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ON THE MULTIDIMENSIONAL PERMANENT AND q-ARY DESIGNS

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Abstract. An $H(n, q, w, t)$ design is a collection of some $(n-w)$-faces of the hypercube Q_q^n that perfectly pierce all $(n-t)$-faces $(n \geq w > t)$. An $A(n, q, w, t)$ design is a collection of some $(n-t)$-faces of Q_q^n that perfectly cover all $(n-w)$-faces. The numbers of H-designs and A-designs are expressed in terms of the multidimensional permanent. Several constructions of H-designs and A-designs are given and the existence of $H(2^{t+1}, s2^t, 2^{t+1}-1, 2^{t+1}-2)$ designs is proven for all $s, t \geq 1$.

Keywords: Steiner system, H-design, perfect matching, clique matching, MDS code, permanent.

1. Introduction

The H-design by Hanani [5] is a generalization of a Steiner system or t-design. The notation of H-design is due to Mills [8]. Let X be a set of points and let $C = \{C_1, \dots, C_n\}$ be a partition of X into n sets of cardinality q. A *transverse* of C is a subset of X meeting each set C_i at most in one point. The set of w-element transverses of C is an *$H(n, q, w, t)$ design* (briefly, H-design) if each t-element transverse of C lies in exactly one transverse of the H-design. We propose another generalization of t-design. A set of t-element transverses of C is an *$A(n, q, w, t)$ design* (briefly, A-design) if each w-element transverse of C contains exactly one transverse of the A-design. We imply everywhere that $n \geq w > t \geq 1$, $q \geq 1$ and all these numbers are integer. The idea of considering the A-designs belongs to S. V. Avgustinovich.

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Put $Q_q = \{0, 1, \dots, q-1\}$ and $Q_{q*} = Q_q \cup \{*\}$. It is clear that each w-transverse of C corresponds to the *codeword* $(*, \dots, a_1, \dots, *, \dots, a_i, \dots, *) \in Q^n_{q*}$ where a_i is the label of the element of C_i that belongs to the w-transverse. Position j of the codeword contains $*$ if and only if the w-transverse does not intersect C_j. Define the *weight* of a codeword from Q^n_{q*} as n minus the number of symbols $*$ contained in the codeword. Then the set H consisting of some vectors $x \in Q^n_{q*}$ of weight w is an $H(n, q, w, t)$ design if each vector $y \in Q^n_{q*}$ of weight t is covered by exactly one $x \in H$ i.e. we can obtain x from y replacing $w - t$ symbols $*$ with numbers. Analogously the set A consisting of vectors $y \in Q^n_{q*}$ of weight t is an $A(n, q, w, t)$ design if each $x \in Q^n_{q*}$ of weight w covers exactly one $y \in A$.

If $q = 1$ then an $H(n, 1, w, t)$ design is just a Steiner system $S(t, w, n)$. In order to get elements of $S(t, w, n)$ from elements of $H(n, 1, w, t)$ we need to replace $*$ by 0 and 0 by 1 (for example $(*, *, 0, *, 0, 0, *) \in H(7, 1, 3, 2)$ is converted to $(0, 0, 1, 0, 1, 1, 0) \in S(2, 3, 7)$). An $A(n, 1, w, t)$ design is just a Steiner system $S(n-w, n-t, n)$ too. In this case $*$ is replaced by 1 (for example $(*, 0, *, 0, 0, 0, *) \in A(7, 1, 5, 4)$ is converted to $(1, 0, 1, 0, 0, 0, 1) \in S(2, 3, 7)$). In [12] an H-design was called a q-ary Steiner system. A set T of $y \in Q^n_{1*}$ of weight t is an (n, w, t)-Turan system, if each $x \in Q^n_{1*}$ of weight w covers at least one $y \in T$. Hence an $A(n, 1, w, t)$ design is a special case of a (n, w, t)-Turan system.

The set Q^n_q is called the *hypercube*. The set of faces of Q^n_q is in one-to-one correspondence with Q^n_{q*} and each k-dimensional face (k-face) corresponds to the codeword with k symbols $*$. Thus an $H(n, q, w, t)$ design is a piercing consisting of $(n-w)$-faces of Q^n_q with the property that each $(n-t)$-face contains exactly one $(n-w)$-face of an H-design; and an $A(n, q, w, t)$ design is a covering consisting of $(n-t)$-faces of Q^n_q with the property that each $(n-w)$-face is contained in exactly one $(n-t)$-face of an A-design.

If $w = n$ then an $H(n, q, w, t)$ design is just an MDS code in Q^n_q with code distance $d = n - t + 1$. If $w = n$ and $t = n - 1$ then an $A(n, q, w, t)$ design is just a tiling of the hypercube by 1-faces. If $q = 2$ then this tiling is equivalent to a perfect matching[1] in Q^n_2. If $q > 2$ then $A(n, q, n, n-1)$ design is called a *perfect clique matching* (see [9]) because the 1-faces of Q^n_q one-to-one correspond to the maximal cliques in the hypercube. It is clear that $H(n, q, n, n-1)$ and $A(n, q, n, t)$ designs exist for all $q \geq 2$ and $n \geq 2$. A set of 1-faces is called a *precise clique matching* if it is both $H(n, q, n-1, n-2)$ design and $A(n, q, n, n-1)$ design. The precise clique matchings (and partitions into precise clique matchings) with $n = 2^{t+1}$ and $q = 2^t$ are constructed in [9].

Mills in [8] showed that for $n > 3$, $n \neq 5$ an $H(n, q, 4, 3)$ design exists if and only if nq is even and $q(n-1)(n-2)$ is divisible by 3. Ji in [6] proved that an $H(5, q, 4, 3)$ exists if q is even, $q \neq 2$, and $q \not\equiv 10, 26 (\mathrm{mod} 48)$.

Consider an $H(n, q, w, t)$ design as a constant-weight code. The Hamming distance[2] between two codewords of an H-design is always greater than $w - t$. The *code distance* of a design is the minimum Hamming distance between two codewords of this design. The code distance of $H(n, q, w, t)$ design is at most $2(w - t + 1)$. An $H(n, q, w, t)$ design that forms a code with minimum Hamming distance $2(w-t+1)$ is called a generalized Steiner system (see [3]). Note that an ordinary Steiner system ($H(n, 1, w, t)$ design) is always a code with Hamming distance $2(w - t + 1)$.

[1] Here we consider Q^n_2 as a minimal Hamming distance graph.

[2] Here we consider elements of H-design as words in alphabet $\{*, 0, \dots, q-1\}$.

Etzion in [3] obtained some series of constructions of generalized Steiner systems that are $H(n, q, 3, 2)$ or $H(n, 2, 4, 3)$ designs. He proved that a generalized Steiner system being an $H(n, 2, 3, 2)$ design exists if and only if $n \equiv 0$ or $1 \pmod 3$, $n \geq 4$, $n \neq 6$.

Similarly we can consider an $A(n, q, w, t)$ design with the maximum code distance. The code distance of an A-design is at most $1 + 2(w - t)$ (but it can be equal to 1). $A(n, 2, n, n - 1)$ designs with Hamming distance 2 were firstly constructed in [4] for every $n \geq 4$. Krotov [7] and Svanström [11] proved (in other terms) that $A(n, 2, n, n - 1)$ designs with Hamming distance 3 exist if and only if $n = 2^t$. It is straightforward that each $A(n, 2, n, t)$ design with Hamming distance $1 + 2(n - t)$ is a perfect ternary constant-weight code.

2. Constructions

In this section we consider some constructions of A- and H-designs and partitions of the set of m-faces into A- and H-designs. Denote by $Q^n_q(w)$ the set of $(n - w)$-faces of Q^n_q. Obviously $|Q^n_q(w)| = q^w \binom{n}{w}$. It is easy to calculate that cardinalities of $A(n, q, w, t)$ and $H(n, q, w, t)$ are equal to $\alpha(n, q, w, t) = q^t \frac{n!(w-t)!}{w!(n-t)!}$. At the same time the cardinality of partition of the set $Q^n_q(w)$ into $H(n, q, w, t)$ designs is equal to $\binom{n-t}{n-w} q^{w-t} = q^w \binom{n}{w} / \alpha(n, q, w, t)$ and the cardinality of partition of the set $Q^n_q(t)$ into $A(n, q, w, t)$ designs is equal to $\binom{w}{t} = q^t \binom{n}{t} / \alpha(n, q, w, t)$.

We propose the following constructions of H-designs.

Construction I. Let $S \subset Q^n_{q*}$ be an $H(n, q, w, t)$ design. For each $\alpha \in S$, we define an $H(w, q', w, t)$ design (MDS code) $R_\alpha \subset Q^w_{q'}$.
Given $\alpha = (*, \dots, a^1, \dots, *, \dots, a^i, \dots, * \dots, a^w, \dots, *) \in S$ and $(b_1, \dots, b_w) \in R_\alpha$ arrange the codeword $(*, \dots, (a^1, b_1), \dots, *, \dots, (a^i, b_i), \dots, *, \dots, (a^n, b_n), \dots, *) \in Q^n_{qq'*}$. Let T be the set of all these codewords.

Proposition 1. *T is an $H(n, qq', w, t)$ design.*

Proof. Take $c^i \in Q_q$ and $d^i \in Q_{q'}$, and let $(*, \dots, (c^1, d^1), \dots, *, \dots, (c^i, d^i), \dots, *, \dots, (c^t, d^t), \dots, *)$ be arbitrary elements of $Q^n_{qq'*}$ with weight t. By the definition of H-design there exists a unique codeword $\alpha \in S$ such that the $(n - w)$-face α is contained in the $(n-t)$-face $(*, \dots, c^1, \dots, *, \dots, c^i, \dots, *, \dots, c^t, \dots, *)$. Convert the codeword $(*, \dots, d^1, \dots, *, \dots, d^i, \dots, *, \dots, d^t, \dots, *) \in Q^n_{q'*}$ to the new word with length w removing a position i if $\alpha = (*, \dots, a^1, \dots, *, \dots, a^i, \dots, *, \dots, a^w, \dots, *)$ has $*$ in position i. So, we form a codeword $\overline{d} \in Q^w_{q'*}$. By the definition of H-design there exists a unique codeword $(b_1, \dots, b_w) \in R_\alpha$ such that $(b_1, \dots, b_w) \subset \overline{d}$. Then the set T is an $H(n, qq', w, t)$ design by definition. □

If we have partitions of the sets $Q^n_q(w)$ and $Q^w_{q'}(w)$ into $H(n, q, w, t)$ and $H(w, q', w, t)$ designs respectively then we obtain a partition of the set $Q^n_{qq'}(w)$ into $H(n, qq', w, t)$ designs by using Construction I for every pairs of $H(n, q, w, t)$ and $H(w, q', w, t)$ designs from this partitions.

As mentioned above, $H(2k, k, 2k - 1, 2k - 2)$ designs exist for $k = 2^t$, $t \geq 1$. Since MDS codes with distance 2 ($H(m, q, m, m - 1)$ designs) exist for all $q \geq 2$ and $m \geq 2$, we get

Corollary 1. *For all $s, t \geq 1$ there exist $H(2^{t+1}, s2^t, 2^{t+1} - 1, 2^{t+1} - 2)$ designs.*

Since partition of the sets $Q_{2^t}^{2^{t+1}}(2^{t+1}-1)$ into $H(2^{t+1},2^t,2^{t+1}-1,2^{t+1}-2)$ designs exists [9] it is possible to construct a partition of the set $Q_{s2^t}^{2^{t+1}}(2^{t+1}-1)$ into $H(2^{t+1},s2^t,2^{t+1}-1,2^{t+1}-2)$ designs for all $s,t\geq 1$.

The number of different $H(m,3,m,m-1)$ designs is $3\times 2^{m-1}$ (see [10]). A doubly exponential lower bound of the number of MDS codes with distance 2 ($q\geq 4$) was established in [10]. Thus we get

Corollary 2. *The number of $H(2^{t+1},s2^t,2^{t+1}-1,2^{t+1}-2)$ designs is double exponential with respect to the dimension 2^{t+1} as $s\geq 3$.*

Construction II. Let $S\subset Q_{q*}^n$ be an $A(n,q,w,t)$ design. For each pair of $(*,\dots,a^1,\dots,*,\dots,a^i,\dots,*,\dots,a^t,\dots,*)\in S$ and $(b_1,\dots,b_t)\in Q_{q'}^t$ we form the codeword $(*,\dots,(a^1,b_1),\dots,*,\dots,(a^i,b_i),\dots,*,\dots,(a^t,b_t),\dots,*)\in Q_{qq'*}^n$. Let U be the set of all these codewords.

Proposition 2. *U is an $A(n,qq',w,t)$ design.*

The proof is similar to that of Proposition 1.

As mentioned above, each Steiner system $S(n-w,n-t,n)$ is equivalent to an $A(n,1,w,t)$ design.

Corollary 3. *If there exists a Steiner system $S(n-w,n-t,n)$ then for each $q\geq 1$ there exists an $A(n,q,w,t)$ design.*

It is easy to construct a partition of the set $Q_q^n(t)$ into $A(n,q,w,t)$ designs from a partition of the layer of Boolean n-dimensional cube into Steiner systems $S(n-w,n-t,n)$.

Construction III. Let $S\subset Q_{q*}^n$ be an $A(n,q,n-1,n-2)$ design. Define $V=(S\times Q_q^n)\cup(Q_q^n\times S)$.

Proposition 3. *V is an $A(2n,q,2n-1,2n-2)$ design.*

Proof. Suppose that $(c_1,\dots,c_{i-1},*,c_{i+1},\dots,c_{2n})$ is a word of weight $2n-1$. If $i\leq n$ then there exists a unique codeword $\overline{a}\in S$ such that $(c_1,\dots,c_{i-1},*,c_{i+1},\dots,c_n)\subset\overline{a}$. It is clear that $(c_1,\dots,c_{i-1},*,c_{i+1},\dots,c_{2n})\subset(\overline{a},\overline{d})$ where $\overline{d}=(c_{n+1},\dots,c_{2n})$. The case $n<i\leq 2n$ is similar. ☐

3. Multidimensional permanent

In [1] Avgustinovich developed a method of counting the number of combinatorial configurations in terms of the multidimensional permanent. Consider a biregular bipartite graph $G=(L,R,E)$ with parts L and R. A set $C\subseteq L$ is called (L,R)-*perfect code* if for each $v\in R$ there exists only one vertex $u\in C$ such that u is adjacent to v. The definition of (R,L)-perfect code is obtained by changing parts L and R. It is easy to see that cardinalities of any (L,R)-perfect code and any (R,L)-perfect code of the same biregular bipartite graph are coincide.

Suppose that $\{C_1,\dots,C_k\}$ is a partition of L into (L,R)-perfect codes. We define the *adjacency array* $M(G,L)=(m_{i_1\dots i_k})$ by the following equation $m_{i_1\dots i_k}=|B_{i_1}^1\cap\dots\cap B_{i_k}^k|$ where $B_{i_j}^j$ is a neighborhood of the i_jth vertex of C_j. If there exists a partition of R consisted of (R,L)-perfect codes then it is possible to define an adjacency array $M(G,R)=(m_{i_1\dots i_k})$ by analogous way.

Let $M=(m_{i_1\dots i_k})$, $i_j\in\{1,\dots,N\}$, $j\in\{1,\dots,k\}$, be a k-dimensional array of real numbers. A k-element subset I of $\{1,\dots,N\}^k$ is called a *diagonal* if every pair of vectors $\bar{i}, \bar{j} \in I$ is distinct in each position that is $i_\sigma \neq j_\sigma$ for all $\sigma \in \{1, \ldots, k\}$. We define the *k-dimensional* permanent of M as

$$\mathrm{per}_k M = \sum_{I \in D_N} \prod_{(i_1,\ldots,i_k) \in I} m_{i_1 \ldots i_k},$$

where D_N is the set of all diagonals. The following statement is straightforward.

Proposition 4. *The number of* (R, L)*-perfect codes of* G *is equal to* $\mathrm{per}_k M(G, L)$.

Consider a k-partite hypergraph G_k containing N vertices in each part C_i, $i = 1, \ldots, k$. Suppose that each k-edge of G_k consists of k vertices, with one vertex in each part of the hypergraph. A set of disjoint k-edges that matches all vertices of the hypergraph is called a *perfect k-matching*. Let the vertices of each part C_i of the hypergraph be enumerated by $1, 2, \ldots, N$. We define the *adjacency array* $M(G_k) = (m_{i_1 \ldots i_k})$ by the following rule: $m_{i_1 \ldots i_k} = 1$ if there exists a k-edge consisting of vertices with numbers i_1 from the first part, i_2 from the second part and so on and $m_{i_1 \ldots i_k} = 0$ otherwise.

It is well known that the permanent of the adjacency matrix of a bipartite graph is equal to the number of perfect matchings of the graph. The following statement is straightforward.

Proposition 5. *The number of perfect* k*-matchings of a hypergraph* G_k *is equal to* $\mathrm{per}_k M(G_k)$.

It is clear that any biregular bipartite graph $G = (L, R, E)$ with partition $\{C_1, \ldots, C_k\}$ of the part L into (L, R)-perfect codes is equivalent to a k-partite regular hypergraph G_k with parts $C_1, \ldots, C_k$. Here k-edges of G_k correspond to vertices of the second part R of G and perfect k-matchings of G_k one-to-one correspond to (R, L)-perfect codes of G.

Given integers w, t $(n \geq w > t \geq 1)$, define the bipartite graph $G(n, q, w, t)$ with the parts $L = Q_q^n(w)$ and $R = Q_q^n(t)$. The pair of vertices $\bar{c} \in Q_q^n(w)$ and $\bar{b} \in Q_q^n(t)$ are connected by an edge in $G(n, q, w, t)$ if and only if $\bar{c} \subset \bar{b}$. By definition each $H(n, q, w, t)$ design is a subset of $Q_q^n(w)$ such that the neighborhoods of its vertices are not mutually intersected but cover $Q_q^n(t)$. We assume that there exists a partition $H = \{H_1, \ldots, H_k\}$, where $k = \binom{n-t}{n-w} q^{w-t}$, of $Q_q^n(w)$ into $H(n, q, w, t)$ designs.

Proposition 6. *The number of different* $A(n, q, w, t)$ *designs is equal to* $\mathrm{per}_k M(G(n, q, w, t), L)$.

Proof. Any A-design $B \subset Q_q^n(t)$ perfectly covers all $(n-w)$-faces. Then B is a (R, L)-perfect code of $G(n, q, w, t)$. Using Proposition 4, we obtain that the number of different $A(n, q, w, t)$ designs is equal to $\mathrm{per}_k M(G(n, q, w, t), L)$. □

By the definition each $A(n, q, w, t)$ design is a subset of $Q_q^n(t)$ such that its faces are not mutually intersected and cover $Q_q^n(w)$. Let us to assume that there exists a partition $A = \{A_1, \ldots, A_m\}$, where $m = \binom{w}{t}$, of $Q_q^n(t)$ into $A(n, q, w, t)$ designs.

Analogously to Proposition 6, we can prove the following

Proposition 7. *The number of different* $H(n, q, w, t)$ *designs is equal to* $\mathrm{per}_m M(G(n, q, w, t), R)$.

The constructions of Section 2 provide examples of the parameters such that there exists a partition into H-designs (A-designs). Thus we can calculate the numbers of H-designs or A-designs with these parameters using the multidimensional permanent.

As mentioned above $H(n,q,n,t)$ designs coincide with MDS codes with distance $d=n-t+1$. For any integers n,q,t there exists a partition of $Q_q^n(t)$ into $A(n,q,n,t)$ designs where every A-design consists of all "parallel"t-faces. Then the problem of existence of MDS codes with distance $d=n-t+1$ in Q_q^n is equivalent to the inequality $\mathrm{per}_m M(G(n,q,n,t),R)>0$. Ball [2] proved that linear MDS code over prime field $\mathbb{F}_q$ has length at most $q+1$ (except for the trivial cases $d=2,n$). The question of existence of nonlinear MDS codes of larger lengths is open.

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