,\theta_{q,v,p},p)$-edge.

Neutral labels correspond, for instance, the formulas $\theta_{p,u,q}(x,y)\vee\theta_{p,v,q}(x,y)$, where $u<0$ and $v\geq 0$.

For types $p_1,p_2,\ldots,p_{k+1}\in S^1(\varnothing)$ and sets $X_1,X_2,\ldots,X_k\subseteq U$ of labels we denote by

$$\mathrm{SI}(p_1,X_1,p_2,X_2,\ldots,p_k,X_k,p_{k+1})$$

the set of all labels $u\in\rho_{\nu(p_1,p_{k+1})}$ corresponding to formulas $\theta_{p_1,u,p_{k+1}}(x,y)$ satisfying, for realizations a of p_1 and some $u_1\in X_1\cap\rho_{\nu(p_1,p_2)},\ldots,u_k\in X_k\cap\rho_{\nu(p_k,p_{k+1})}$, the following condition:

$$\theta_{p_1,u,p_{k+1}}(a,y)\vdash\theta_{p_1,u_1,p_2,u_2,\ldots,p_k,u_k,p_{k+1}}(a,y),$$

where

$$\begin{gathered}\theta_{p_1,u_1,p_2,u_2,\ldots,p_k,u_k,p_{k+1}}(x,y)\rightleftharpoons\\ \rightleftharpoons\exists x_2,x_3,\ldots x_k(\theta_{p_1,u_1,p_2}(x,x_2)\wedge\theta_{p_2,u_2,p_3}(x_2,x_3)\wedge\ldots\\ \ldots\wedge\theta_{p_{k-1},u_{k-1},p_k}(x_{k-1},x_k)\wedge\theta_{p_k,u_k,p_{k+1}}(x_k,y)).\end{gathered}$$

In view of transitivity of semi-isolation, each formula $\theta_{p_1,u_1,p_2,u_2,\ldots,p_k,u_k,p_{k+1}}(x,y)$ has a label in $\rho_{\nu(p_1,p_{k+1})}$.

Thus the Boolean $\mathcal{P}(U)$ of U is the universe of an *algebra* $\mathfrak{A}$ *of distributions of binary semi-isolating formulas* with k-ary operations

$$\mathrm{SI}(p_1,\cdot,p_2,\cdot,\ldots,p_k,\cdot,p_{k+1}),$$

where $p_1,\ldots,p_{k+1}\in S^1(\varnothing)$.[1] This algebra has a natural restriction to any family $R\subseteq S^1(\varnothing)$ as well as to the algebras of distributions of binary *isolating* formulas [17]. Besides, if U_0 is a subalphabet of U then the restriction of the universe of $\mathfrak{A}$

[1]Later (Section 7) it will be shown that it is sufficient to consider only SI$(p_1,\cdot,p_2,\cdot,p_3)$.

to the set $\mathcal{P}(U_0)$ and the restrictions for values of operations to the set U_0 forms, possibly partial, algebra $\mathfrak{A} \upharpoonright U_0$.

Note that if some set X_i is disjoint with $\rho_{\nu(p_i,p_{i+1})}$, in particular, if it is empty then

$$\mathrm{SI}(p_1, X_1, p_2, X_2, \dots, p_k, X_k, p_{k+1}) = \varnothing,$$

and if each X_i has common elements with $\rho_{\nu(p_i,p_{i+1})}$ then

$$\mathrm{SI}(p_1, X_1, p_2, X_2, \dots, p_k, X_k, p_{k+1}) \neq \varnothing.$$

Note also that if $X_i \not\subseteq \rho_{\nu(p_i,p_{i+1})}$ for some i then

$$\mathrm{SI}(p_1, X_1, p_2, X_2, \dots, p_k, X_k, p_{k+1}) =$$

$$= \mathrm{SI}(p_1, X_1 \cap \rho_{\nu(p_1,p_2)}, p_2, X_2 \cap \rho_{\nu(p_2,p_3)}, \dots, p_k, X_k \cap \rho_{\nu(p_k,p_{k+1})}, p_{k+1}).$$

In view of the previous equality, below, considering values

$$\mathrm{SI}(p_1, X_1, p_2, X_2, \dots, p_k, X_k, p_{k+1}),$$

we shall assume that $X_i \subseteq \rho_{\nu(p_i,p_{i+1})}$, $i = 1, \dots, k$.

If each set X_i is a singleton consisting of an element u_i then we use u_i instead of X_i in $\mathrm{SI}(p_1, X_1, p_2, X_2, \dots, p_k, X_k, p_{k+1})$ and write

$$\mathrm{SI}(p_1, u_1, p_2, u_2, \dots, p_k, u_k, p_{k+1}).$$

By the definition the following equality holds:

$$\mathrm{SI}(p_1, X_1, p_2, X_2, \dots, p_k, X_k, p_{k+1}) =$$

$$= \cup\{\mathrm{SI}(p_1, u_1, p_2, u_2, \dots, p_k, u_k, p_{k+1}) \mid u_1 \in X_1, \dots, u_k \in X_k\}.$$

Hence the specification of $\mathrm{SI}(p_1, X_1, p_2, X_2, \dots, p_k, X_k, p_{k+1})$ is reduced to the specifications of $\mathrm{SI}(p_1, u_1, p_2, u_2, \dots, p_k, u_k, p_{k+1})$. Note also that $\mathrm{SI}(p, X, q) = X$ for any $X \subseteq \rho_{\nu(p,q)}$.

Clearly, if $u_i = 0$ then $p_i = p_{i+1}$ for nonempty sets

$$\mathrm{SI}(p_1, u_1, p_2, u_2, \dots, p_i, 0, p_{i+1}, \dots, p_k, u_k, p_{k+1})$$

and the following conditions hold:

$$\mathrm{SI}(p_1, 0, p_1) = \{0\},$$

$$\mathrm{SI}(p_1, u_1, p_2, u_2, \dots, p_i, 0, p_{i+1}, u_{i+1}, p_{i+2}, \dots, p_k, u_k, p_{k+1}) =$$
$$= \mathrm{SI}(p_1, u_1, p_2, u_2, \dots, p_i, u_{i+1}, p_{i+2}, \dots, p_k, u_k, p_{k+1}).$$

If all types p_i equal to a type p then we write $\mathrm{SI}_p(X_1, X_2, \dots, X_k)$ and $\mathrm{SI}_p(u_1, u_2, \dots, u_k)$ as well as $\lceil X_1, X_2, \dots, X_k \rceil_p$ and $\lceil u_1, u_2, \dots, u_k \rceil_p$ instead of

$$\mathrm{SI}(p_1, X_1, p_2, X_2, \dots, p_k, X_k, p_{k+1})$$

and

$$\mathrm{SI}(p_1, u_1, p_2, u_2, \dots, p_k, u_k, p_{k+1})$$

respectively. We omit the index $\cdot_p$ if the type p is fixed. In this case, we write $\theta_{u_1,u_2,\dots,u_k}(x, y)$ instead of $\theta_{p,u_1,p,u_2,\dots,p,u_k,p}(x, y)$.

Proposition 1.2. (1) *If $p, q \in S^1(T)$ are principal types then $\rho_{\nu(p,q)} \cup \rho_{\nu(q,p)} \subseteq U^{\geq 0}$.*

(2) *If $p, q \in S^1(T)$, p is a principal type and q is a non-principal type then $\rho_{\nu(p,q)} = \varnothing$ and $\rho_{\nu(q,p)} \subseteq U^{-}$.*

Proof. (1) If $\rho_{\nu(p,q)}$ contains a label $u \notin U^{\geq 0}$ then there are realizations a and b of p and q respectively such that $(a, b) \in \mathrm{SI}_{\{p,q\}}$ and $(b, a) \notin \mathrm{SI}_{\{p,q\}}$. But since $p(x)$

contains a principal formula $\varphi(x)$, this formula witnesses that $(b,a)\in \mathrm{SI}_{\{p,q\}}$. The contradiction implies that $\rho_{\nu(p,q)}\subseteq U^{\geq 0}$. Similarly we obtain $\rho_{\nu(q,p)}\subseteq U^{\geq 0}$.

(2) Let $\varphi(x)$ be a principal formula of $p(x)$. If $\models p(a)$, $\models q(b)$, and $(a,b)\in \mathrm{SI}_{\{p,q\}}$ that witnessed by a formula $\theta_u(x,y)$, the formula $\exists x(\varphi(x)\wedge\theta_u(x,y))$ isolates $q(y)$. Since q is not isolated we obtain $\rho_{\nu(p,q)}=\varnothing$. By the same reason, $\rho_{\nu(q,p)}\subseteq U^-$. □

Corollary 1.3. *If $p(x)$ is a principal type then $\rho_{\nu(p)}\subseteq U^{\geq 0}$.*

Proposition 1.4. *Let $p_1,p_2,\dots,p_{k+1}$ be types in $S^1(\varnothing)$. The following assertions hold.*

(1) *If $u_i\in\rho_{\nu(p_i,p_{i+1})}$, $i=1,\dots,k$, and some u_i is negative then*

$$\mathrm{SI}(p_1,u_1,p_2,u_2,\dots,p_k,u_k,p_{k+1})\subseteq U^-.$$

(2) *If $u_i\in\rho_{\nu(p_i,p_{i+1})}\cap U^{\geq 0}$, $i=1,\dots,k$, then*

$$\mathrm{SI}(p_1,u_1,p_2,u_2,\dots,p_k,u_k,p_{k+1})\subseteq U^{\geq 0}.$$

(3) *If $u_i\in\rho_{\nu(p_i,p_{i+1})}\cap(U^{\geq 0}\cup U')$, $i=1,\dots,k$, and some u_i belongs to U' then*

$$\mathrm{SI}(p_1,u_1,p_2,u_2,\dots,p_k,u_k,p_{k+1})\subseteq U'.$$

(4) *If $u_i\in\rho_{\nu(p_i,p_{i+1})}\cap U^{\geq 0}$, $i=1,\dots,k$, then all elements of the set $X\rightleftharpoons \mathrm{SI}(p_1,u_1,p_2,u_2,\dots,p_k,u_k,p_{k+1})$ are invertible and the set $X^{-1}\rightleftharpoons\cup\{v^{-1}\mid v\in X\}$ is contained in $\mathrm{SI}(p_{k+1},u_k^{-1},p_k,u_{k-1}^{-1},\dots,p_2,u_1^{-1},p_1)$.*

Proof. (1)–(3) follow by transitivity of semi-isolation [15].

(4) All elements in X are invertible by (2). Let v' be an element in $v^{-1}\subseteq X^{-1}$. Then for any $(p_1,\theta_{p_1,v,p_{k+1}},p_{k+1})$-edge $[a,b]$ such that $[b,a]$ is a $(p_{k+1},\theta_{p_{k+1},v',p_1},p_1)$-edge, there are realizations a_i of p_i, $i=1,\dots,k+1$, such that $a_0=a$, $a_{k+1}=b$, $\models\theta_{p_i,u_i,p_{i+1}}(a_i,a_{i+1})$, $i=1,\dots,k$.

Since $[a_{i+1},a_i]$ is an u'_i-edge for some $u'_i\in u_i^{-1}$, $i=1,\dots,k$, then

$$\theta_{p_{k+1},v',p_1}(b,x)\vdash\theta_{p_{k+1},u'_k,p_k,u'_{k_1}\dots,p_2,u'_1,p_1}(b,x),$$

whence, $v'\in\mathrm{SI}(p_{k+1},u_k^{-1},p_k,u_{k-1}^{-1},\dots,p_2,u_1^{-1},p_1)$. □

Note that the inclusion $X^{-1}\subseteq\mathrm{SI}(p_{k+1},u_k^{-1},p_k,u_{k-1}^{-1},\dots,p_2,u_1^{-1},p_1)$ can be strict since labels in u_i^{-1} may compose new (with respect to X^{-1}) labels for $\mathrm{SI}(p_{k+1},u_k^{-1},p_k,u_{k-1}^{-1},\dots,p_2,u_1^{-1},p_1)$.

Indeed, by the definition, X^{-1} should consist of labels whose formulas satisfy all edges for $\mathrm{SI}(p_1,u_1,p_2,u_2,\dots,p_k,u_k,p_{k+1})$, whereas labels in $\mathrm{SI}(p_{k+1},u_k^{-1},p_k,u_{k-1}^{-1},\dots,p_2,u_1^{-1},p_1)$ may correspond to formulas disjoint with these edges. Taking, for instance, $k=2$ and a disjunction $\theta_{p_3,v_2,p_2,v_1,p_1}(x,y)\vee\theta_{p_3,w_2,p_2,w_1,p_1}(x,y)$, where the label v for $\theta_{p_3,v_2,p_2,v_1,p_1}(x,y)$ belongs to X^{-1} and $\theta_{p_3,w_2,p_2,w_1,p_1}$ does not satisfy edges for $\mathrm{SI}(p_1,u_1,p_2,u_2,p_3)$, we have $w\in\mathrm{SI}(p_3,u_2^{-1},p_2,u_1^{-1},p_1)\setminus X^{-1}$, where w is the label for $\theta_{p_3,w_2,p_2,w_1,p_1}(x,y)$.

Corollary 1.5. *Restrictions of U to the sets $U^{\leq 0}$, $U^{\geq 0}$, and $U^{\geq 0}\cup U'$ form subalgebras of the algebra of distributions of binary semi-isolating formulas. The operation of inversion is coordinated with the operations of the algebra.*

2. Preordered algebras of distributions of binary semi-isolating formulas

For the set U of labels in the algebra $\mathfrak{A}$ of binary semi-isolating formulas of theory T, we define the following relation $\trianglelefteq$: if $u, v \in U$ then $u \trianglelefteq v$ if and only if $u = v$, or $u, v \in \rho_{\nu(p,q)}$ for some types $p, q \in S^1(\varnothing)$ and $\theta_{p,u,q}(a, y) \vdash \theta_{p,v,q}(a, y)$ for some (any) realization a of p. If $u \trianglelefteq v$ and $u \neq v$ we write $u \vartriangleleft v$.

By the definition the relation $\trianglelefteq$ is reflexive and transitive. It is antisymmetric since distinct labels correspond to non-equivalent formulas.

Below we consider some properties for the substructures $\langle \rho_{\nu(p,q)}; \trianglelefteq \rangle$ of the partially ordered set $\langle U; \trianglelefteq \rangle$.

Proposition 2.1. (1) *For any types* $p, q \in S^1(\varnothing)$, *the partially ordered set* $\langle \rho_{\nu(p,q)}; \trianglelefteq \rangle$ *forms a upper semilattice.*

(2) *An element* $u \in \rho_{\nu(p,q)}$ *is* $\trianglelefteq$*-minimal if and only if for a realization* a *of* p, *the formula* $\theta_{p,u,q}(a, y)$ *is isolating.*

(3) (monotony) *If* $u, v \in \rho_{\nu(p,q)}$ *and* $u \trianglelefteq v$ *then* $v \in U^\delta$, $\delta \in \{-, +\}$, *implies* $u \in U^\delta$, *and if* $u \in U'$ *then* $v \in U'$.

Proof. (1) If $u_1, u_2 \in \rho_{\nu(p,q)}$ then for the formulas $\theta_{p,u_1,q}(x, y)$ and $\theta_{p,u_2,q}(x, y)$ the label v for the formula $\theta_{p,u_1,q}(x, y) \vee \theta_{p,u_2,q}(x, y)$ is the supremum for the labels u_1 and u_2.

(2) If $\theta_{p,u,q}(a, y)$ is an isolating formula then the label u is $\trianglelefteq$-minimal by the definition. If the formula $\theta_{p,u,q}(a, y)$ is not isolating then there is a formula $\varphi(a, y)$ such that the semi-isolating formulas $\theta_{p,u,q}(a, y) \wedge \varphi(a, y)$ and $\theta_{p,u,q}(a, y) \wedge \neg\varphi(a, y)$ are consistent. For the labels v_1 and v_2 of these formulas, we have $v_1 \neq v_2$, $v_1 \vartriangleleft u$, and $v_2 \vartriangleleft u$.

(3) If $v \in \rho_{\nu(p,q)} \cap U^-$ then for any solution b of the formula $\theta_{p,v,q}(a, y)$, where $\models p(a)$, the pair (a, b) is an irreversible arc. Hence, for any solution b of $\theta_{p,u,q}(a, y)$, where $u \trianglelefteq v$, the pair (a, b) is also an irreversible arc and so u belongs to U^-. Replacing arcs by edges, the same arguments show that $u \trianglelefteq v$ and $v \in U^+$ imply $u \in U^+$. If $u \in U'$ then the set of pairs (a, b) for the formula $\theta_{p,u,q}(a, y)$ contains both irreversible and reversible arcs. This property is preserved for any label v with $u \trianglelefteq v$, whence $v \in U'$. □

The partial order $\trianglelefteq$ has a natural extension to a preorder on the set $\mathcal{P}(U)$: for any sets $X, Y \in \mathcal{P}(U)$ we put $X \trianglelefteq Y$ if $X = \varnothing$, or for any $x \in X$ there is $y \in Y$ with $x \trianglelefteq y$ and for any $y \in Y$ there is $x \in X$ with $x \trianglelefteq y$. Thus, the algebra $\mathfrak{A}$ is transformed to the preordered algebra $\langle \mathfrak{A}; \trianglelefteq \rangle$ with the monotonic property with respect to its restrictions to the sets $U^{\leq 0}$, $U^{\geq 0}$, and U'.

Another natural expansion of the already preordered algebra $\langle \mathfrak{A}; \trianglelefteq \rangle$ is based on the properties mentioned that if $u_1, u_2 \in \rho_{\nu(p,q)}$ and $v \in \rho_{\nu(q,r)}$ then the formulas $\theta_{p,u_1,q,v,r}(x, y)$ and $\theta_{p,u_1,q}(x, y) \vee \theta_{p,u_2,q}(x, y)$ as well as $\theta_{p,u_1,q}(x, y) \wedge \theta_{p,u_2,q}(x, y)$ and $\theta_{p,u_1,q}(x, y) \wedge \neg\theta_{p,u_2,q}(x, y)$ (if the formulas $\theta_{p,u_1,q}(a, y) \wedge \theta_{p,u_2,q}(a, y)$ and $\theta_{p,u_1,q}(a, y) \wedge \neg\theta_{p,u_2,q}(a, y)$ are consistent for a with $\models p(a)$) have labels in U. We denote these labels by $u_1 \circ v$, $u_1 \vee u_2$, $u_1 \wedge u_2$, and $u_1 \wedge \neg u_2$ respectively. The last label is also denoted by $\neg u_2 \wedge u_1$. The label $u_1 \circ v$ is the *composition* of labels u_1 and v; $u_1 \vee u_2$ is the *union* or the *disjunction* of labels u_1 and u_2; $u_1 \wedge u_2$ is their *intersection* or *conjunction*; $u_1 \wedge \neg u_2$ is the *relative complement* of u_2 in u_1.

Clearly, $u_1 \trianglelefteq u_1 \vee u_2$, $u_2 \trianglelefteq u_1 \vee u_2$, $u_1 \wedge u_2 \trianglelefteq u_1$, $u_1 \wedge u_2 \trianglelefteq u_2$, $u_1 \wedge \neg u_2 \trianglelefteq u_1$.

We set

$$(p,(u_1\circ v),r)\rightleftharpoons\left\{\begin{array}{ll}\{u_1\circ v\}, & \text{if } u_1\in\rho_{\nu(p,q)} \text{ and } v\in\rho_{\nu(q,r)},\\ \varnothing, & \text{if } u_1\notin\rho_{\nu(p,q)} \text{ or } v\notin\rho_{\nu(q,r)},\end{array}\right.$$

$$(p,(u_1\vee u_2),q)\rightleftharpoons\left\{\begin{array}{ll}\{u_1\vee u_2\}, & \text{if } u_1\in\rho_{\nu(p,q)} \text{ and } u_2\in\rho_{\nu(p,q)},\\ \{u_1\}, & \text{if } u_1\in\rho_{\nu(p,q)} \text{ and } u_2\notin\rho_{\nu(p,q)},\\ \{u_2\}, & \text{if } u_1\notin\rho_{\nu(p,q)} \text{ and } u_2\in\rho_{\nu(p,q)},\\ \varnothing, & \text{if } u_1\notin\rho_{\nu(p,q)} \text{ and } u_2\notin\rho_{\nu(p,q)},\end{array}\right.$$

$$(p,(u_1\wedge u_2),q)\rightleftharpoons\left\{\begin{array}{ll}\{u_1\wedge u_2\}, & \text{if } u_1\in\rho_{\nu(p,q)}, u_2\in\rho_{\nu(p,q)}\\ & \text{and } \models\exists y(\theta_{p,u_1,q}(a,y)\wedge\theta_{p,u_2,q}(a,y)),\\ \varnothing, & \text{otherwise},\end{array}\right.$$

$$(p,(u_1\wedge\neg u_2),q)\rightleftharpoons\left\{\begin{array}{ll}\{u_1\wedge\neg u_2\}, & \text{if } u_1\in\rho_{\nu(p,q)}, u_2\in\rho_{\nu(p,q)}\\ & \text{and } \models\exists y(\theta_{p,u_1,q}(a,y)\wedge\neg\theta_{p,u_2,q}(a,y)),\\ \varnothing, & \text{otherwise},\end{array}\right.$$

$$(p,(X_1\,\tau\,X_2),q)\rightleftharpoons\cup\{(p,(u_1\,\tau\,u_2),q)\mid u_1\in X_1,u_2\in X_2\},\ \tau\in\{\circ,\vee,\wedge\},$$

$$(p,(X_1\wedge\neg X_2),q)\rightleftharpoons(p,(\neg X_2\wedge X_1),q)\rightleftharpoons$$

$$\rightleftharpoons\cup\{(p,(u_1\wedge\neg u_2),q)\mid u_1\in X_1,u_2\in X_2\},\ X_1,X_2\in\mathcal{P}(U).$$

Labels u_1 and u_2 are *consistent* if $u_1\wedge u_2\in U$. If $u_1\wedge u_2=\varnothing$ the labels u_1 and u_2 are called *inconsistent*.

The preordered algebra $\langle\mathfrak{A};\trianglelefteq\rangle$ equipped with binary operations $(p,(\cdot\,\tau\,\cdot),q)$, $\tau\in\{\vee,\wedge,\circ\}$, and $(p,(\cdot\wedge\neg\,\cdot),q)$, $p,q\in S^1(\varnothing)$, is called a *preordered algebra with relative set-theoretic operations and the composition* or briefly a POSTC-*algebra*.

For any types $p,q\in S^1(\varnothing)$ the structure $\langle\rho_{\nu(p,q)}\cup\{\varnothing\};\vee,\wedge,\varnothing\rangle$ with operations $\vee$ and $\wedge$ on labels, being extended by equalities $u\vee\varnothing=u$, $u\wedge\varnothing=\varnothing$, where $u\in\rho_{\nu(p,q)}\cup\{\varnothing\}$, is an Ershov algebra, i. e., a *distributive lattice with zero* $\varnothing$ *and relative complements* [5] such that for any $u,v\in\rho_{\nu(p,q)}$ if $u\trianglelefteq v$ and $u'=\neg u\wedge v$ is a label then $u\wedge u'=\varnothing$ and $u\vee u'=v$, and if the label u' does not exist then $u=v$.

A label $u\in U$ is an *atom* or an *atomic label* if u is a $\trianglelefteq$-minimal element in U, i. e., for any label $v\in U$ if $v\trianglelefteq u$ then $v=u$.

By Proposition 2.1, the set of atoms equals the set of isolating labels and, thus, each atom $u\in\rho_{\nu(p,q)}$ is represented by an isolated formula $\theta_{p,u,q}(a,y)$, where $\models p(a)$.

Let R be a nonempty family of types in $S^1(\varnothing)$, $\mathfrak{A}_R$ be a restriction of POSTC-algebra $\mathfrak{A}$ to the family R. The structure $\mathfrak{A}_R$ is *atomic* if for any types $p,q\in R$ and for any label $u\in\rho_{\nu(p,q)}$ there is an atom $v\in\rho_{\nu(p,q)}$ such that $v\trianglelefteq u$. The POSTC-algebra $\mathfrak{A}$ is called R-*atomic* if $\mathfrak{A}_R$ is atomic. If $R=S^1(\varnothing)$ then the R-atomic POSTC-algebra is called *atomic*.

Using the definition of atomic structure, of R-atomic POSTC-algebra, and of small theory we obtain the following assertions.

Proposition 2.2. *If R is a nonempty family of types in $S^1(\varnothing)$ and for any type $p\in R$, there is an atomic model $\mathcal{M}_p$ over a realization of p, then the* POSTC-*algebra $\mathfrak{A}$ is R-atomic.*

Corollary 2.3. *If T is a small theory then the* POSTC-*algebra $\mathfrak{A}$ is atomic.*

3. Ranks and degrees of semi-isolation

The following definition is a local variation of Morley rank [14].

Definition 3.1. For triples (p, u, q), where $p, q \in S^1(\varnothing)$, $u \in U \cup \{\varnothing\}$, we define inductively the *rank* $\mathrm{si}(p, u, q)$ *of semi-isolation*:

(1) $\mathrm{si}(p, u, q) = 0$ if $u \notin \rho_{\nu(p,q)}$;

(2) $\mathrm{si}(p, u, q) \geq 1$ if $u \in \rho_{\nu(p,q)}$;

(3) for a positive ordinal α, $\mathrm{si}(p, u, q) \geq \alpha + 1$ if there is a set $\{v_i \mid i \in \omega\}$ of pairwise inconsistent labels such that $v_i \lhd u$ and $\mathrm{si}(p, v_i, q) \geq \alpha$, $i \in \omega$;

(4) for a limit ordinal α, $\mathrm{si}(p, u, q) \geq \alpha$ if $\mathrm{si}(p, u, q) \geq \beta$ for any $\beta \in \alpha$.

As usual, we write $\mathrm{si}(p, u, q) = \alpha$ if $\mathrm{si}(p, u, q) \geq \alpha$ and $\mathrm{si}(p, u, q) \not\geq \beta$ for $\alpha \in \beta$; $\mathrm{si}(p, u, q) \rightleftharpoons \infty$ if $\mathrm{si}(p, u, q) \geq \alpha$ for any ordinal α.

If types p and q are fixed, we write $\mathrm{si}(u)$ instead of $\mathrm{si}(p, u, q)$ and this value is said to be the *rank of semi-isolation* or the si-*rank* of the label u or of the element $u = \varnothing$ (with respect to the pair (p, q)). For a formula $\theta_{p,u,q}(x, y)$ we set $\mathrm{si}(\theta_{p,u,q}(x, y)) \rightleftharpoons \mathrm{si}(u)$.

Clearly, if the theory is small then the si-rank of each label is a countable ordinal (having a label u with $\mathrm{si}(p, u, q) \geq \omega_1$, we get continuum many complete types $r(x, y) \supset p(x) \cup q(y)$).

By the definition we have the following inequality for any formula $\theta_{p,u,q}(x, y)$ and any realization a of p giving a low bound for the Morley rank of the formula $\theta_{p,u,q}(a, y)$ by the si-rank:

$$\mathrm{si}(\theta_{p,u,q}(x, y)) \leq \mathrm{MR}(\theta_{p,u,q}(a, y)) + 1. \tag{1}$$

The inequality (1) implies

Remark 3.2. If a theory T has a finite Morley rank then si-ranks of labels in $\bigcup\limits_{p,q\in S^1(T)} \rho_{\nu(p,q)}$ are bounded by the value $\mathrm{MR}(x \approx x) + 1$.

We set $\mathrm{si}(p, q) \rightleftharpoons \sup\{\mathrm{si}(p, u, q) \mid u \in U \cup \{\varnothing\}\}$, $\mathrm{si}(p) \rightleftharpoons \mathrm{si}(p, p)$. For a nonempty family R of 1-types, we put $\mathrm{si}(R) \rightleftharpoons \sup\{\mathrm{si}(p, q) \mid p, q \in R\}$. A family R is called si-*minimal* if $\mathrm{si}(R) = 1$. The value $\mathrm{si}(p, q)$ is said to be the *rank of semi-isolation* or the si-*rank* of pair (p, q), and $\mathrm{si}(R)$ is the *rank of semi-isolation* or the si-*rank* of the family R.

Since there are $|T|$ formulas of a theory T and the inequality (1) holds we obtain

Proposition 3.3. *Each* si-*rank in a theory T is either equal to ∞ or less than* $\min\{|T|^+, (\mathrm{MR}(x \approx x)+1)^+\}$. *If the Morley rank* $\mathrm{MR}(x \approx x)$ *is equal to an ordinal α then any* si-*rank in T is not more than $\alpha + 1$.*

The estimation for si-ranks in Proposition 3.3 can be far from exact. For instance, si-ranks in ω-categorical theories are finite while there are non-ω-stable ω-categorical theories.

Proposition 3.4. *For any types $p, q \in S^1(\varnothing)$ the following assertions are satisfied.*

(1) *If $u, v \in \rho_{\nu(p,q)} \cup \{\varnothing\}$ and $u \unlhd v$ then* $\mathrm{si}(u) \leq \mathrm{si}(v)$.

(2) *If $u, v \in \rho_{\nu(p,q)} \cup \{\varnothing\}$ then* $\mathrm{si}(u \vee v) = \max\{\mathrm{si}(u), \mathrm{si}(v)\}$ *and* $\mathrm{si}(u \wedge v) \leq \min\{\mathrm{si}(u), \mathrm{si}(v)\}$. *The last inequality is transformed to the equality if and only if there is a label v' such that $v' \unlhd u$, $v' \unlhd v$, and* $\mathrm{si}(v') = \mathrm{si}(u)$ *or* $\mathrm{si}(v') = \mathrm{si}(v)$.

(3) *The equality* $\mathrm{si}(p, q) = 0$ *holds if and only if there is no realization of p semi-isolating realizations of q.*

(4) *The equality* si$(p,q) = 1$ *holds if and only if there is a* $(p \to q)$*-formula witnessing that a realization of* p *semi-isolates a realization of* q *and each such a* $(p \to q)$*-formula* $\varphi(x,y)$ *is equivalent to a disjunction of formulas* $\varphi_i(x,y)$ *such that each formula* $\varphi_i(a,y)$ *is isolating, where* $\models p(a)$.

Proof is obvious. □

Proposition 3.5. *For any nonempty family* $R \subseteq S^1(\varnothing)$ *the following assertions are satisfied.*

(1) si$(R) \geq 1$.

(2) *The family* R *is* si*-minimal if and only if for any types* $p, q \in R$ *each* $(p \to q)$*-formula* $\varphi(x,y)$ *is equivalent to a disjunction of formulas* $\varphi_i(x,y)$ *such that each formula* $\varphi_i(a,y)$ *is isolating, where* $\models p(a)$.

Proof. (1) is implied by the inequality si$(p) \geq 1$ for any type $p \in S^1(\varnothing)$ since the formula $(a \approx y)$ witnesses that a semi-isolates itself, where $\models p(a)$. (2) is an obvious corollary of (1) and Proposition 3.4, (4). □

Remark 3.6. Since for a strongly minimal theory T the set of solutions for any formula $\varphi(a,y)$ is finite or cofinite, any semi-isolating formula $\psi(a,y)$ is represented as a finite disjunction of some isolating formulas $\psi_i(a,y)$ or as a negation of a finite disjunction of isolating formulas $\psi_i(a,y)$. If $\psi(a,y) \vdash p(y)$ and $p(y)$ is a non-principal type then the representation of $\psi(a,y)$ is possible only as a finite disjunction of isolating formulas. It means that si$(p) = 1$. If $p(y)$ is a principal type and there are finitely many pairwise non-equivalent isolating formulas $\psi(a,y)$ with $\models p(a)$ and $\psi(a,y) \vdash p(y)$ then si$(p) = 1$, too. If there are infinitely many these pairwise non-equivalent isolating formulas $\psi(a,y)$ then si$(p) = 2$.

Definition 3.7. Let α be a positive ordinal, u_1 and u_2 be labels in $\rho_{\nu(p,q)}$ such that si$(u_1) =$ si$(u_2) = \alpha$. The labels u_1 and u_2 are *α-almost identic* or *$\sim_\alpha$-equivalent* (denoted by $u_1 \sim_\alpha u_2$) if si$(u_1 \div u_2) < \alpha$, where $u_1 \div u_2 \rightleftharpoons (u_1 \wedge \neg u_2) \vee (\neg u_1 \wedge u_2)$.

Proposition 3.8. *The relation* $\sim_\alpha$ *is an equivalence relation for any set of labels in* $\rho_{\nu(p,q)}$ *having the* si*-rank* α.

Proof. Clearly the relation $\sim_\alpha$ is reflexive and symmetric. For checking transitivity we assume that $u_1 \sim_\alpha u_2$ and $u_2 \sim_\alpha u_3$. Since $(u_1 \wedge \neg u_2 \wedge u_3) \trianglelefteq (u_1 \wedge \neg u_2) \trianglelefteq (u_1 \div u_2)$ we have

$$\mathrm{si}(u_1 \wedge \neg u_2 \wedge u_3) \leq \mathrm{si}(u_1 \div u_2) < \alpha.$$

As $u_1 \wedge u_3 = (u_1 \wedge u_2 \wedge u_3) \vee (u_1 \wedge \neg u_2 \wedge u_3)$ and si$(u_1 \wedge \neg u_2 \wedge u_3) < \alpha$, for $u_1 \sim_\alpha u_3$, it is enough to prove that si$(u_1 \wedge u_2 \wedge u_3) = \alpha$. Suppose on contrary that si$(u_1 \wedge u_2 \wedge u_3) < \alpha$. Then si$(u_1 \wedge u_2) = \alpha$ and

$$u_1 \wedge u_2 = (u_1 \wedge u_2 \wedge u_3) \vee (u_1 \wedge u_2 \wedge \neg u_3)$$

imply si$(u_1 \wedge u_2 \wedge \neg u_3) = \alpha$. But $(u_1 \wedge u_2 \wedge \neg u_3) \trianglelefteq (u_2 \wedge \neg u_3) \trianglelefteq (u_2 \div u_3)$, and si$(u_2 \div u_3) < \alpha$ gives si$(u_1 \wedge u_2 \wedge \neg u_3) < \alpha$. The obtained contradiction means that $u_1 \sim_\alpha u_3$. □

By the definition, for any label $u \in \rho_{\nu(p,q)}$ having the si-rank α, there is a greatest number $n \in \omega \setminus \{0\}$ of pairwise inconsistent (or, that equivalent, of pairwise non-$\sim_\alpha$-equivalent) labels $u_1, \ldots, u_n$ such that $u_i \trianglelefteq u$ and si$(u_i) = \alpha$, $i = 1, \ldots, n$. This number n is called the *degree of semi-isolation* or the si*-degree* of the label u and it is denoted by deg(p,u,q) or by deg(u). We have si$(\varnothing) = 0$ and put deg$(\varnothing) \rightleftharpoons 1$.

Proposition 3.9. (1) *If* $u \in \rho_{\nu(p,q)}$ *and* $\mathrm{si}(u) = \alpha$ *then* $\deg(u)$ *is equal to the number of pairwise inconsistent labels* $u_1, \ldots, u_n \in \rho_{\nu(p,q)}$ *having the* si*-rank* α*, the* si*-degree* 1*, and such that* $u = u_1 \vee \ldots \vee u_n$*.*

(2) *If* $u, v \in \rho_{\nu(p,q)}$*,* $\mathrm{si}(u) = \mathrm{si}(v)$*, and* $u \trianglelefteq v$ *then* $\deg(u) \le \deg(v)$*.*

(3) *If* $u, v \in \rho_{\nu(p,q)}$ *and* $\mathrm{si}(u) = \mathrm{si}(v)$ *then*

$$\deg(u \vee v) \le \deg(u) + \deg(v).$$

The equality in this inequality holds if and only if $\mathrm{si}(u \wedge v) < \mathrm{si}(u)$*. If* $\mathrm{si}(u \wedge v) = \mathrm{si}(u)$ *then*

$$\deg(u \vee v) = \deg(u) + \deg(v) - \deg(u \wedge v).$$

(4) *If* $u \in \rho_{\nu(p,q)}$ *is a label for an isolating formula, i. e.,* u *is an atom, then* $\mathrm{si}(u) = 1$ *and* $\deg(u) = 1$*.*

(5) *If for a label* $u \in \rho_{\nu(p,q)}$*,* $\mathrm{si}(u) = 1$ *and* $\deg(u) = 1$*, then* u *is not neutral.*

(6). *If* $u \in \rho_{\nu(p,q)}$ *and* $\mathrm{si}(u) = 1$ *then* $\deg(u)$ *is equal to the number of pairwise inconsistent labels* $u_1, \ldots, u_n \in \rho_{\nu(p,q)}$ *for isolating formulas such that* $u = u_1 \vee \ldots \vee u_n$*.*

Proof is obvious. □

If there is a label $u \in \rho_{\nu(p,q)}$ with $\mathrm{si}(p,q) = \mathrm{si}(u)$ then the *degree of semi-isolation* or the si-*degree* $\deg(p,q)$ of the pair (p,q) is

$$\sup\{\deg(u) \mid u \in \rho_{\nu(p,q)}, \mathrm{si}(p,q) = \mathrm{si}(u)\},$$

$\deg(p) \rightleftharpoons \deg(p,p)$.

If for a nonempty family R of 1-types there is a label $u \in \rho_{\nu(p,q)}$, $p, q \in R$, with $\mathrm{si}(R) = \mathrm{si}(u)$ then the *degree of semi-isolation* or the si-*degree* $\deg(R)$ of R is

$$\sup\{\deg(u) \mid u \in \rho_{\nu(R)}, \mathrm{si}(R) = \mathrm{si}(u)\}.$$

Clearly, if $\deg(p,q)$ or $\deg(R)$ exist then these values are positive natural numbers or equal ω.

For an ordinal α, a natural number $n \ge 1$, and a set $X \in \{U, U \cup \{\varnothing\}\}$ we put

$$X \restriction (\alpha, n) \rightleftharpoons \{u \in X \mid \mathrm{si}(u) \le \alpha \text{ and if } \mathrm{si}(u) = \alpha \text{ then } \deg(u) < n\},$$

$$X \restriction (\alpha, \omega) \rightleftharpoons X \restriction \alpha \rightleftharpoons \bigcup_{n \ge 1} X \restriction (\alpha, n).$$

Clearly, if $\alpha = \beta + 1$ then $X \restriction (\alpha, 1) = X \restriction \beta$, and if α is a limit ordinal then $X \restriction (\alpha, 1) = \bigcup\limits_{\beta < \alpha} X \restriction \beta$.

For ordinals α, β, where $\beta \in (\omega + 1) \setminus \{0\}$, and for the algebra $\mathfrak{A}$ of distributions of binary semi-isolating formulas of a theory T as well as for expansions and restrictions $\mathfrak{A}'$ of $\mathfrak{A}$, defined in the previous sections, we denote by $\mathfrak{A} \restriction (\alpha, \beta)$ and $\mathfrak{A}' \restriction (\alpha, \beta)$ as well as by $\mathfrak{A}_{\alpha,\beta}$ and $\mathfrak{A}'_{\alpha,\beta}$ the restrictions of these algebras to the set $(U \cup \{\varnothing\}) \restriction (\alpha, \beta)$. If $\beta = \omega$, these restrictions are denoted by $\mathfrak{A} \restriction \alpha$, $\mathfrak{A}' \restriction \alpha$, $\mathfrak{A}_\alpha$, and $\mathfrak{A}'_\alpha$. The restrictions are called the (α, β)*-restrictions* and the α*-restrictions* respectively.

Since the si-rank of each label is positive, non-trivial restrictions (i. e., with nonempty sets of used labels) are only the restrictions of algebras with $\alpha > 0$. If $\mathrm{si}(S^1(\varnothing)) = \alpha_0$ then, taking into consideration the inequality $\alpha > 0$, all essential (i. e., reflecting links of sets of labels of semi-isolating formulas with respect to their si-ranks) restrictions of these algebras are formed only for $0 < \alpha < \alpha_0$.

By Proposition 3.9, we obtain

Proposition 3.10. *The algebra of distributions of binary isolating formulas of theory T coincides with the algebra $\mathfrak{A}\upharpoonright(1,2)$. The algebra $\mathfrak{A}\upharpoonright 1$ consists of labels being disjunctions of labels of isolating formulas.*

4. Monoid of distributions of binary semi-isolating formulas on a set of realizations of a type

Consider a complete theory T, a type $p(x)\in S(T)$, a regular labelling function $\nu(p)$: $\mathrm{SICF}(p)/\mathrm{SICE}(p)\to U$, and a family of sets $\mathrm{SI}_p(u_1,\ldots,u_k)$ of labels of binary semi-isolating formulas, $u_1,\ldots,u_k\in\rho_{\nu(p)}$, $k\in\omega$.

Below we show some basic properties for sets

$$\lceil u_1,\ldots,u_k\rceil \rightleftharpoons \mathrm{SI}_p(u_1,\ldots,u_k).$$

Proposition 4.1. (Associativity). *For any $u_1,u_2,u_3\in\rho_{\nu(p)}$, the following equalities hold:*

$$\lceil\lceil u_1,u_2\rceil,u_3\rceil=\lceil u_1,u_2,u_3\rceil=\lceil u_1,\lceil u_2,u_3\rceil\rceil.$$

Proof of inclusions $\lceil\lceil u_1,u_2\rceil,u_3\rceil\subseteq\lceil u_1,u_2,u_3\rceil$ and $\lceil u_1,\lceil u_2,u_3\rceil\rceil\subseteq\lceil u_1,u_2,u_3\rceil$ is identical to the proof of [17, Proposition 3.1, 4].

The reverse inclusions are satisfied since, taking labels v_1 and v_2 for the formulas $\theta_{u_1,u_2}(x,y)$ and $\theta_{u_2,u_3}(x,y)$, we obtain, for $\models p(a)$, that the formulas $\theta_{v_1,u_3}(a,y)$, $\theta_{u_1,u_2,u_3}(a,y)$, and $\theta_{u_1,v_2}(a,y)$ are pairwise equivalent, i. e.,

$$\lceil v_1,u_3\rceil=\lceil u_1,u_2,u_3\rceil=\lceil u_1,v_2\rceil.$$

□

In view of associativity, using the induction on the number of brackets, we prove that all operations $\lceil\cdot,\cdot,\ldots,\cdot\rceil$ acting on sets in $\mathcal{P}(\rho_{\nu(p)})\setminus\{\varnothing\}$ are generated by the binary operation $\lceil\cdot,\cdot\rceil$ on the set $\mathcal{P}(\rho_{\nu(p)})\setminus\{\varnothing\}$ and the values $\lceil X_1,X_2,\ldots,X_k\rceil$, $X_1,X_2,\ldots,X_k\subseteq\rho_{\nu(p)}$, do not depend on the sequence of adding of brackets for

$$X_{i,i+1,\ldots,i+m+n}\rightleftharpoons\lceil X_{i,i+1,\ldots,i+m},X_{i+m+1,i+m+2,\ldots,i+m+n}\rceil,$$

where $X_{1,2,\ldots,k}=\lceil X_1,X_2,\ldots,X_k\rceil$.

Thus the structure $\mathfrak{SI}_{\nu(p)}\rightleftharpoons\langle\mathcal{P}(\rho_{\nu(p)})\setminus\{\varnothing\};\lceil\cdot,\cdot\rceil\rangle$ is a semigroup admitting the representation of all operations $\lceil\cdot,\cdot,\ldots,\cdot\rceil$ by terms of the language $\lceil\cdot,\cdot\rceil$. Below the operation $\lceil\cdot,\cdot\rceil$ will be denoted also by $\cdot$ and we shall use the record uv instead of $u\cdot v$.

Since by the choice of the label 0 for the formula $(x\approx y)$ the equalities $X\cdot\{0\}=X$ and $\{0\}\cdot X=X$ are true for any $X\subseteq\rho_{\nu(p)}$, the semigroup $\mathfrak{SI}_{\nu(p)}$ has the unit $\{0\}$, and it is a monoid. We have

$$Y\cdot Z=\bigcup\{yz\mid y\in Y,z\in Z\}$$

for any sets $Y,Z\in\mathcal{P}(\rho_{\nu(p)})\setminus\{\varnothing\}$ in this structure.

Thus the following proposition holds.

Proposition 4.2. *For any complete theory T, any type $p\in S^1(T)$, and the regular labelling function $\nu(p)$, any operation $\mathrm{SI}_p(\cdot,\cdot,\ldots,\cdot)$ on the set $\mathcal{P}(\rho_{\nu(p)})\setminus\{\varnothing\}$ interpretable by a term of the monoid $\mathfrak{SI}_{\nu(p)}$.*

The monoid $\mathfrak{SI}_{\nu(p)}$ is called the *monoid of binary semi-isolating formulas over the labelling function* $\nu(p)$ or the $\mathrm{SI}_{\nu(p)}$*-monoid*.

By Propositions 1.4 and 4.1, we obtain

Proposition 4.3. *For any complete theory T, any type $p \in S^1(T)$, and the regular labelling function $\nu(p)$, the restriction $\mathfrak{SI}^{\leq 0}_{\nu(p)}$ (respectively $\mathfrak{SI}^{\geq 0}_{\nu(p)}$, $\mathfrak{SI}^{\geq 0,\mathrm{neu}}_{\nu(p)}$) of the monoid $\mathfrak{SI}_{\nu(p)}$ to the set $U^{\leq 0}$ ($U^{\geq 0}$, $U^{\geq 0} \cup U'$) is a submonoid of $\mathfrak{SI}_{\nu(p)}$.*

By Proposition 3.10, the $(1,2)$-restriction of the monoid $\mathfrak{SI}_{\nu(p)}$ coincides with the $I_{\nu(p)}$-groupoid $\mathfrak{P}_{\nu(p)}$ [17]. Besides, the $(1,2)$-restrictions of monoids $\mathfrak{SI}^{\leq 0}_{\nu(p)}$ and $\mathfrak{SI}^{\geq 0}_{\nu(p)}$ equal respectively to the groupoid $\mathfrak{P}^{\leq 0}_{\nu(p)}$ and the monoid $\mathfrak{P}^{\geq 0}_{\nu(p)}$.

5. α-deterministic and almost α-deterministic $\mathrm{SI}_{\nu(p)}$-monoids

In the following definition, we generalize the notions of deterministic and almost deterministic structure $\mathfrak{P}_{\nu(p)}$ proposed in [17].

Definition 5.1. Let U_0 be a subalphabet of the alphabet U, α be a positive ordinal, and $n \geq 1$ be a natural number. We put

$$\rho_{\nu(p),\alpha,n} \rightleftharpoons \{u \in \rho_{\nu(p)} \mid \mathrm{si}(u) \leq \alpha, \deg(u) < n \text{ for } \mathrm{si}(u) = \alpha\},$$

$$\rho_{\nu(p),\alpha} \rightleftharpoons \bigcup_{n\in\omega} \rho_{\nu(p),\alpha,n}.$$

The partial subalgebra $\mathfrak{SI}_{\nu(p)} \upharpoonright U_0$ of the monoid $\mathfrak{SI}_{\nu(p)}$ is called (α,n)*-deterministic* if for any labels $u_1, u_2 \in \rho_{\nu(p),\alpha,n} \cap U_0$, the set $\lceil u_1, u_2\rceil \cap U_0$ consists of labels having the si-ranks $\leq \alpha$ and contains less than n pairwise non-$\sim_\alpha$-equivalent labels of si-rank α.

The partial subalgebra $\mathfrak{SI}_{\nu(p)} \upharpoonright U_0$ of the monoid $\mathfrak{SI}_{\nu(p)}$ is called α*-deterministic* if $\mathfrak{SI}_{\nu(p)} \upharpoonright U_0$ is $(\alpha, 2)$-deterministic.

The partial subalgebra $\mathfrak{SI}_{\nu(p)} \upharpoonright U_0$ of the monoid $\mathfrak{SI}_{\nu(p)}$ is called *almost* α*-deterministic* or (α,ω)*-deterministic* if for any labels $u_1, u_2 \in \rho_{\nu(p),\alpha} \cap U_0$, the set $\lceil u_1, u_2\rceil \cap U_0$ consists of labels having the si-ranks $\leq \alpha$ and contains finitely many pairwise non-$\sim_\alpha$-equivalent labels of si-rank α.

By the definition, each (α,ω)-deterministic structure $\mathfrak{SI}_{\nu(p)} \upharpoonright U_0$ is a union of its (α,n)-deterministic substructures, $n \geq 1$. So each α-deterministic structure $\mathfrak{SI}_{\nu(p)} \upharpoonright U_0$ is almost α-deterministic.

If $U_0 = U$ we shall not point out restrictions to the set U_0 for considering structures.

Below we show some basic properties of (almost) α-deterministic partial algebras $\mathfrak{SI}_{\nu(p)} \upharpoonright U_0$.

Proposition 5.2 (Monotony). *If a structure $\mathfrak{SI}_{\nu(p)} \upharpoonright U_0$ is (almost) α-deterministic and β is a positive ordinal then the structure $(\mathfrak{SI}_{\nu(p)} \upharpoonright U_0) \upharpoonright \beta$ is also (almost) α-deterministic.*

Proof is obvious. □

Proposition 5.3. *For any monoid $\mathfrak{SI}_{\nu(p)}$ and ordinals α, β, where $\alpha, \beta > 0$, $\beta \in \omega + 1$, the following conditions are equivalent:*

(1) *the monoid $\mathfrak{SI}_{\nu(p)}$ is (α, β)-deterministic;*

(2) $\mathrm{si}(u_1 \circ u_2) \leq \alpha$ *for any labels* $u_1, u_2 \in \rho_{\nu(p),\alpha,\beta}$ *and if* $\mathrm{si}(u_1 \circ u_2) = \alpha$ *then* $\deg(u_1 \circ u_2) < \beta$.

Proof. The implication (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1). Consider arbitrary labels $u_1, u_2 \in \rho_{\nu(p),\alpha,\beta}$. Since, by the hypothesis, $\mathrm{si}(u_1 \circ u_2) \leq \alpha$ and $v \trianglelefteq (u_1 \circ u_2)$ for any label $v \in \lceil u_1, u_2 \rceil$, so $\lceil u_1, u_2 \rceil$ consists of labels of si-ranks $\leq \alpha$, and if $\mathrm{si}(v) = \mathrm{si}(u_1 \circ u_2) = \alpha$ then $\deg(v) \leq \deg(u_1 \circ u_2) < \beta$. Thus, the monoid $\mathfrak{S}\mathfrak{I}_{\nu(p)}$ is (α, β)-deterministic. □

Proposition 5.3 immediately implies

Corollary 5.4. *For any monoid* $\mathfrak{S}\mathfrak{I}_{\nu(p)}$ *and a positive ordinal* α *the following conditions are equivalent:*

(1) *the monoid* $\mathfrak{S}\mathfrak{I}_{\nu(p)}$ *is almost* α*-deterministic;*

(2) $\mathrm{si}(u_1 \circ u_2) \leq \alpha$ *for any labels* $u_1, u_2 \in \rho_{\nu(p),\alpha}$.

Corollary 5.5. *If* $\mathrm{si}(p)$ *is an ordinal then the monoid* $\mathfrak{S}\mathfrak{I}_{\nu(p)}$ *is almost* $\mathrm{si}(p)$*-deterministic.*

Proposition 5.6. *If a monoid* $\mathfrak{S}\mathfrak{I}_{\nu(p)}$ *is* (α, β)*-deterministic then the structure* $\mathfrak{S}\mathfrak{I}_{\nu(p),\alpha} \rightleftharpoons \mathfrak{S}\mathfrak{I}_{\nu(p)} \upharpoonright \alpha$ *is also an* (α, β)*-deterministic monoid.*

Proof. Since for any α-restriction the associativity, the presence of unit $\{0\}$, and the (α, β)-determinacy is preserved, it is enough to note that for any labels u_1 and u_2 in $\mathfrak{S}\mathfrak{I}_{\nu(p),\alpha,\beta}$ there is a label v in $\mathfrak{S}\mathfrak{I}_{\nu(p),\alpha,\beta}$ belonging $\lceil u_1, u_2 \rceil$. We can take $u_1 \circ u_2$ for v since, by the hypothesis, $\mathrm{si}(u_1 \circ u_2) \leq \alpha$ and if $\mathrm{si}(u_1 \circ u_2) = \alpha$ then $\deg(u_1 \circ u_2) < \beta$. □

Proposition 5.7. *If* $\mathrm{si}(p)$ *is an ordinal then the monoid* $\mathfrak{S}\mathfrak{I}_{\nu(p)}$ *is* $\mathrm{si}(p)$*-deterministic if and only if the value* $\deg(p)$ *is not defined or equals* 1.

Proof. If $\deg(p)$ is not defined the ordinal $\alpha = \mathrm{si}(p)$ is limit and can not be achieved by labels in $\rho_{\nu(p)}$. In particular, for any $u_1, u_2 \in \rho_{\nu(p)}$ the set $\lceil u_1, u_2 \rceil$ does not contain labels of si-rank α. If $\deg(p) \geq 2$ then there are non-$\sim_\alpha$-equivalent labels $u_1, u_2 \in \rho_{\nu(p)}$ of si-rank α. Then $\lceil u_1 \vee u_2, 0 \rceil$ contains the labels u_1 and u_2, whence the monoid $\mathfrak{S}\mathfrak{I}_{\nu(p)}$ is not α-deterministic. If $\deg(p) = 1$ then there is unique, up to $\sim_\alpha$-equivalence, label in $\rho_{\nu(p)}$ having the si-rank α. Since such a label is unique, the monoid $\mathfrak{S}\mathfrak{I}_{\nu(p)}$ is α-deterministic. □

Proposition 5.8. *The structure* $\mathfrak{P}_{\nu(p)}$ *is* (*almost*) *deterministic if and only if the structure* $\mathfrak{S}\mathfrak{I}_{\nu(p),1,2}$ *is* (*almost*) 1*-deterministic.*

Proof follows by the equality $\mathfrak{S}\mathfrak{I}_{\nu(p),1,2} = \mathfrak{P}_{\nu(p)}$. □

Proposition 5.9. *Let* $p(x)$ *be a complete* 1*-type of a theory* T, $\nu(p)$ *be a regular labelling function, and* $\mathrm{si}(p) < \omega$. *The following conditions are equivalent:*

(1) *the monoid* $\mathfrak{S}\mathfrak{I}_{\nu(p)}$ *is* $(1, n)$*-deterministic for some* $n \in \omega$*;*

(2) *the set* $\rho_{\nu(p)}$ *is finite;*

(3) *the set* $\rho_{\nu(p),1}$ *finite;*

(4) *the set* $\rho_{\nu(p),1,2}$ (*consisting of all atoms* $u \in \rho_{\nu(p)}$) *is finite.*

Proof. If $\mathrm{si}(p) > 1$ then, by $\mathrm{si}(p) < \omega$, the set $\rho_{\nu(p),1}$ is infinite and so the set $\rho_{\nu(p)}$ is also infinite. Since each label in $\rho_{\nu(p),1}$ is a disjunction of labels in $\rho_{\nu(p),1,2}$ and for any labels $u_1, \ldots, u_n \in \rho_{\nu(p),1}$ the label $u_1 \vee \ldots \vee u_n$ belongs also to $\rho_{\nu(p),1}$, the set $\rho_{\nu(p),1,2}$ is infinite and the monoid $\mathfrak{S}\mathfrak{I}_{\nu(p)}$ is not $(1, n)$-deterministic for $n \in \omega$. Thus, none of the conditions (1)–(4) is not satisfied.

If $\mathrm{si}(p) = 1$ then each label in $\rho_{\nu(p)}$ has the si-rank 1 and is represented as a disjunction of labels in $\rho_{\nu(p),1,2}$. Thus, the conditions (2)–(4) are equivalent. If the set $\rho_{\nu(p),1,2}$ contains $m \in \omega$ labels then there are $2^m - 1$ labels forming the set $\rho_{\nu(p)}$. Hence, the monoid $\mathfrak{S}\mathfrak{I}_{\nu(p)}$ is $(1, 2^m - 1)$-deterministic. If the set $\rho_{\nu(p),1,2}$ is infinite then, for pairwise distinct labels $u_1, \ldots, u_m \in \rho_{\nu(p),1,2}$, the set $\lceil u_1 \vee \ldots u_m, 0\rceil$ contains $2^m - 1$ labels and, since m is not bounded, the monoid $\mathfrak{S}\mathfrak{I}_{\nu(p)}$ is not $(1, n)$-deterministic for any n. Thus, the condition (1) is equivalent to each of the conditions (2)–(4). □

Proposition 5.9 and [17, Corollary 7.4] imply

Corollary 5.10. *Let $p(x)$ be a complete 1-type of a theory T, $\nu(p)$ be a regular labelling function,* $\mathrm{si}(p) < \omega$, *and $\mathfrak{S}\mathfrak{I}_{\nu(p)}$ is a $(1, n)$-deterministic monoid, for some $n \in \omega$, having a negative label. Then the groupoid $\mathfrak{S}\mathfrak{I}_{\nu(p),1,2}$ generates the strict order property.*

Definition 5.11 [18, 19]. Let $p(x)$ be a 1-type in $S(T)$. A type $q(x_1, \ldots, x_n) \in S(T)$ is called a *(n, p)-type* if $q(x_1, \ldots, x_n) \supseteq \bigcup\limits_{i=1}^{n} p(x_i)$. The set of all (n, p)-types of T is denoted by $S_{n,p}(T)$ and elements of the set $S_p(T) \rightleftharpoons \bigcup\limits_{n\in\omega\setminus\{0\}} S_{n,p}(T)$ are called *p-types*.

A type $q(\bar{y})$ in $S_p(T)$ is called *p-principal* if there is a formula $\varphi(\bar{y}) \in q(\bar{y})$ such that $\cup\{p(y_i) \mid y_i \in \bar{y}\} \cup \{\varphi(\bar{y})\} \vdash q(\bar{y})$.

Lemma 5.12 [18, 19]. *For any type p and a natural number $n \geq 1$ the following conditions are equivalent:*

(1) *the set of (n, p)-types with a tuple $(x_1, \ldots, x_n)$ of free variables is infinite;*
(2) *there is a non-p-principal (n, p)-type.*

Proposition 5.9 and Lemma 5.12 imply

Corollary 5.13. *If $p(x)$ is a complete type of a theory T, $\nu(p)$ is a regular labelling function, and all $(2, p)$-types are p-principal, then the monoid $\mathfrak{S}\mathfrak{I}_{\nu(p)}$ is $(1, n)$-deterministic for some $n \in \omega$.*

By Corollaries 5.10 and 5.13, we obtain

Corollary 5.14. *Let $p(x)$ be a complete type of a theory T, $\nu(p)$ be a regular labelling function, $\rho_{\nu(p)} \cap U^- \neq \varnothing$, and all $(2, p)$-types be p-principal. Then the groupoid $\mathfrak{S}\mathfrak{I}_{\nu(p),1,2}$ generates the strict order property.*

For a type $p(x)$ and a positive ordinal α, we denote by $\mathrm{SI}_{p,\alpha}$ (in a model $\mathcal{M}$ of T) the relation of semi-isolation (over $\varnothing$) on a set of realizations of p restricted to the set of formulas of si-rank $\leq \alpha$:

$$\mathrm{SI}_{p,\alpha} \rightleftharpoons \{(a, b) \mid \mathcal{M} \models p(a) \wedge p(b) \text{ and } a \text{ semi-isolates } b$$
$$\text{by a formula } \theta_u(x, y) \text{ with a si-rank } \leq \alpha\}.$$

Clearly, $I_p = \mathrm{SI}_{p,1}$ for any type $p \in S^1(\varnothing)$. Seeing this equality and $\mathfrak{S}\mathfrak{I}_{\nu(p),1,2} = \mathfrak{P}_{\nu(p)}$ the following proposition generalizes Proposition 4.3 in [17].

Proposition 5.15. *Let $p(x)$ be a complete 1-type of a theory T, $\nu(p)$ be a regular labelling function, and α be a positive ordinal. The following conditions are equivalent:*

(1) *the relation* $\mathrm{SI}_{p,\alpha}$ *(on a set of realizations of* p *in any model* $\mathcal{M} \models T$*) is transitive;*

(2) *the structure* $\mathfrak{S}\mathfrak{I}_{\nu(p),\alpha}$ *is an almost* α*-deterministic monoid.*

Proof. Let a, b, and c be realizations of p such that $(a,b) \in \mathrm{SI}_{p,\alpha}$ and $(b,c) \in \mathrm{SI}_{p,\alpha}$ by semi-isolating formulas $\theta_{u_1}(a,y)$ and $\theta_{u_2}(b,y)$ respectively. If the structure $\mathfrak{S}\mathfrak{I}_{\nu(p),\alpha}$ is an almost α-deterministic monoid then $\mathrm{si}(u_1 \circ u_2) \leq \alpha$ and the pair (a,c) belongs to $\mathrm{SI}_{p,\alpha}$ by the semi-isolating formula $\theta_{u_1,u_2}(x,y)$. Since elements a, b, and c are arbitrary we have (2) $\Rightarrow$ (1).

Assume now that for some $u_1, u_2 \in \rho_{\nu(p),\alpha}$ the set $\mathrm{SI}_p(u_1,u_2)$ contains a label u such that $\mathrm{si}(u) > \alpha$. Then by compactness the set

$$q(a,y) \rightleftharpoons \{\theta_{u_1,u_2}(a,y)\} \cup \{\neg\theta_v(a,y) \mid v \in \mathrm{SI}_p(u_1,u_2), \mathrm{si}(v) \leq \alpha\}$$

is consistent, where $\models p(a)$. Consider realizations b and c of p such that $\models \theta_{u_1}(a,b) \wedge \theta_{u_2}(b,c)$ and $\models q(a,c)$. We have $(a,b) \in \mathrm{SI}_{p,\alpha}$ and $(b,c) \in \mathrm{SI}_{p,\alpha}$ but $(a,c) \notin \mathrm{SI}_{p,\alpha}$ by the construction of q. Thus, the relation $\mathrm{SI}_{p,\alpha}$ is not transitive and the implication (1) $\Rightarrow$ (2) holds. □

Note that for any ordinal $\alpha > 0$, there are no (p,θ_u,p)-edges, linking distinct realizations of p and satisfying the conditions $u > 0$, $\mathrm{si}(u) \leq \alpha$, and $\mathrm{si}(u^{-1}) \leq \alpha$, if and only if the relation $\mathrm{SI}_{p,\alpha}$ is antisymmetric. Since $\mathrm{SI}_{p,\alpha}$ is reflexive, the definition of $\nu(p)$ and Propositions 1.4, 5.15 imply

Corollary 5.16. *Let* $p(x)$ *be a complete* 1*-type of a theory* T*,* $\nu(p)$ *be a regular labelling function, and* α *be a positive ordinal. The following conditions are equivalent:*

(1) *the relation* $\mathrm{SI}_{p,\alpha}$ *is a partial order on a set of realizations of* p *in any model* $\mathcal{M} \models T$*;*

(2) *the structure* $\mathfrak{S}\mathfrak{I}_{\nu(p),\alpha}$ *is an almost* α*-deterministic monoid and* $\rho_{\nu(p),\alpha} \subseteq U^{\leq 0}$.

This partial order $\mathrm{SI}_{\nu(p),\alpha}$ *is identical if and only if* $\rho_{\nu(p),\alpha} = \{0\}$*. If* $\mathrm{SI}_{p,\alpha}$ *is not identical, it has infinite chains.*

Propositions 1.4 and 5.15 also imply

Corollary 5.17. *Let* $p(x)$ *be a complete* 1*-type of a theory* T*,* $\nu(p)$ *be a regular labelling function, and* α *be a positive ordinal. The following conditions are equivalent:*

(1) *the relation* $\mathrm{SI}_{p,\alpha}$ *is an equivalence relation on the set of realizations of* p *in any model* $\mathcal{M} \models T$*;*

(2) *the structure* $\mathfrak{S}\mathfrak{I}_{\nu(p),\alpha}$ *is an almost* α*-deterministic monoid and consists of labels in* $U^{\geq 0}$.

Recall [17] that an element $u \in \rho_{\nu(p)}$ is called (*almost*) *deterministic* if for any/some realization a of p the formula $\theta_u(a,y)$ has unique solution (has finitely many solutions).

Since each semi-isolating formula $\theta_u(a,y)$ with finitely many solutions is equivalent to a disjunction of isolating formulas $\theta_{u_i}(a,y)$, each almost deterministic element has the si-rank 1 and so belongs to the set of labels in the structure $\mathfrak{S}\mathfrak{I}_{\nu(p),1,n+1}$, where n is the number of solutions for $\theta_u(a,y)$. In particular, each deterministic element belongs to the set of labels in the structure $\mathfrak{S}\mathfrak{I}_{\nu(p),1,2}$.

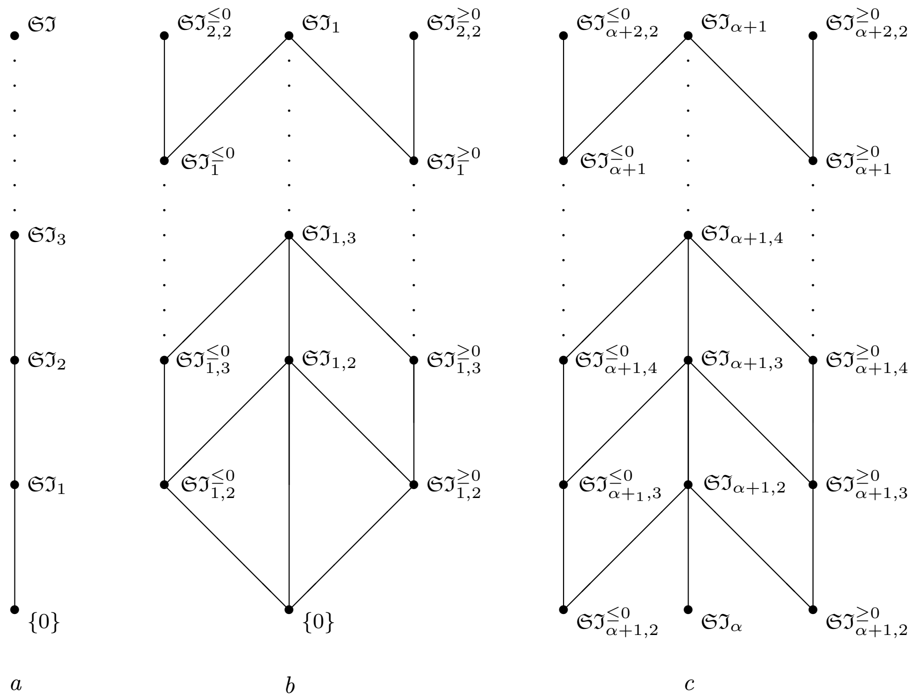

Fig. 1

It is shown in [17, Proposition 4.7] that if elements u and v are (almost) deterministic then each element v' in $u \cdot v$ is also (almost) deterministic. Hence, the si-rank 1 is preserved for compositions $u \circ v$ of (almost) deterministic elements u and v. Moreover, the si-degree 1 is preserved for compositions of deterministic elements.

In Figure 1, the fragments of Hasse diagram are presented illustrating the links of the structure $\mathfrak{SI} \rightleftharpoons \mathfrak{SI}_{\nu(p)}$ with structures above, being restrictions of $\mathfrak{SI}$ to subalphabets of U. Here the superscripts $\cdot^{\leq 0}$ and $\cdot^{\geq 0}$ point out on restrictions of $\mathfrak{SI}$ to the sets $U^{\leq 0}$ and $U^{\geq 0}$ respectively, and the subscripts to the upper estimates for si-ranks and si-degrees of labels. In Figure 1, a, a hierarchy of structures $\mathfrak{SI}_\alpha$, $\alpha \leq \mathrm{si}(p)$, is depicted starting with the trivial substructure; in Figure 1, b, links between substructures of $\mathfrak{SI}_{\nu(p),1}$ are presented; in Figure 1, c, links between substructures of $\mathfrak{SI}_{\alpha+1}$ for $1 \leq \alpha < \mathrm{si}(p)$ are shown. For a limit ordinal $\beta \leq \mathrm{si}(p)$, the Hasse diagram for substructures of $\mathfrak{SI}_\beta$ is obtained by union of presented diagrams for $\alpha < \beta$. If an ordinal $\beta \leq \mathrm{si}(p)$ is not limit, the Hasse diagram corresponds to the union of presented diagrams for $\alpha < \beta$ with the removal of structures $\mathfrak{SI}^{\leq 0}_{\beta+1,2}$ and $\mathfrak{SI}^{\geq 0}_{\beta+1,2}$.

6. POSTC-monoids

In this Section, we shall consider both the monoids $\mathfrak{SI}_{\nu(p)}$ and their expansions (with the addition of empty set to the universe such that $X \cdot \varnothing = \varnothing \cdot X = \varnothing$ for $X \in \mathcal{P}(\rho_{\nu(p)})$) by operations and relations of POSTC-algebras containing these monoids. These expansions

$$\mathfrak{M}_{\nu(p)} \rightleftharpoons \langle \mathcal{P}(\rho_{\nu(p)});\ \cdot, \trianglelefteq, \vee, \wedge, (\cdot\ \wedge\ \neg\,\cdot), \circ \rangle$$

are called *preordered monoids with relative set-theoretic operations and compositions* over regular labelling functions $\nu(p)$, or briefly POSTC$_{\nu(p)}$*-monoids.*

We collect basic structural properties of POSTC$_{\nu(p)}$-monoids and show that any expanded monoid $\mathfrak{SI}$, satisfying the following list of properties, coincides with some POSTC$_{\nu(p)}$-monoid $\mathfrak{M}_{\nu(p)}$.

Let $U = U^- \dot\cup \{0\} \dot\cup U^+ \dot\cup U'$ be an alphabet consisting of a set U^- of *negative elements*, a set U^+ of *positive elements*, a set U' of *neutral elements*, and zero 0. As above, we write $u < 0$ for any element $u \in U^-$, $u > 0$ for any element $u \in U^+$, and $u \cdot v$ instead of $\{u\} \cdot \{v\}$ considering an operation $\cdot$ on the set $\mathcal{P}(U)$; $U^{\leq 0} \rightleftharpoons U^- \cup \{0\}$, $U^{\geq 0} \rightleftharpoons U^+ \cup \{0\}$.

A structure $\mathfrak{M} = \langle \mathcal{P}(U);\ \cdot, \trianglelefteq, \vee, \wedge, (\cdot\ \wedge\ \neg\,\cdot), \circ \rangle$ is called a POSTC-*monoid* if it satisfies the following conditions:

• the operation $\cdot$ of the monoid $\langle \mathcal{P}(U) \setminus \{\varnothing\};\ \cdot \rangle$ with the unit $\{0\}$ is generated by the function $\cdot$ on elements in U such that each elements $u, v \in U$ define a nonempty set $(u \cdot v) \subseteq U$: for any sets $X, Y \in \mathcal{P}(U) \setminus \{\varnothing\}$ the following equality holds:

$$X \cdot Y = \bigcup \{u \cdot v \mid u \in X, v \in Y\};$$

if $X \in \mathcal{P}(U)$ then $X \cdot \varnothing = \varnothing \cdot X = \varnothing$;

• the relation $\trianglelefteq$ on the set $\mathcal{P}(U)$ is a preorder with the least element $\varnothing$; this preorder is induced by the partial order $\trianglelefteq'$ on the set U of labels (forming a upper semilattice) by the following rule: if $X, Y \in \mathcal{P}(U)$ then $X \trianglelefteq Y$ if and only if $X = \varnothing$, or for any label $u \in X$ there is a label $v \in Y$ with $u \trianglelefteq' v$ and for any label $v \in Y$ there is a label $u \in X$ with $u \trianglelefteq' v$;

• a label $u \in U$ is called an *atom* if $v \trianglelefteq u$ implies $v = u$ for any label $v \in U$; only labels in $U^- \dot\cup \{0\} \dot\cup U^+$ may be atoms; the label 0 is an atom; some labels in $U^{\geq 0}$ lay under each label in U', moreover, if only labels $v \in U^{\geq 0}$ lay under a label $u \in U'$ then there are no greatest labels among labels v; only labels in U' lay over each label in U';

• the operations $\vee, \wedge, (\cdot\ \wedge\ \neg\,\cdot)$ on the set $U \cup \{\varnothing\}$ form a distributive lattice with relative complements, moreover, for any elements $u, v \in U \cup \{\varnothing\}$,

$$u \trianglelefteq' v \Leftrightarrow u \wedge v = u \Leftrightarrow u \vee v = v,$$

$$(u \wedge \neg v) = \varnothing \Leftrightarrow u \trianglelefteq v;$$

• the operation $\circ$ is defined on the set U such that for any labels $u, v \in U$ the label $u \circ v$ is the greatest element of the set $u \cdot v$;

• the operations $\vee, \wedge, \circ$ on the set $\mathcal{P}(U)$ are induced by the corresponding operations on the set $U \cup \{\varnothing\}$: if $X, Y \in \mathcal{P}(U)$ and $\tau \in \{\vee, \wedge, \circ\}$ then $X\,\tau\,Y = \{u\,\tau\,v \mid u \in X, v \in Y\}$; the operation $(\cdot\ \wedge\ \neg\,\cdot)$ on the set $\mathcal{P}(U)$ is also induced by the corresponding operation on the set $U \cup \{\varnothing\}$: if $X, Y \in \mathcal{P}(U)$ then $X \wedge \neg Y = \{u \wedge \neg v \mid u \in X, v \in Y\}$;

• the sets $U^- \cup \{\varnothing\}$ and $U^{\geq 0} \cup \{\varnothing\}$ are closed with respect to the operations $\vee, \wedge, (\cdot \wedge \neg \cdot)$; the set U' is closed under the operation $\vee$; $u \in U'$ and $v \in U$ then $(u \vee v) \in U'$;

• repeating the definition in Section 2, for each label $u \in U$, the *rank of semi-isolation* $\mathrm{si}(u) \geq 1$ and the *degree of semi-isolation* $\deg(u)$ of label u is defined inductively, $\mathrm{si}(\varnothing) = 0$, $\deg(\varnothing) = 1$, as well as equivalence relations $\sim_\alpha$, restrictions X_α, $X_{\alpha,\beta}$ of sets $X \in \{U, U \cup \{\varnothing\}\}$ and restrictions $\mathfrak{M}'_\alpha$, $\mathfrak{M}'_{\alpha,\beta}$ for restrictions $\mathfrak{M}'$ of $\mathfrak{M}$ to sets of labels of si-ranks $\leq \alpha$, and for labels of si-rank α to sets of labels of si-degree $< \beta$;

• the restriction $\langle \mathcal{P}(U) \setminus \{\varnothing\}; \cdot \rangle_{1,2}$ of the monoid $\langle \mathcal{P}(U) \setminus \{\varnothing\}; \cdot \rangle$ is a I-groupoid;

• if $u < 0$ then sets $u \cdot v$ and $v \cdot u$ consist of negative elements for any $v \in U$;

• if $u > 0$ and $v > 0$ then $(u \cdot v) \subseteq U^{\geq 0}$;

• if $u, v \in U^{\geq 0} \cup U'$, and $u \in U'$ or $v \in U'$, then $(u \cdot v) \subseteq U'$;

• for any element $u > 0$ there is a nonempty set u^{-1} of *inverse* elements $u' > 0$ such that $0 \in (u \cdot u') \cap (u' \cdot u)$; in this case if $u \trianglelefteq' v$ and $v \in U^+$ then $u^{-1} \subseteq v^{-1}$;

• if a positive element u belongs to a set $v_1 \cdot v_2$, where $v_1 \circ v_2 \in U^+$, then $u^{-1} \subseteq v_2^{-1} \cdot v_1^{-1}$.

By the definition each POSTC-monoid $\mathfrak{M}$ contains POSTC-submonoids $\mathfrak{M}^{\leq 0}$ and $\mathfrak{M}^{\geq 0}$ with the universes $\mathcal{P}(U^- \cup \{0\})$ and $\mathcal{P}(U^+ \cup \{0\})$ respectively, being also POSTC-monoids (with $U^+ \cup U' = \varnothing$ and $U^- \cup U' = \varnothing$ respectively).

A POSTC-monoid $\mathfrak{M}$ is called *atomic* if for any label $u \in U$ there is an atom $v \in U$ such that $v \trianglelefteq u$.

Note that POSTC-monoids with ordinals $\sup\{\mathrm{si}(u) \mid u \in U\}$ are atomic.

Theorem 6.1. *For any* POSTC-*monoid* $\mathfrak{M}$ *there is a theory* T *with a type* $p(x) \in S^1(T)$ *and a regular labelling function* $\nu(p)$ *such that* $\mathfrak{M}_{\nu(p)} = \mathfrak{M}$. *If the alphabet is at most countable and the operations of* $\mathfrak{M}$ *do not force continuum many types then* T *is small.*

Proof follows the same scheme as the proof of [17, Theorem 6.1] and, for the structure $\langle \mathcal{P}(U) \setminus \{\varnothing\}; \cdot \rangle_{1,2}$, it is identical to this proof word for word. Since the proof of [17, Theorem 6.1] is voluminous we only point out the distinctive features leading to the proof of this theorem.

1. A binary predicate Q_u is defined for each element $u \in U \cup \{\varnothing\}$. This predicate links only elements of the same colors if $u \geq 0$, and defines a Q_u-ordered coloring Col if $u \in U^- \cup U'$; $Q_\varnothing = \varnothing$.

2. For any elements $u, v \in U \cup \{\varnothing\}$ the following condition is satisfied: $u \trianglelefteq' v \Leftrightarrow Q_u \subseteq Q_v$.

3. For any elements $u, v \in U \cup \{\varnothing\}$ the following conditions hold:

$$u_1 \vee u_2 = v \Leftrightarrow Q_{u_1} \cup Q_{u_2} = Q_v,$$

$$u_1 \wedge u_2 = v \Leftrightarrow Q_{u_1} \cap Q_{u_2} = Q_v,$$

$$u_1 \wedge \neg u_2 = v \Leftrightarrow Q_{u_1} \setminus Q_{u_2} = Q_v,$$

$$u_1 \circ u_2 = v \Leftrightarrow Q_{u_1} \circ Q_{u_2} = Q_v.$$

In particular, the predicates Q_{u_1} and Q_{u_2} are disjoint if and only if $u_1 \wedge u_2 = \varnothing$. $\square$

Remark 6.2. Since labels $u \in \rho_{\nu(p)}$ for semi-isolating formulas admit complements in $\rho_{\nu(p)}$ only for principal types p (and these complements are defined relative to the isolating formula of p), unlike I-groupoids, if a POSTC-monoid $\mathfrak{M}$ is constructed by a set $U^{\geq 0}$, it admits a representation in a transitive theory T with a (unique) type $p(x) \in S(T)$ and a regular labelling function $\nu(p)$ such that $\mathfrak{M}_{\nu(p)} = \mathfrak{M}$ if and only if the set-theoretic operations in $\mathfrak{M}$ form a Boolean algebra.

7. Partial POSTC-monoid on a set of realizations for a family of 1-types of a complete theory

In this section, the results above for a structure of a type, as well as results in [17] for isolating formulas, are generalized for a structure on the set of all realizations for a family of types.

Let R be a nonempty family of types in $S^1(T)$. We denote by $\nu(R)$ a regular family of labelling functions

$$\nu(p,q)\colon \mathrm{SICF}(p,q)/\mathrm{SICE}(p,q) \to U, \ \ p,q \in R,$$

$$\rho_{\nu(R)} \rightleftharpoons \bigcup_{p,q\in R} \rho_{\nu(p,q)}.$$

As in Proposition 4.1, the partial (for $|R| > 1$) function SI on the set $R \times \mathcal{P}(U) \times R$, which maps each tuple of triples $(p_1,u_1,p_2),\ldots,(p_k,u_k,p_{k+1})$, where $u_1 \in \rho_{\nu(p_1,p_2)} \cup \{\varnothing\},\ldots,u_k \in \rho_{\nu(p_k,p_{k+1})} \cup \{\varnothing\}$, to the set of triples (p_1,v,p_{k+1}), where $v \in \mathrm{SI}(p_1,u_1,p_2,u_2,\ldots,p_k,u_k,p_{k+1})$, is associative:

$$\begin{aligned}\mathrm{SI}(\mathrm{SI}(p_1,u_1,p_2,u_2,p_3),u_3,p_4) &= \mathrm{SI}(p_1,u_1,p_2,u_2,p_3,u_3,p_4) = \\ &= \mathrm{SI}(p_1,u_1,\mathrm{SI}(p_2,u_2,p_3,u_3,p_4))\end{aligned} \tag{2}$$

for $u_1 \in \rho_{\nu(p_1,p_2)} \cup \{\varnothing\}$, $u_2 \in \rho_{\nu(p_2,p_3)} \cup \{\varnothing\}$, $u_3 \in \rho_{\nu(p_3,p_4)} \cup \{\varnothing\}$.

Consider the structure

$$\mathfrak{M}_{\nu(R)} \rightleftharpoons \langle R \times \mathcal{P}(U) \times R;\ \cdot, \trianglelefteq, \vee, \wedge, (\cdot\ \wedge\ \neg\,\cdot), \circ\rangle$$

with the partial operations $\cdot$ and $\circ$ such that

$$(p_1,X_1,p_2)\cdot(p_2,X_2,p_3) = \bigcup\{(p_1,u_1,p_2)\cdot(p_2,u_2,p_3) \mid u_1 \in X_1, u_2 \in X_2\},$$

$$(p_1,u_1,p_2)\cdot(p_2,u_2,p_3) = \{(p_1,v,p_3) \mid v \in \mathrm{SI}(p_1,u_1,p_2,u_2,p_3)\},$$

$$(p_1,X_1,p_2)\circ(p_2,X_2,p_3) = \bigcup\{(p_1,u_1,p_2)\circ(p_2,u_2,p_3) \mid u_1 \in X_1, u_2 \in X_2\},$$

$$(p_1,u_1,p_2)\circ(p_2,u_2,p_3) = \{(p_1,u\circ v,p_3)\},$$

$$u_1 \in \rho_{\nu(p_1,p_2)} \cup \{\varnothing\}, u_2 \in \rho_{\nu(p_2,p_3)} \cup \{\varnothing\},$$

as well as the relation $\trianglelefteq$ of preorder, being induced by the partial order, of the same name, on the set of labels and the partial operations $\vee, \wedge, (\cdot\ \wedge\ \neg\,\cdot)$ such that

$$(p,X,q)\vee(p,Y,q) = \bigcup\{(p,u,q)\vee(p,v,q) \mid u \in X, v \in Y\},$$

$$(p,u,q)\vee(p,v,q) = \{(p,u\vee v,q)\},$$

$$(p,X,q)\wedge(p,Y,q) = \bigcup\{(p,u,q)\wedge(p,v,q) \mid u \in X, v \in Y\},$$

$$(p,u,q)\wedge(p,v,q) = \{(p,u\wedge v,q)\},$$

$$(p,X,q)\wedge\neg(p,Y,q) = \bigcup\{(p,u,q)\wedge\neg(p,v,q) \mid u \in X, v \in Y\},$$

$$(p,u,q)\wedge\neg(p,v,q) = \{(p,u\wedge\neg v,q)\},$$

$$u, v \in \rho_{\nu(p,q)} \cup \{\varnothing\}.$$

The POSTC-monoids $\mathfrak{M}_{\nu(p)}$, $p \in R$, are naturally embeddable into this structure. The structure $\mathfrak{M}_{\nu(R)}$ is called a *join of* POSTC-*monoids* $\mathfrak{M}_{\nu(p)}$, $p \in R$, relative to the family $\nu(R)$ of labelling functions and it is denoted by $\bigoplus\limits_{p\in R} \mathfrak{M}_{\nu(p)}$. If $\rho_{\nu(p,q)} = \varnothing$ for all $p \neq q$ the join $\bigoplus\limits_{p\in R} \mathfrak{M}_{\nu(p)}$ is *free*, it is represented as the disjoint union of POSTC-monoids $\mathfrak{M}_{\nu(p)}$ and denoted by $\bigsqcup\limits_{p\in R} \mathfrak{M}_{\nu(p)}$.

By (2) we have

Proposition 7.1. *For any complete theory T, for any nonempty family $R \subset S(T)$ of 1-types, and for any regular family $\nu(R)$ of labelling functions, each n-ary partial operation* $\mathrm{SI}(p_1, \cdot, p_2, \cdot, p_3 \ldots, p_n, \cdot, p_{n+1})$ *on the set $\mathcal{P}(U)$ is interpretable by a term of the structure $\bigoplus\limits_{p\in R} \mathfrak{M}_{\nu(p)}$ with fixed types $p_1, \ldots, p_{n+1} \in R$.*

Denote by $\mathfrak{SI}_{\nu(R)}$ the restriction of $\mathfrak{M}_{\nu(R)}$ to the partial operation $\cdot$.

Using Proposition 1.4 we obtain the following analogue of Proposition 4.3.

Proposition 7.2. *For any complete theory T, for any nonempty family $R \subset S(T)$ of 1-types, and for any regular family $\nu(R)$ of labelling functions, the restriction of the structure $\mathfrak{SI}_{\nu(R)}$ to the set $U^{\leq 0}$ (respectively $U^{\geq 0}$, $U^{\geq 0} \cup U'$) is closed under the partial operation $\cdot$.*

By Proposition 7.2, the structure $\mathfrak{SI}_{\nu(R)}$ has substructures $\mathfrak{SI}^{\leq 0}_{\nu(R)}$, $\mathfrak{SI}^{\geq 0}_{\nu(R)}$ and $\mathfrak{SI}^{\geq 0,\mathrm{neu}}_{\nu(R)}$, generated by triples (p, u, q) with $u \leq 0$, $u \geq 0$, and $u \in U^{\geq 0} \cup U'$ respectively, $p, q \in R$. Here, for any triple (p, u, q) in $\mathfrak{SI}^{\geq 0}_{\nu(R)}$ the triple (q, u^{-1}, p) is also attributed to $\mathfrak{SI}^{\geq 0}_{\nu(R)}$.

Replacing for the definition in Section 5 the function $\nu(p)$ to the family $\nu(R)$ of functions we obtain the notions of (α, n)-*deterministic*, α-*deterministic, almost* α-*deterministic*, and (α, ω)-*deterministic* structures $\mathfrak{SI}_{\nu(R)} \upharpoonright U_0$.

Below we formulate a series of assertions that immediately transformed from the class of structures $\mathfrak{SI}_{\nu(p)}$ to the class of structures $\mathfrak{SI}_{\nu(R)}$.

Proposition 7.3 (Monotony). *If a structure $\mathfrak{SI}_{\nu(R)} \upharpoonright U_0$ is (almost) α-deterministic and β is a positive ordinal then the structure $(\mathfrak{SI}_{\nu(R)} \upharpoonright U_0) \upharpoonright \beta$ is also (almost) α-deterministic.*

Proposition 7.4. *For a structure $\mathfrak{SI}_{\nu(R)}$ and ordinals α, β, where $\alpha, \beta > 0$, $\beta \in \omega + 1$, the following conditions are equivalent:*

(1) *the structure $\mathfrak{SI}_{\nu(R)}$ is (α, β)-deterministic;*

(2) *for any types $p, q, r \in R$ and labels $u_1 \in \rho_{\nu(p,q),\alpha,\beta}$, $u_2 \in \rho_{\nu(q,r),\alpha,\beta}$, the inequality* $\mathrm{si}(u_1 \circ u_2) \leq \alpha$ *holds, and if* $\mathrm{si}(u_1 \circ u_2) = \alpha$ *then* $\deg(u_1 \circ u_2) < \beta$.

Corollary 7.5. *For a structure $\mathfrak{SI}_{\nu(R)}$ and a positive ordinal α, the following conditions are equivalent:*

(1) *the structure $\mathfrak{SI}_{\nu(R)}$ is almost α-deterministic;*

(2) $\mathrm{si}(u_1 \circ u_2) \leq \alpha$ *for any types $p, q, r \in R$ and labels $u_1 \in \rho_{\nu(p,q),\alpha}$, $u_2 \in \rho_{\nu(q,r),\alpha}$.*

Corollary 7.6. *If* $\mathrm{si}(R)$ *is an ordinal then the structure $\mathfrak{SI}_{\nu(R)}$ is almost* $\mathrm{si}(R)$-*deterministic.*

Proposition 7.7. *If a structure $\mathfrak{SI}_{\nu(R)}$ is (α,β)-deterministic then $\mathfrak{SI}_{\nu(R),\alpha} \rightleftharpoons \mathfrak{SI}_{\nu(R)} \upharpoonright \alpha$ is also (α,β)-deterministic.*

Proposition 7.8. *If $\mathrm{si}(R)$ is an ordinal then the structure $\mathfrak{SI}_{\nu(R)}$ is $\mathrm{si}(R)$-deterministic if and only if the value $\deg(R)$ is not defined or equals 1.*

Proposition 7.9. *A structure $\mathfrak{P}_{\nu(R)}$ is (almost) deterministic if and only if $\mathfrak{SI}_{\nu(R),1,2}$ is (almost) 1-deterministic.*

Let R be a nonempty family of complete 1-types of a theory T, $\nu(R)$ be a regular family of labelling functions, and α be an ordinal, $\alpha > 0$. The structure $\mathfrak{SI}_{\nu(R)}$ is called *locally α-deterministic* if for any nonempty finite set $R_0 \subseteq R$ there is a natural number $n \geq 2$ such that the structure $\mathfrak{SI}_{\nu(R_0)}$ is (α, n)-deterministic.

Repeating the proof of Proposition 5.9 we obtain

Proposition 7.10. *Let R be a nonempty family of complete 1-types of a theory T, $\nu(R)$ be a regular family of labelling functions, $\mathrm{si}(R) < \omega$. The following conditions are equivalent:*

(1) *the structure $\mathfrak{SI}_{\nu(R)}$ is locally 1-deterministic;*
(2) *the set $\rho_{\nu(p,q)}$ is finite for any $p, q \in R$;*
(3) *the set $\rho_{\nu(p,q),1}$ is finite for any $p, q \in R$;*
(4) *the set $\rho_{\nu(p,q),1,2}$ (consisting of all atoms $u \in \rho_{\nu(p,q)}$) is finite for any $p, q \in R$.*

The notion of an (n,p)-type is generalized in the following definition.

Definition 7.11 (K. Ikeda, A. Pillay, A. Tsuboi [11]). Let $p_1(x_1), \ldots, p_n(x_n)$ be 1-types in $S(T)$ with disjoint free variables. A type $q(x_1, \ldots, x_n) \in S(T)$ is said to be a *$(p_1, \ldots, p_n)$-type* if $q(x_1, \ldots, x_n) \supseteq \bigcup\limits_{i=1}^{n} p_i(x_i)$. The set of all $(p_1, \ldots, p_n)$-types of T is denoted by $S_{p_1,\ldots,p_n}(T)$. A theory T is *almost ω-categorical* if for any types $p_1(x_1), \ldots, p_n(x_n) \in S(T)$ there are only finitely many types $q(x_1, \ldots, x_n) \in S_{p_1,\ldots,p_n}(T)$.

Definition 7.12 (B. S. Baizhanov, S. V. Sudoplatov, V. V. Verbovskiy [1]). A type $q(\bar{x})$ in $S_{p_1,\ldots,p_n}(T)$ is said to be *$(p_1, \ldots, p_n)$-principal* if there is a formula $\varphi(\bar{y}) \in q(\bar{x})$ such that

$$\cup\{p_i(x_i) \mid i = 1, \ldots, n\} \cup \{\varphi(\bar{x})\} \vdash q(\bar{x}).$$

The following lemma obviously generalizes Lemma 5.12.

Lemma 7.13 [1]. *For any types $p_1(x_1), \ldots, p_n(x_n) \in S(\varnothing)$ the following conditions are equivalent:*

(1) *the set of $(p_1, \ldots, p_n)$-types with free variables in $(x_1, \ldots, x_n)$ is finite;*
(2) *any $(p_1, \ldots, p_n)$-type is $(p_1, \ldots, p_n)$-principal.*

By Lemma 7.13, a theory T is almost ω-categorical if and only if for any types $p_1(x_1), \ldots, p_n(x_n) \in S^1(T)$, each $(p_1, \ldots, p_n)$-type is $(p_1, \ldots, p_n)$-principal.

Proposition 7.10 and Lemma 7.13 imply

Corollary 7.14. *If R is a nonempty family of complete 1-types of a theory T, $\nu(R)$ is a regular family of labelling functions, and all (p_1, p_2)-types, where $p_1, p_2 \in R$, are (p_1, p_2)-principal then the structure $\mathfrak{SI}_{\nu(R)}$ is locally 1-deterministic.*

Corollary 7.15. *If T is an almost ω-categorical theory and $\nu(S^1(\varnothing))$ is a regular family of labelling functions then the structure $\mathfrak{SI}_{\nu(S^1(\varnothing))}$ is locally 1-deterministic.*

For a nonempty family R of 1-types in $S(T)$ and a positive ordinal α, we denote by $\mathrm{SI}_{R,\alpha}$ (in a model $\mathcal{M}$ of T) the restriction of SI_R to the set of formulas of si-ranks $\leq \alpha$:

$$\mathrm{SI}_{R,\alpha} \rightleftharpoons \{(a,b) \mid \mathrm{tp}(a), \mathrm{tp}(b) \in R \text{ and } a \text{ semi-isolates } b$$
$$\text{by a formula } \theta_{\mathrm{tp}(a),u,\mathrm{tp}(b)}(x,y), \text{ with a si-rank } \leq \alpha\}.$$

Clearly, $I_R = \mathrm{SI}_{R,1}$ for any nonempty family R of 1-types. Considering this equality and the equality $\mathfrak{SI}_{\nu(R),1,2} = \mathfrak{P}_{\nu(R)}$, the following proposition generalizes Proposition 5.15 as well as Propositions 4.3 and 8.3 in [17].

Proposition 7.16. *Let R be a nonempty family of complete 1-types of a theory T, $\nu(R)$ be a regular family of labelling functions, and α be a positive ordinal. The following conditions are equivalent:*

(1) *the relation $\mathrm{SI}_{R,\alpha}$ (on a set of realizations of types $p \in R$ in any model $\mathcal{M} \models T$) is transitive;*

(2) *the structure $\mathfrak{SI}_{\nu(R),\alpha}$ is almost α-deterministic.*

Proof. is identical to the proof of Proposition 5.15 almost word for word. □

Propositions 1.4 and 7.16 imply the following assertions.

Corollary 7.17. *Let R be a nonempty family of complete 1-types of a theory T, $\nu(R)$ be a regular family of labelling functions, and α be a positive ordinal. The following conditions are equivalent:*

(1) *the relation $\mathrm{SI}_{R,\alpha}$, in any model $\mathcal{M} \models T$, is a partial order;*

(2) *the structure $\mathfrak{SI}_{\nu(R),\alpha}$ is almost α-deterministic and $\rho_{\nu(R),\alpha} \subseteq U^{\leq 0}$.*

The partial order $\mathrm{SI}_{R,\alpha}$ is identical if and only if $\rho_{\nu(R),\alpha} = \{0\}$. The non-identical partial order $\mathrm{SI}_{R,\alpha}$ has infinite chains if and only if $|\rho_{\nu(p),\alpha}| > 1$ for some type $p \in R$ or there is a sequence p_n, $n \in \omega$, of pairwise distinct types in R such that $|\rho_{\nu(p_n,p_{n+1}),\alpha}| \geq 1$, $n \in \omega$, or $|\rho_{\nu(p_{n+1},p_n)}| \geq 1$, $n \in \omega$.

Corollary 7.18. *Let R be a nonempty family of complete 1-types of a theory T, $\nu(R)$ be a regular family of labelling functions, and α be a positive ordinal. The following conditions are equivalent:*

(1) *the relation $\mathrm{SI}_{R,\alpha}$ on a set of realizations of types $p \in R$ in any model $\mathcal{M} \models T$ is an equivalence relation;*

(2) *the structure $\mathfrak{SI}_{\nu(R),\alpha}$ is almost α-deterministic and $\rho_{\nu(R),\alpha} \subseteq U^{\geq 0}$.*

The results above substantiate that the diagram in Figure 1 admits the transformation replacing the type p by a nonempty family $R \subseteq S^1(\varnothing)$.

8. POSTC$_{\mathcal{R}}$-STRUCTURES

Definition 8.1. Let $\mathcal{R}$ be a nonempty set,

$$U = U^- \,\dot\cup\, \{0\} \,\dot\cup\, U^+ \,\dot\cup\, U'$$

be an alphabet consisting of a set U^- of *negative elements*, a set U^+ of *positive elements*, a set U' of *neutral elements*, and zero 0. If p and q are elements in $\mathcal{R}$, we write $u < 0$ and $(p,u,q) < 0$ for any element $u \in U^-$, $u > 0$ and $(p,u,q) > 0$ for any element $u \in U^+$; $U^{\leq 0} \rightleftharpoons U^- \cup \{0\}$, $U^{\geq 0} \rightleftharpoons U^+ \cup \{0\}$. For the set $\mathcal{R}^2$ of all pairs (p,q), $p,q \in \mathcal{R}$, we consider a *regular* family $\mu(\mathcal{R})$ of sets $\mu(p,q) \subseteq U$ such that

- $0 \in \mu(p,q)$ if and only if $p = q$;
- $\mu(p,p) \cap \mu(q,q) = \{0\}$ for $p \neq q$;

- $\mu(p,q) \cap \mu(p',q') = \varnothing$ if $p \neq q$ and $(p,q) \neq (p',q')$;
- $\bigcup\limits_{p,q\in\mathcal{R}} \mu(p,q) = U$.

Below we write $\mu(p)$ instead of $\mu(p,p)$, and considering a partial operation $\cdot$ on the set $\mathcal{R} \times \mathcal{P}(U) \times \mathcal{R}$ we shall write, as above, $(p,u,q)\cdot(q,v,r)$ instead of $(p,\{u\},q)\cdot(q,\{v\},r)$.

A structure

$$\mathfrak{M} = \langle \mathcal{R} \times \mathcal{P}(U) \times \mathcal{R};\ \cdot, \trianglelefteq, \vee, \wedge, (\cdot\ \wedge\ \neg\,\cdot), \circ\rangle$$

with a regular family $\mu(\mathcal{R})$ of sets is said to be a POSTC$_{\mathcal{R}}$-*structure* if the following conditions hold:

• the partial operation $\cdot$ of the structure $\langle \mathcal{R} \times \mathcal{P}(U) \times \mathcal{R};\ \cdot\rangle$ has values $(p,X,q)\cdot(p',Y,q')$ only for $p' = q$, $X \subseteq \mu(p,q)$, $Y \subseteq \mu(p',q')$, and it is generated by the function $\cdot$ for elements in U: for any sets $X, Y \in \mathcal{P}(U)$, $\varnothing \neq X \subseteq \mu(p,q)$, $\varnothing \neq Y \subseteq \mu(q,r)$, the following equality is satisfied:

$$(p,X,q)\cdot(q,Y,r) = \bigcup\{(p,x,q)\cdot(q,y,r) \mid x \in X, y \in Y\},$$

and if some of X, Y is empty then $(p,X,q)\cdot(q,Y,r) = \varnothing$;

• each restriction $\mathfrak{M}_{\mu(p)}$ of $\mathfrak{M}$ to $\{p\} \times \mathcal{P}(\mu(p)) \times \{p\}$ is isomorphic to a POSTC-monoid with the universe $\mathcal{P}(\mu(p))$, $p \in \mathcal{R}$; atoms $u \in \mu(p)$ in $\mathfrak{M}_{\mu(p)}$ are called p-*atoms*;

• each restriction $\mathfrak{M}_{\mu(p,q)}$, $p \neq q$, of $\mathfrak{M}$ to $\{p\}\times\mathcal{P}(\mu(p,q))\times\{q\}$ has empty partial operations $\cdot$ and $\circ$; the restriction of $\mathfrak{M}_{\mu(p,q)}$ to the relation $\trianglelefteq$ is a preordered set $\langle \{p\} \times \mathcal{P}(\mu(p,q)) \times \{q\};\ \trianglelefteq_{p,q}\rangle$ with the least element $(p,\varnothing,q)$, the preorder $\trianglelefteq_{p,q}$ of this structure is induced by the partial order $\trianglelefteq'_{p,q}$ on the set $\mu(p,q)$ of labels (forming a upper semilattice if $\mu(p,q) \neq \varnothing$) by the following rule: if $X, Y \in \mathcal{P}(\mu(p,q))$ then $X \trianglelefteq_{p,q} Y$ if and only if $X = \varnothing$, or for any label $u \in X$ there is a label $v \in Y$ with $u \trianglelefteq_{p,q} v$ and for any label $v \in Y$ there is a label $u \in X$ with $u \trianglelefteq_{p,q} v$;

• a label $u \in \mu(p,q)$, where $p \neq q$, is said to be a (p,q)-*atom* if $v \trianglelefteq_{p,q} u$ implies $v = u$ for any label $v \in \mu(p,q)$; only labels in $\mu(p,q) \cap (U^- \cup U^+)$ may be (p,q)-atoms; some labels in $\mu(p,q) \cap U^{\geq 0}$ lay under each label in $\mu(p,q) \cap U'$, moreover, if only labels $v \in \mu(p,q) \cap U^{\geq 0}$ lay under a label $u \in \mu(p,q) \cap U'$ then there are no greatest labels among labels v; only labels in $\mu(p,q) \cap U'$ lay over each label in $\mu(p,q) \cap U'$;

• the operations $\vee, \wedge, (\cdot \wedge \neg\,\cdot)$ are defined on each set $\mu(p,q)\cup\{\varnothing\}$ in the structure $\mathfrak{M}_{\mu(p,q)}$ and form a distributive lattice with relative complements on $\mu(p,q) \cup \{\varnothing\}$, moreover, for any elements $u, v \in \mu(p,q) \cup \{\varnothing\}$,

$$u \trianglelefteq_{p,q} v \Leftrightarrow u \wedge v = u \Leftrightarrow u \vee v = v \Leftrightarrow u \wedge \neg v = \varnothing;$$

• the relation $\trianglelefteq$ on the set $\mathcal{R} \times \mathcal{P}(U) \times \mathcal{R}$ is a preorder with minimal elements $(p,\varnothing,q)$, $p, q \in \mathcal{R}$; this preorder is induced by the union $\trianglelefteq_U$ of preorders $\trianglelefteq_p$ in the structures $\mathfrak{M}_{\mu(p)}$, $p \in \mathcal{R}$, and of preorders $\trianglelefteq_{p,q}$ in the structures $\mathfrak{M}_{\mu(p,q)}$, $p, q \in \mathcal{R}$, $p \neq q$, on sets of labels in these structures: if $X, Y \in \mathcal{P}(U)$ then $(p,X,q) \trianglelefteq (p',Y,q')$ if and only if $p = p'$, $q = q'$, and $X = \varnothing$ or for any label $u \in X$ there is a label $v \in Y$ with $u \trianglelefteq_U v$ and for any label $v \in Y$ there is a label $u \in X$ with $u \trianglelefteq_U v$;

• the partial operations $\vee,\wedge,(\cdot\wedge\neg\cdot)$ are defined on the set $\mathcal{R}\times(U\cup\{\varnothing\})\times\mathcal{R}$ in the structure $\mathfrak{M}$ being unions of corresponding operations on the sets $\mu(p)\cup\{\varnothing\}$ in $\mathfrak{M}_{\mu(p)}$ and on the sets $\mu(p,q)\cup\{\varnothing\}$ in $\mathfrak{M}_{\mu(p,q)}$, $p\neq q$;

• the partial operation $\circ$ is defined on the set $\mathcal{R}\times(U\cup\{\varnothing\})\times\mathcal{R}$ in the structure $\mathfrak{M}$ being obtained from the union of corresponding operations in the structures $\mathfrak{M}_{\mu(p)}$, $p\in\mathcal{R}$, by the following extension: if $u_1\in\mu(p,q)$ and $u_2\in\mu(q,r)$ then there is unique element $v\in\mu(p,r)$ such that $(p,u_1,q)\circ(q,u_2,r)=(p,v,r)$; this element v is the $\trianglelefteq_{p,r}$-greatest label in the set $(p,u_1,q)\cdot(q,u_2,r)$, it is called a *composition* of elements u_1 and u_2 and it is denoted by $u_1\circ u_2$;

$$(p,u_1,q)\circ(q,\varnothing,r)=(p,\varnothing,q)\cdot(q,u_2,r)=(p,\varnothing,q)\cdot(q,\varnothing,r)=(p,\varnothing,r);$$

• the partial operations $\vee,\wedge,\circ$ on the set $\mathcal{R}\times\mathcal{P}(U)\times\mathcal{R}$ are induced by the corresponding partial operations on the set $\mathcal{R}\times(U\cup\{\varnothing\})\times\mathcal{R}$: if $(p,X,q),(p',Y,q')\in\mathcal{R}\times\mathcal{P}(U)\times\mathcal{R}$ and $\tau\in\{\vee,\wedge,\circ\}$ then the value $(p,X,q)\,\tau\,(p',Y,q')$ is not defined or it is defined and coincides with the set $\{(p,u,q)\,\tau\,(p',v,q')\mid u\in X,v\in Y\}$, in which all values are defined; the partial operation $(\cdot\wedge\neg\cdot)$ on the set $\mathcal{R}\times\mathcal{P}(U)\times\mathcal{R}$ is also induced by the corresponding partial operation on the set $\mathcal{R}\times(U\cup\{\varnothing\})\times\mathcal{R}$: if $(p,X,q),(p',Y,q')\in\mathcal{R}\times\mathcal{P}(U)\times\mathcal{R}$ then the value $(p,X,q)\wedge\neg(p',Y,q')$ is defined only for $p=p'$, $q=q'$, $X,Y\subseteq\mu(p,q)$ and it is equal to $\{(p,u,q)\wedge\neg(p,v,q)\mid u\in X,v\in Y\}$;

• each of the sets $U^-\cup\{\varnothing\}$ and $U^{\geq 0}\cup\{\varnothing\}$ is closed under operations $\vee,\wedge,(\cdot\wedge\neg\cdot)$; the set U' is closed under the operation $\vee$; if $u\in U^-$ and $v\in U^{\geq 0}$ then $(u\vee v)\in U'$;

• repeating the definition in Section 3, each label $u\in U$ obtains inductively the *rank of semi-isolation* $\mathrm{si}(u)\geq 1$ and the *degree of semi-isolation* $\deg(u)$, $\mathrm{si}(\varnothing)=0$, $\deg(\varnothing)=1$, as well as the following attributes are defined: the equivalence relations $\sim_\alpha$, restrictions X_α and $X_{\alpha,\beta}$ of sets $X\in\{U,U\cup\{\varnothing\}\}$, and restrictions $\mathfrak{M}'_\alpha$, $\mathfrak{M}'_{\alpha,\beta}$ for restrictions $\mathfrak{M}'$ of the structure $\mathfrak{M}$ to the set of labels of si-ranks $\leq\alpha$, and for labels of si-rank α to the set of labels of si-degree $<\beta$;

• the restriction $\langle\mathcal{R}\times(\mathcal{P}(U)\setminus\{\varnothing\})\times\mathcal{R};\ \cdot\rangle_{1,2}$ of the structure $\mathfrak{M}$ is an $I_{\mathcal{R}}$-structure;

• if $u\in\mu(p,q)$ and $u<0$ then the set $(p,u,q)\cdot(q,v,r)$ and $(r,v',p)\cdot(p,u,q)$ consist of negative elements for any $v\in\mu(q,r)$ and $v'\in(r,p)$;

• if $u\in\mu(p,q)$, $v\in\mu(q,r)$, $u>0$, and $v>0$, then the set $(p,u,q)\cdot(q,v,r)$ consists of elements in $U^{\geq 0}$;

• if $u\in\mu(p,q)\cap(U^{\geq 0}\cup U')$, $v\in\mu(q,r)\cap(U^{\geq 0}\cup U')$, and $u\in U'$ or $v\in U'$, then $(p,u,q)\cdot(q,v,r)\subseteq U'$;

• for any element $u\in\mu(p,q)$ with $u>0$ there is a nonempty set u^{-1} of *inverse* elements $u'>0$ such that $(p,0,p)\in(p,u,q)\cdot(q,u',p)$ and $(q,0,q)\in(q,u',p)\cdot(p,u,q)$, moreover, if $u\trianglelefteq v$ and $v\in U^+$ then $u^{-1}\subseteq v^{-1}$;

• if an element (p,u,r), where $u>0$, belongs to a set $(p,v_1,q)\cdot(q,v_2,r)$, where $v_1\circ v_2\in U^+$, then $(r,u^{-1},p)\subseteq(r,v_2^{-1},q)\cdot(q,v_1^{-1},p)$.

By the definition, each $\mathrm{POSTC}_{\mathcal{R}}$-structure $\mathfrak{M}$ contains $\mathrm{POSTC}_{\mathcal{R}}$-substructures $\mathfrak{M}^{\leq 0}$ and $\mathfrak{M}^{\geq 0}$ being restrictions of $\mathfrak{M}$ to the sets $U^{\leq 0}$ and $U^{\geq 0}$ respectively.

A $\mathrm{POSTC}_{\mathcal{R}}$-structure $\mathfrak{M}$ is called *atomic* if for any label $u\in\mu(p)$, $p\in\mathcal{R}$, there is a p-atom $v\in U$ such that $v\trianglelefteq_p u$, and for any label $u\in\mu(p,q)$, $p,q\in\mathcal{R}$, $p\neq q$, there is a (p,q)-atom $v\in U$ such that $v\trianglelefteq_{p,q} u$.

Combining the proof of Theorems 6.1 and 9.1 in [17] as well as the proof of Theorem 6.1, we obtain the following theorem.

Theorem 8.1. *For any* $\mathrm{POSTC}_{\mathcal{R}}$*-structure* $\mathfrak{M}$ *there is a theory* T *with a family* $R \subset S(T)$ *of* 1*-types and a regular family* $\nu(R)$ *of labelling functions such that* $\mathfrak{M}_{\nu(R)} = \mathfrak{M}$*. If the alphabet and the family* $\mathcal{R}$ *are at most countable, and the operations of* $\mathfrak{M}$ *do not force continuum many types, then* T *is small.*

In conclusion, we note that, using the operation $\cdot^{\mathrm{eq}}$, the constructions above can be transformed for an arbitrary family of types in $S(T)$.

Acknowledgment. The author thanks Evgeniy A. Palyutin and Vadim G. Puzarenko for useful remarks. The author is grateful to the anonymous referee for the detailed reading of the text and comments that helped to improve the exposition.

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