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UNIFICATION PROBLEM IN NELSON'S LOGIC N4

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Abstract. We consider the unification problem for formulas with coefficients in the Nelson's paraconsitent logic **N4**. By presence coefficients (parameters) the problem is quite not trivial and challenging (yet what makes the problem for **N4** to be peculiar is missing of replacement equivalents rule in this logic). It is shown that the unification problem in **N4** is decidable for $\sim$-free formulas. We also show that there is an algorithm which computes finite complete sets of unifiers (so to say – all best unifiers) for unifiable in **N4** $\sim$-free formulas (i.e. any unifier is equivalent to a substitutional example of a unifier from this complete set). Though the unification problem for all formulas (not $\sim$-free formulas) remains open.

Keywords: Nelson's logic, strong negation, unification, complete sets of unifiers, decidability, Vorob'ev translation

1. Introduction

Unification problem is a contemporary area on the border of Mathematics, in particular, - Universal Algebra (as problems of solving algebraic equations in free algebras) and Computer Science (e.g. - term rewriting). In present time it is an active area in non-classical logic and knowledge representation (cf., for instance, Baader and Ghilardi [6]).

Logical unification (being, in particular, a base for logic programming, e.g. Prolog language) is interesting both for logical community and computer science (cf. F. Baa-der and W. Snyder [3], F. Baader et al [5], S. Ghilardi [8, 9, 10, 11], D. Gabbay and U. Reyle [12], A. Oliart et al [18], J. Levi et al [19]). Standard logical unification problem (whether a formula can be unified in a given logic) is, in fact,

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a particular case of more complicated problem: the substitution problem: whether a formula can be made a theorem after replacing a part of variables (keeping the same value for coefficients — parameters). This problem was studied and solved, V. Rybakov [23, 24, 25]) for intuitionistic logic and modal logics $S4$ and Grz (but only to determine if a solution exits and to compute a particular one if yes).

In present decade, S.Ghilardi [8, 9, 10] studied extensively the unification in propositional modal logics over $K4$ and intuitionistic logic (using ideas from projective algebras and applying a technique of special projective formulas) with aim to describe all possible unifiers. In these works the problem of construction complete sets of unifiers (those, where all other unifiers are some substitutional examples of these ones), in logics under consideration, was solved and the algorithms for computation best unifiers were constructed. Using the technique of projective formulas, Dzik and Wojtylak [7] recently showed that the modal logic $S4.3$ has unitary unification type (again computational algorithms for building complete sets of unifiers were offered). Solutions of unification problem for formulas with coefficients in transitive modal logics and intuitionistic logic (with constructions of algorithms computing finite complete sets of unifiers) were offered in Rybakov [26] and Rybakov [27].

Recently, Odintsov and Rybakov [17], found an approach of dealing with unification problem for formulas with coefficients in Johansson's paraconsistent logic, and the solution of unification problem in this logic (as well as in the positive intuitionistic logic $\mathbf{IPC}^{+}$) was obtained. This our work will essentially use (better to say – to be based on) technique and results from this paper.

Concerning possible negative solutions of the unification problem, Wolter and Zakharyaschev [30] showed that, if we consider modal logics with additional universal modality operation, then even the standard unification problem for any modal logic from the interval from K to $K4$ is undecidable.

In this paper we consider the unification problem for formulas with coefficients in the Nelson's paraconsitent logic **N4**. This logic appeared as attempts of refinement the fact of non-constructivity of intuitionistic negation in D. Nelson [14]. There the concept of constructible falsity was suggested. It assumes that the falsity of atomic statements is given explicitly, and the falsity of complex statements is reduced to the truth or falsity of its constituents via a constructive procedure. Subsequently, his system of constructive logic with strong negation, traditionally denoted by **N3**, was axiomatized by Vorob'ev [28, 29] and studied algebraically by Helena Rasiowa [20, 21]. The concept of constructible falsity agrees well with that of paraconsistency. If the falsity of an atom p represented as $\sim p$, the strong negation of p, is given explicitly, we may admit that both p and $\sim p$ are true. The paraconsistent Nelson's logic **N4** is obtained by deleting the "explosive" axiom $\sim p \rightarrow (p \rightarrow q)$ from the axiomatics of **N3**. From the early 1970s several versions of **N4** were studied independently by R. Routley (later R. Sylvan) in the propositional case in [22], by López-Escobar in [13] and by Nelson himself in [1], both in the first-order case. Algebraic semantics for **N4** was suggested in [15].

In this our paper we consider only **N4** with the aim to solve unification problem for formulas with coefficients. By presence coefficients (parameters) the problem is quite not trivial and challenging (yet what makes the problem for **N4** to be peculiar is missing of replacement of equivalents rule in this logic). It is shown that the unification problem in **N4** is decidable for $\sim$-free formulas. We also show that there is an algorithm which computes finite complete sets of unifiers (so to say, – all best unifiers) for unifiable in **N4** $\sim$-free formulas (i.e. any unifier is equivalent to a substitutional example of a unifier from this complete set). Though the unification problem for all (not $\sim$-free formulas with coefficients) remains open.

2. Nelson's logic N4

We proceed by recalling definitions, notation and basic facts necessary for reading this paper. Any propositional logic is usually based on a countable set of propositional variables (letters) $Prop = \{p_1, p_2, \ldots, p_n, \ldots\}$.

Any propositional language $\mathcal{L}$ is assumed to contain a finite set of connectives with indicated arities. The set of formulas $For(\mathcal{L})$ of a propositional language $\mathcal{L}$ is obtained in a usual way from variables of $Prop$ with the help of connectives of $\mathcal{L}$.

By a *logic* in a propositional language $\mathcal{L}$ with $\to \in \mathcal{L}$ we mean a set of formulas closed under the rules of substitution and *modus ponens*.

In this approach, we can define *positive (intuitionistic) logic* $\mathbf{IPC}^+$ as the least logic in the language $\mathcal{L}^+ = \langle \vee, \wedge, \to \rangle$ containing the following axioms:

(1) $p \to (q \to p)$
(2) $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$
(3) $(p \wedge q) \to p$
(4) $(p \wedge q) \to q$
(5) $(p \to q) \to ((p \to r) \to (p \to (q \wedge r)))$
(6) $p \to (p \vee q)$
(7) $q \to (p \vee q)$
(8) $(p \to r) \to ((q \to r) \to ((p \vee q) \to r))$

Intuitionistic logic **IPC** is the least logic in the language $\mathcal{L}^\perp = \langle \vee, \wedge, \to, \perp \rangle$ containing axioms of positive logic and the axiom $\perp \to p$.

In what follows we use the abbreviation $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$.

Paraconsistent Nelson's logic **N4** is the least logic in the language $\mathcal{L}^\sim = \langle \vee, \wedge, \to, \sim \rangle$, where $\sim$ is a symbol for the strong negation connective (operation), containing axioms of positive logic and the following axioms for the strong negation:

($1^\sim$) $\sim (p \vee q) \leftrightarrow (\sim p \wedge \sim q)$
($2^\sim$) $\sim (p \wedge q) \leftrightarrow (\sim p \vee \sim q)$
($3^\sim$) $\sim (p \to q) \leftrightarrow (p \wedge \sim q)$
($4^\sim$) $\sim\sim p \leftrightarrow p$

Let Fm^+ to be the set of all $\sim$-free formulas in the language of **N4**. Recall that $\mathbf{IPC}^+$ coincides with the positive fragment of logics **IPC** and **N4**, i.e.,

$$\mathbf{IPC}^+ = \mathbf{IPC} \cap Fm^+ = \mathbf{N4} \cap Fm^+.$$

A Kripke style semantics for Nelson's logic can be defined in a similar way to that for intuitionistic logic. In more details, a *frame* is a pair $\mathcal{W} = \langle W, \leq \rangle$, where W is a non-empty set (of possible worlds), $\leq$ is a partial order on W. A subset R of W is called a *cone of* $\mathcal{W}$ (*cone w.r.t.* $\leq$) if it is upward closed w.r.t. $\leq$, i.e., for every $x \in R$ and $y \in W$, if $x \leq y$, then $y \in R$. We say that the cone R is *sharp* if there is an element $a \in W$ such that $R = \{b \in W \mid a \leq b\}$. In this case we will use the denotation $R = [a]$ and $\langle a \rangle = [a] \setminus \{a\}$. For a subset U of W, we denote $U\downarrow = \{x \in W \mid x \leq y \text{ for some } y \in U\}$.

An **N4**-*model* $\mathcal{M} = \langle W, \leq, V \rangle = \langle \mathcal{W}, V \rangle$ is a frame $\mathcal{W}$ augmented with a valuation $V : Lit(S) \to 2^W$ such that S is some set of propositional variables,

$Lit(S) = S \cup \{\sim p \mid p \in S\}$, and $V(p)$ and $V(\sim p)$ are cones w.r.t. $\leq$ for all $p \in S$. In this case we say that $\mathcal{M}$ is a *model over* $\mathcal{W}$.

Validity of formulas at worlds of the model $\mathcal{M}$ is defined by induction. For $p \in S$, φ, ψ to be formulas and $x \in W$ we put:

- $\mathcal{M}, x \Vdash p$ iff $x \in V(p)$, $\quad \mathcal{M}, x \Vdash\sim p$ iff $x \in V(\sim p)$;
- $\mathcal{M}, x \Vdash \varphi \wedge \psi$ iff $\mathcal{M}, x \Vdash \varphi$ and $\mathcal{M}, x \Vdash \psi$;
- $\mathcal{M}, x \Vdash \varphi \vee \psi$ iff $\mathcal{M}, x \Vdash \varphi$ or $\mathcal{M}, x \Vdash \psi$;
- $\mathcal{M}, x \Vdash \varphi \to \psi$ iff $\forall y \geq x$ $(\mathcal{M}, y \nVdash \varphi$ or $\mathcal{M}, y \Vdash \psi)$;
- $\mathcal{M}, x \nVdash \bot$, $\quad \mathcal{M}, x \Vdash\sim \bot$ (for the case when $\bot$ is in the language);
- $\mathcal{M}, x \Vdash\sim (\varphi \wedge \psi)$ iff $\mathcal{M}, x \Vdash\sim \varphi$ or $\mathcal{M}, x \Vdash\sim \psi$;
- $\mathcal{M}, x \Vdash\sim (\varphi \vee \psi)$ iff $\mathcal{M}, x \Vdash\sim \varphi$ and $\mathcal{M}, x \Vdash\sim \psi$;
- $\mathcal{M}, x \Vdash\sim (\varphi \to \psi)$ iff $\mathcal{M}, x \Vdash \varphi$ and $\mathcal{M}, x \Vdash\sim \psi$;
- $\mathcal{M}, x \Vdash\sim\sim \varphi$ iff $\mathcal{M}, x \Vdash \varphi$.

If $\mathcal{M}, x \Vdash \varphi$ and $x \leq y$, then $\mathcal{M}, y \Vdash \varphi$.

A formula φ is *valid in* $\mathcal{M}$, $\mathcal{M} \Vdash \varphi$, if $\mathcal{M}, x \Vdash \varphi$ for all $x \in W$. A formula φ is *valid in* $\mathcal{W}$, $\mathcal{W} \Vdash \varphi$, if $\mathcal{M} \Vdash \varphi$ for all models $\mathcal{M}$ over $\mathcal{W}$. For a set of formulas Γ, we write $\mathcal{M}, x \Vdash \Gamma$ $(\mathcal{M} \Vdash \Gamma)$ if $\mathcal{M}, x \Vdash \psi$ $(\mathcal{M} \Vdash \psi)$ for all $\psi \in \Gamma$. In what follows we assume an agreement that if a model is denoted by a calligraphic latter, its set of worlds is denoted by the same italic letter, $\mathcal{M} = \langle M, \leq, V \rangle$.

It is known that **N4** is strongly complete w.r.t. the class of all frames and complete w.r.t. the class of all finite frames. That is, a formula $\varphi \in$ **N4** iff φ is valid at all finite frames.

3. Unification

The concept of unification and unifiers for usual propositional logics L looks as follows. Let For be the set of all formulas in the language of L and ε be a mapping (we will refer to ε as a substitution) of a set of letters $Dom(\varepsilon)$ in For. Any such mapping ε can be extended to the set of all formulas φ in letters from $Dom(\varepsilon)$ by

$$\varepsilon(\varphi(x_1, \ldots, x_n)) := \varphi(\varepsilon(x_1), \ldots, \varepsilon(x_n)).$$

For a given formula φ, φ is unifiable in a logic L if there is a substitution ε (which is called a unifier for φ), such that $\varepsilon(\varphi) \in L$.

If a logic L is decidable and $\top, \bot$ are definable in it, to check just unifiability of a formula in L is usually an easy task: it is sufficient to use only ground substitutions, that is mappings of letters in the set $\{\bot, \top\}$[1] (ground substitutions; but, note that, the question how to describe all possible unifiers is yet not simple at all). But we consider a task more general than just usual unification: unification of formulas with coefficients (meta-variables, parameters).

Thus, we will consider formulas constructed out of two sorts of letters: letters x_i from potentially infinite set of letters VL (we call them variable letters) and letters p_j from a potentially infinite set of letters MVL (which we call meta-variable letters or coefficients or parameters). Before to comment essence and use of the division the letters in variable-letters and coefficient-letters we give some definitions.

Any substitution ε, in what follows, always maps any coefficient (meta-variable) to itself, i.e. $\varepsilon(p_j) = p_j$, so lets them intact.

[1]This way is impossible however for the logic **N4**, because the constants $\bot$ and $\top$ neither belong to, nor are definable in the language of **N4**.

Definition 1. *A formula φ with coefficients is unifiable in a logic L if there is a substitution ε (which is called a unifier for φ) (which, we recall, maps any coefficient to itself) such that $\varepsilon(\varphi) \in L$.*

Definition 2. *The unification problem for a logic L is* **decidable***, if for any given formula (with coefficients) φ we may compute if φ is unifiable in L.*

If we consider general unification - unification for formulas with coefficients, the unification problem is *non-trivial even for decidable logics with well known decision algorithms* (since we cannot use ground substitutions (or similar ones), and need, in principal to test all formula constructed out of coefficients as possible unifiers).

If we wish to describe all possible unifiers for any given formula, the concept of more general unifiers may be applied.

Definition 3. *A unifier ε (for a formula φ in a logic L) is more general than a unifier ε_1 iff there is a substitution δ such that for any letter x, $[\varepsilon_1(x) \leftrightarrow \delta(\varepsilon(x))] \in L$.*

Definition 4. *A set of unifiers SU for a formula φ (in a logic L) is said to be a* **complete set of unifiers** *if, for any unifier σ (for φ in L), there is a unifier from SU which is more general than φ.*

The division of letters in variable-letters and coefficient letters aims to consider the task more general than usual unification. Say, if we consider the equation in the language of rational numbers: $2 * x = y$, the solution :- $x = y/2$ – gives us a unique particular solution (for $x \neq 0$) for just this unique equation. But, if we solve the equation in form $a * x = y$ (where a is a coefficient - actually a meta-variable letter), the solution $x = y/a$ gives us infinite series of all possible solutions for any given y and any given value (different from 0) of a.

In terms of algebraic logic, usual unification is finding solutions of equations in the free algebra from the variety of algebras corresponding to the logic L in standard signature. Unification with coefficients is finding solutions for equations in the free algebras in the signature extended by constants for meta-variables.

If we work with Nelson's logic **N4** the task of unification became to be harder and peculiar since **N4** does not have the rule of replacement of equivalents. We illustrate it below with an example.

Proposition 1. *The logic* **N4** *is closed under the positive replacement rule:*

$$\frac{\varphi \leftrightarrow \psi}{\chi(\varphi) \leftrightarrow \chi(\psi)},$$

where $\chi(p)$ is a $\sim$-free formula.

Proof easily follows because **N4** contains all axioms of $\mathbf{IPC}^+$, so the proof is the same as for $\mathbf{IPC}^+$.

Proposition 2. *The logic* **N4** *is not closed under the replacement of equivalents rule.*

Proof. The following holds: $\sim (p \to q) \leftrightarrow (p \wedge \sim q) \in \mathbf{N4}$. Assume that $\sim\sim (p \to q) \leftrightarrow \sim (p \wedge \sim q) \in \mathbf{N4}$. The left side of this equivalence is equivalent to $p \to q$ by axiom $(4^\sim)$, the right side is equivalent to $\sim p \vee q$ by $(2^\sim)$, $(4^\sim)$, and the positive replacement rule. Thus, $(p \to q) \leftrightarrow (\sim p \vee q) \in \mathbf{N4}$. This fact obviously contradicts to semantical characterization of **N4** described in previous section. □

Since this observation, we have to be very accurate with constructions and proofs, because we cannot apply equivalency of formulas in usual way.

We will need also normal negative forms of formulas with strong negation and embeddings of Nelson's logics into positive and intuitionistic logics described below.

We say that a formula φ is in a *negative normal form* (*nnf*) if the strong negation connective $\sim$ may occur in φ only in front of letters (variables or coefficients).

The fact that Nelson's logics are closed under the positive replacement rules and the strong negation axioms allow us to prove that every formula can be reduced to a negative normal form. More precisely, for every formula φ, there is formula in a *nnf* ψ such that

$$\varphi \leftrightarrow \psi \in \mathbf{N4}.$$

In what follows we assume that we have fixed an algorithm assigning to a formula φ its negative normal form $\varphi^{\natural}$.

Let $\varphi = \varphi(\overline{x}, \overline{p})$ be a formula and

$$\varphi^{\natural} = \varphi'(\overline{x}, \overline{p}, \sim \overline{x}, \sim \overline{p}) =$$
$$= \varphi'(x_1, \ldots, x_m, p_1, \ldots, p_n, \sim x_1, \ldots, \sim x_m, \sim p_1, \ldots, \sim p_n),$$

where φ' contains no $\sim$.

To every variable $x_i \in VL$ (metavariable $p_j \in MVL$) we assign a new variable $x_i^{\dagger}$ (metavariable $p_j^{\dagger}$) and will consider formulas of the language $\mathcal{L}^{+}$ constructed from these extended sets of variables and metavariables.

For a formula $\varphi = \varphi(\overline{x}, \overline{p})$, put

$$\varphi^{\mathbf{4}} := \varphi'(\overline{x}, \overline{p},\ \overline{x}^{\dagger}, \overline{p}^{\dagger}) =$$
$$= \varphi'(x_1, \ldots, x_m,\ p_1, \ldots, p_n,\ x_1^{\dagger}, \ldots, x_m^{\dagger},\ p_1^{\dagger}, \ldots, p_n^{\dagger}).$$

We have just defined embedding of Nelson's logic into positive intuitionistic logic:

Theorem 1. *For every φ, the following equivalence holds:*

$$\varphi \in \mathbf{N4} \quad \textit{iff} \quad \varphi^{\mathbf{4}} \in \mathbf{IPC}^{+}.$$

For the logic **N3** a similar statement was originally proved by Vorob'ev [28]. Here to prove the theorem we may use the similar argument.

Now on, we have all instruments ready to prove our main statements.

Theorem 2. *Unification problem in logic* **N4** *for $\sim$-free formulas is decidable. There is an algorithm which for any $\sim$-free formula*

$$\varphi(x_1, \ldots, x_m, p_1, \ldots, p_n)$$

verifies if this formula is unifiable in **N4** *and if yes constructs a unifier.*

Proof. Let a formula $\varphi(x_1, \ldots, x_m, p_1, \ldots, p_n)$ be unifiable in **N4** and $\varepsilon(x_i) = \psi_i$ be its unifier. Let ψ_i^1 be the formulas in *nnf* equivalent to the formulas ψ_i. Since φ is in *nnf* and ε is a unifier, we have

$$\psi := \varphi(\psi_1^1, \ldots, \psi_m^1, p_1, \ldots, p_n) \in \mathbf{N4}.$$

By Theorem 1 it follows that

$$\psi^{\mathbf{4}} = \varphi((\psi_1^1)^{\mathbf{4}}, \ldots, (\psi_m^1)^{\mathbf{4}}, p_1, \ldots, p_n) \in \mathbf{IPC}^{+}.$$

That is the formula

$$\varphi(x_1,\dots,x_m,p_1,\dots,p_n)$$

is unifiable in $\mathbf{IPC}^+$. If it is the case, then by Theorem 5.6 from [17] we may determine if this holds and if yes to construct a unifier in $\mathbf{IPC}^+$. The same unifier will unify the formula $\varphi(x_1,\dots,x_m,p_1,\dots,p_n)$ in **N4**, since positive fragments in $\mathbf{IPC}^+$ and **N4** coincide.

If the algorithm will show that the formula $\varphi(x_1,\dots,x_m,\ p_1,\dots,p_n)$ is not unifiable in $\mathbf{IPC}^+$, then this formula cannot be unifiable in **N4** as well. □

Theorem 3. *The logic* **N4** *has a finitary unification type for* $\sim$*-free formulas with coefficients. There is an algorithm which, for any unifiable in* **N4** *formula* φ*, constructs a finite complete set of unifiers in* **N4** *for* φ*.*

Proof. First, in accordance with previous theorem, we may verify if a given formula φ is unifiable. Then, consider any unifier $x_i \mapsto \psi_i$, as in previous theorem we may use ψ_i^1 instead ψ_i, and we will have

$$\psi := \varphi(\psi_1^1,\dots,\psi_m^1,p_1,\dots,p_n) \in \mathbf{N4}.$$

By Theorem 1 we obtain that

$$\varphi((\psi_1^1)^{\mathbf{4}},\dots,(\psi_m^1)^{\mathbf{4}},p_1,\dots,p_n) \in \mathbf{IPC}^+.$$

That is, the formula $\varphi(x_1,\dots,x_m,\ p_1,\dots,p_n)$ is unifiable in $\mathbf{IPC}^+$ and by Theorem 5.6. from [17], using suggested there algorithm, we may construct a finite complete set S of unifiers for $\varphi(x_1,\dots,x_m,\ p_1,\dots,p_n)$ in $\mathbf{IPC}^+$. In particular, all $(\psi_j^1)^{\mathbf{4}}$ are equivalent in $\mathbf{IPC}^+$ to certain substitutional examples α_j^s of some formulas α_j from the computed complete set S. Then, in particular, we have

$$\varphi(\alpha_1,\dots,\alpha_m,p_1,\dots,p_n) \in \mathbf{IPC}^+$$

and respectively

$$\varphi(\alpha_1,\dots,\alpha_m,p_1,\dots,p_n) \in \mathbf{N4}.$$

That is all α_j from S give us a unifier in **N4**. Besides, an evident (back) substitution makes ψ_j^1 from $(\psi_j^1)^{\mathbf{4}}$ (where $(\psi_j^1)^{\mathbf{4}}$ are equivalent to α_j^s in $\mathbf{IPC}^+$ and respectively at **N4**).

So, any ψ_j is equivalent in **N4** to ψ_j^1 which is a substitutional example of $(\psi_j^1)^{\mathbf{4}}$ which is equivalent to α_j^s. Since $(\psi_j^1)^{\mathbf{4}}$ and α_j^s are $\sim$-free, ψ_j is equivalent in **N4** to a substitutional example of α_j^s. So, S is a complete set of unifiers for $\varphi(x_1,\dots,x_m,p_1,\dots,p_n)$ in **N4**. □

We are not ready yet to solve the unification problem for the logic **N4** in total, but concluding the paper we would like to point a strengthening of the unification problem in positive intuitionistic logic which is exactly equivalent to the unification problem in Nelson's logic **N4**.

For a formula φ in the positive language with extended set of variables and metavariables, we define its dual formula φ^d as follows:

$$\begin{array}{rclrcl} x_i^d & := & x_i^\dagger & (x_i^\dagger)^d & := & x_i \\ p_j^d & := & p_j^\dagger & (p_j^\dagger)^d & := & p_j \\ (\psi \wedge \chi)^d & := & \psi^d \vee \chi^d & (\psi \vee \chi)^d & := & \psi^d \wedge \chi^d \\ (\psi \to \chi)^d & := & \psi \wedge \chi^d & & & \end{array}$$

Definition 5. *Let $\overline{x}$ and $\overline{y}$ be tuples of variables of the same length. A formula $\varphi = \varphi(\overline{x}, \overline{y}, \overline{p})$ with coefficients $\overline{p}$ is d-unifiable in the logic* $\mathbf{IPC}^+$ *if it is unifiable in* $\mathbf{IPC}^+$ *and there is a unifier ε of φ in* $\mathbf{IPC}^+$ *such that $\varepsilon(y_i) = (\varepsilon(x_i))^d$ for all $y_i \in \overline{y}$.*

In other words, a formula $\varphi(\overline{x}, \overline{y}, \overline{p})$ is d-unifiable if there is a tuple of formulas $\overline{\xi} = \xi_1, \ldots, \xi_m$ such that

$$\varphi(\overline{\xi}, \overline{\xi}^d, \overline{p}) \in \mathbf{IPC}^+,$$

where $\overline{\xi}^d$ denotes a tuple $\xi_1^d, \ldots, \xi_m^d$.

The notion of d-unification of a formula is a strengthening of that of unification. If we are interested whether a formula φ is d-unifiable, we assume that there is a dependence between variables of a formula, $\varphi = \varphi(\overline{x}, \overline{y}, \overline{p})$, and we are looking not for an arbitrary unifier of this formula, but for a unifier, which replace dependent variables x_i and y_i by a formula and its dual, ξ_i and ξ_i^d. It looks quite natural to be interested not in arbitrary unifiers, but in unifiers of a special kind. We may also consider different kinds of dependences between variables of a formula and different combinations of formulas substituted for dependent variables.

For formulas in *nnf* the unification problem in **N4** can be reduced to the d-unification problem in $\mathbf{IPC}^+$.

Proposition 3. *Let $\varphi(\bar{x}, \bar{y}, \bar{p}, \bar{q})$ be a positive formula. The formula*

$$\varphi(\bar{x}, \sim \bar{x}, \bar{p}, \sim \bar{p})$$

with parameters $\bar{p}$ is unifiable in **N4** *iff the formula*

$$\varphi(\bar{x}, \bar{y}, \bar{p}, \bar{p}^\dagger)$$

with parameters $\bar{p}$ and $\bar{p}^\dagger$ is d-unifiable in $\mathbf{IPC}^+$.

Proof. Assume that $\bar{\xi}$ is such that $\varphi(\bar{\xi}, \sim \bar{\xi}, \bar{p}, \sim \bar{p}) \in \mathbf{N4}$. According to Theorem 1 we conclude $(\varphi(\bar{\xi}, \sim \bar{\xi}, \bar{p}, \sim \bar{p}))^{\mathbf{4}} \in \mathbf{IPC}^+$. Taking into account that the translation $(\cdot)^{\mathbf{4}}$ commutes with positive connectives we obtain $\varphi((\bar{\xi})^{\mathbf{4}}, (\sim \bar{\xi})^{\mathbf{4}}, \bar{p}, \bar{p}^\dagger) \in \mathbf{IPC}^+$. By induction on the structure of formulas we can easily check that $(\sim \theta)^{\mathbf{4}} = (\theta^{\mathbf{4}})^d$, which allows us to conclude that

$$\varphi(\bar{\xi}^{\mathbf{4}}, (\bar{\xi}^{\mathbf{4}})^d, \bar{p}, \bar{p}^\dagger) \in \mathbf{IPC}^+.$$

We have thus proved that unifiability in **N4** implies d-unifiability in $\mathbf{IPC}^+$.

To prove the inverse implication assume that $\varphi(\bar{\xi}, \bar{\xi}^d, \bar{p}, \bar{p}^\dagger) \in \mathbf{IPC}^+$ for a tuple $\bar{\xi}$ of positive formulas.

For a positive formula θ, we denote by θ' the result of replacement of every variable $x_i^\dagger$ by $\sim x_i$ of every metavariable $p_j^\dagger$ by $\sim p_j$. It is clear that $(\theta')^{\mathbf{4}} = \theta$. Using this fact we can prove by induction on the structure of formulas that for every positive formula θ, we have $(\sim \theta')^{\mathbf{4}} = \theta^d$. From this fact and the definition of translation $(\cdot)^{\mathbf{4}}$ we obtain

$$(\varphi(\bar{\xi}', \sim \bar{\xi}', \bar{p}, \sim \bar{p}))^{\mathbf{4}} = \varphi(\bar{\xi}, \bar{\xi}^d, \bar{p}, \bar{p}^\dagger).$$

Consequently, $\varphi(\bar{\xi}', \sim \bar{\xi}', \bar{p}, \sim \bar{p}) \in \mathbf{N4}$. We have thus proved that the formula $\varphi(\bar{x}, \sim \bar{x}, \bar{p}, \sim \bar{p})$ is unifiable in **N4**. □

Every formula is equivalent in **N4** to a formula in *nnf*, and every formula in *nnf* has the form $\varphi(\bar{x}, \sim \bar{x}, \bar{p}, \sim \bar{p})$, where φ is a positive formula. This, we proved that the unification problem in **N4** is equivalent to the d-unification problem in $\mathbf{IPC}^+$.

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