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ON A FINITE 2,3-GENERATED GROUP OF PERIOD 12

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Abstract. Using calculations in computer algebra systems along with some theoretic results, we construct the largest finite group of period 12 generated by an element of order 2 and an element of order 3. In particular, we prove that this group has order $2^{66} \cdot 3^7$.

Keywords: periodic groups, Burnside problem

1. Introduction

A group G has period n if $x^n = 1$ for all $x \in G$ or, equivalently, if $\exp(G)$ is finite and divides n. Groups of period 12 are of interest in light of the Burnside problem.

There has been a recent progress in proving local finiteness of certain classes of groups of period 12. For example, groups of period 12 in which the product of every two involutions has order distinct from 6 (respectively, from 4) are locally finite by [1] and [2]. Groups of period 12 without elements of order 12 are locally finite by [3].

A group is 2,3-*generated* if it is a quotient of the free product $\mathbb{Z}_2 * \mathbb{Z}_3$. A 2,3-generated group of period 12 will be called a (2,3; 12)-*group*. It is not known if the free (2,3; 12)-group B is finite, and our aim is to study the finite quotients of B. The importance of this study lies in the fact that it represents the smallest unknown case among the 2-generator groups of period 12.

From the positive solution of the restricted Burnside problem, it easily follows that there exists a unique maximal finite (2,3; 12)-group, which we denote by $B_0(2,3;\,12)$. Every finite (2,3; 12)-group is a quotient of $B_0(2,3;\,12)$. The principal result is as follows.

Theorem 1. *Let $B_0 = B_0(2,3;\,12)$. Then the structure of B_0 is known. In particular, the following facts hold.*

- $|B_0| = 161\,372\,117\,156\,811\,157\,536\,768 = 2^{66} \cdot 3^7$.

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- *B_0 is solvable of derived length 4 and Fitting length 3; $Z(B_0) = 1$.*
- *The quotients of the derived series for B_0 are $\mathbb{Z}_6$, $\mathbb{Z}_{12}^2$, $\mathbb{Z}_2^{61}$, $\mathbb{Z}_3^4$.*
- *A Sylow 2-subgroup of B_0 has nilpotency class 5 and rank 7.*
- *A Sylow 3-subgroup of B_0 has nilpotency class 2 and rank 4.*
- *$\mathrm{O}_2(B_0)$ has order 2^{62} and nilpotency class 2; $|\Phi(B_0)| = 2^{54}$.*
- *$\mathrm{O}_{2,3}(B_0)/\mathrm{O}_2(B_0) \cong \mathbb{Z}_3^6$.*
- *$B_0/\mathrm{O}_{2,3}(B_0) \cong \mathrm{SL}_2(3) \circ \mathbb{Z}_4$; in particular, the 3-length of B_0 is 2.*
- *$\mathrm{O}_3(B_0) \cong \mathbb{Z}_3^4$.*
- *$\mathrm{O}_{3,2}(B_0)/\mathrm{O}_3(B_0)$ has order 2^{65} and nilpotency class 4.*
- *$B_0/\mathrm{O}_{3,2}(B_0) \cong 3^{1+2} : 2$; in particular, the 2-length of B_0 is 2.*

By knowing the structure of B_0 we mean that this group is constructed explicitly in the computer systems GAP [4] and Magma [5], and that we are able to prove that this group is indeed $B_0(2,3;\,12)$. We obtain B_0 as a homomorphic image of the finitely presented group

$$G = \langle\, a, b \mid 1 = a^2 = b^3 = w_1^{12} = \ldots = w_{21}^{12} \,\rangle, \tag{1}$$

where $w_1, \ldots, w_{21}$ are explicitly given words (4). The open question of whether G is finite is of interest, because a positive answer would readily imply that $B = B_0$.

A code for Magma that confirms various computation steps made in the paper can be downloaded from [6].

2. The algorithm

The idea is to first construct a certain "large" finite (2,3;12)-group and then prove its maximality. The construction part uses the "Solvable Quotient" algorithms available in GAP and Magma.

Let $F = \langle x, y\rangle$ be a free 2-generator group. The elements of F will be viewed as words in $\{x, y\}$. Let

$$B = B(2,3;\,12) = \langle a, b \mid 1 = a^2 = b^3 = w(a,b)^{12},\ \forall w \in F\rangle \tag{2}$$

be the free (2,3; 12)-group. Clearly, every (2,3; 12)-group is a quotient of B. The first observation is that we can simplify slightly the set of relators for B by leaving only the words w that can be expressed as monoid words in $s = ab$ and $t = ab^2$ and discarding some equivalent words, i. e., cyclic permutations and inversions, which do not change the group. Powers $w^2, w^3, \ldots$ of a fixed word w can also be discarded, since we know that $w^{12} = 1$. The remaining set of words ordered by length begins as follows

$$L = \{s; st; s^2t; s^3t, s^2t^2; s^4t, s^3t^2, (st)^2s; \ldots\} \tag{3}$$

and may be substituted for F in (2). Semicolons in (3) separate words of different lengths. The advantage of using the set L in place of F is that it is less redundant and straightforward to construct. We only sketch the proof of the above observation.

Proof. For every $w \in F$, we have

$$w(a,b) = b^{\varepsilon_0}ab^{\varepsilon_1}a \ldots ab^{\varepsilon_n} \sim ab^{\varepsilon_1}a \ldots b^{\varepsilon_{n-1}}ab^{\varepsilon'_n} = v(s,t)a^{\varepsilon}$$

for some $n \geqslant 0$, where $\varepsilon_0, \varepsilon_n = 0,1,2$; $\varepsilon_1, \ldots \varepsilon_{n-1} = 1,2$; $\varepsilon'_n \equiv \varepsilon_n + \varepsilon_0 \pmod 3$; v is a suitable element of F; the symbol "$\sim$" means a cyclic permutation; and $\varepsilon = 0, 1$. The trailing letter "a", if present, can be eliminated by a (repeated) cyclic permutation that moves first letter in $v(s,t)$ to the end and then an application of one of the identities

$$sas = t, \quad tat = s, \quad sat = a, \quad tas = a.$$

This is always possible, unless we are left with one of the words a, sa, ta, which are equivalent to either a or b and hence can be discarded as well. On the remaining words in $\{s,t\}$ of a given length l, a dihedral group D_{2l} acts by effecting cyclic permutations and inversion followed by conjugation by a. (Group inversion alone does not map the set of monoid words in s and t to itself.) Orbit representatives of this action for $l \geqslant 1$, except for proper powers, are collected in (3). □

We now briefly explain the idea behind the algorithm for finding a large finite $(2,3;\,12)$-group. Starting off with the group

$$G = \langle a, b \mid 1 = a^2 = b^3 = r^{12},\ \forall r \in R\rangle,$$

where R is initially an empty set, and a known finite $(2,3;\,12)$-quotient G_0 of G, say $G_0 = \mathbb{Z}_6$, we apply the "Solvable Quotient" method to find a bigger quotient G_1 of order $2^e \cdot |G_0|$ successively for $e = 1, 2, \dots$ until we either succeed or e exceeds the maximal exponent max_e. If the found quotient has period 12 then we replace G_0 with G_1 and start over. Otherwise, we search through the above list L for a word w such that $w^{12} \neq 1$ in G_1 and add this word in R to start with a new group G. The previously found G_0 is still a quotient of G, but G_1 no longer is, because of the new relator w^{12} for G. A more formalized version of this algorithm written in a meta-language is given below.

```
Input:
  l := 10; // Maximal length in the alphabet {s,t} of elements in L
  L := [s, s*t, s^2*t, ... ]; // Candidates for relators of length at most l
  d0 := 6; // Known order of a (2,3;12)-quotient G_0 of G
  R:=[]; // Found words w for relators w^12
Multiplier:
  p:=2 or 3;  e:=1; // Searching for a new quotient of G_0 of order d_0 * p^e
  max_e:=10; max_t:=100 h; max_m:=16 G; // Maximal exponent e, time, memory
Start:
  G := < a,b | 1 = a^2 = b^3 = w^12, w in R >;
Quotient:
  d1 := d0 * p^e; // New order of a quotient to search for
  G1 := SolvableQuotient(G, d1); // Invoking "Solvable Quotient" routine
  If time > max_t Or memory > max_m  Then -> Output;
  If G1 = fail Then { e := e+1; If  e > max_e Then -> Output;
                                Else -> Quotient };
  If period of G1 is 12  Then { G0 := G1; d0 := d1; e := 1;
                                          -> Quotient }
  Else {
Search:
  find w in L : w(a,b)^12 <> 1 in G1;
  If found Then { add w to R; e := 1; -> Start }
  Else { l:=l+1;  add words of length l to L; -> Search } };
Output:
  return G0;
```

When the search stops (which happens either if the maximal exponent `max_e` is exceeded or due to memory/time reasons), we may continue, if necessary, from section `Multiplier` switching the value of p to 3, or back to 2.

This algorithm, despite its limitations, allowed us to construct a large $(2,3;\,12)$-group. The calculations in GAP yielded a group G_0 of order $2^{24}\cdot 3^7$ but exceeded maximal time `max_t` when trying to find a quotient of order $2\cdot|G_0|$. The calculations in Magma yielded a group G_0 of order $2^{66}\cdot 3^7$ and found no larger quotient of order up to $2^{10}\cdot|G_0|$, but exceeded maximal memory `max_m` when searching for a quotient of order $3\cdot|G_0|$. We assume henceforth that G_0 is the latter group of order $2^{66}\cdot 3^7$.

Observe that the set R of found relators consists of the 12th powers of following 21 words:

$$\begin{gathered}\{\ s;\ st;\ s^3t,\ s^2t^2;\ s^4t,\ s^3t^2,\ (st)^2s;\ s^4t^2,\ s(st)^2t;\ s^5t^2,\\ s^3(st)^2,\ s^2(st)^2t,\ s^3t^2st;\ s^4t^4,\ s^2(st)^2t^2,\ s^2(st^2)^2;\\ s^4(st)^2t,\ s^3(st)^3,\ s^5t^2st,\ s^2(st)^3t;\ s^2(st)^2t^2st\ \},\end{gathered} \tag{4}$$

where $s = ab$ and $t = ab^2$.

3. Maximality

We now address the problem of proving the maximality of the above $(2,3;\,12)$-group G_0 of order $2^{66}\cdot 3^7$. Suppose that there is a larger finite $(2,3;\,12)$-group E. Due to the uniqueness mentioned before Theorem 1, we may assume that E is a 2, 3-*extension*

$$1 \to V \to E \xrightarrow{\pi} G_0 \to 1, \tag{5}$$

which means that the 2, 3-generating pair of E is mapped by π to the chosen 2, 3-generating pair of G_0. Moreover, we may assume that V is an elementary abelian p-group (with $p = 2, 3$) and is irreducible as an $\mathbb{F}_pG_0$-module in a natural way. We want to reduce the possibilities for V.

Let p a prime. A group X has *p-period p^l* if every p-element of X has order dividing p^l. An *invariant section* of X is a section of a normal series for X. The order of $x \in X$ is denoted by $|x|$.

Lemma 1. *Let H be a periodic group of p-period p^l and let V be a p-elementary abelian invariant section of H viewed as an $\mathbb{F}_pH$-module. Then, for every $h \in H$ of order p^l, the element*

$$h_0 = 1 + h + h^2 + \ldots + h^{|h|-1}$$

effects a zero linear transformation of V.

Proof. Let $V = K/N$ for suitable $K, N \trianglelefteqslant H$. Assume to the contrary that $\overline{v}h_0 \neq 0$ for some $\overline{v} \in V$. Observe that $\langle h\rangle \cap K = 1$. Indeed, otherwise, the element $h^{p^{l-1}} \in K$ induces the identity transformation of V and $h_0 = p\cdot h_1 = 0$, where $h_1 = 1+h+\ldots+h^{p^{l-1}-1}$, contrary to the assumption. Now, let $v \in K$ be a preimage of $\overline{v}$. Since $\langle h\rangle \cap K = 1$, we see that $|hv| = |h|\cdot|v_0|$, where

$$v_0 = (hv)^{|h|} = v^{|h|-1}\cdot v^{|h|-2}\cdot\ldots\cdot v.$$

Clearly, the image of v_0 in V is $\overline{v}h_0 \neq 0$, which implies that $|v_0|$ is a multiple of p. Therefore, $|hv|$ is a multiple of p^{l+1}, a contradiction. $\square$

Getting back to the extension (5) of G_0, we have the following restriction on the $\mathbb{F}_p G_0$-module V.

Corollary 1. *(i) In case $p = 3$, the action of all elements of G_0 of order 3 on V is* quadratic*, i.e. annihilated by the polynomial $x^2 + x + 1$.*

(ii) In case $p = 2$, the action of all elements of G_0 of order 4 on V is cubic*, i.e. annihilated by the polynomial $x^3 + x^2 + x + 1$.*

Let $p = 2$ and let $\mathcal{X}$ be a representation of G_0 corresponding to the module V. Then $\mathrm{O}_2(G_0) \leqslant \mathrm{Ker}\mathcal{X}$ and so V is naturally a $G/\mathrm{O}_2(G_0)$-module. The quotient $G/\mathrm{O}_2(G_0)$ has small enough order, $2^4 \cdot 3^7$, to make it possible to find explicitly all irreducible modules over $\mathbb{F}_2$ and check which of them have cubic action of element of order 4. There are five such modules $V_i^{(2)}$, $i = 1, \dots, 5$ listed in the first part of Table 1.

TABLE 1. $\mathbb{F}_p G_0$-modules with quadratic and cubic action

p	2					3		
V	$V_1^{(2)}$	$V_2^{(2)}$	$V_3^{(2)}$	$V_4^{(2)}$	$V_5^{(2)}$	$V_1^{(3)}$	$V_2^{(3)}$	$V_3^{(3)}$
$\dim V$	1	2	2	4	6	1	1	4
absol. irred.	+	−	+	−	+	princ.	+	−
$\dim H^2(G_0, V)$	14	24	12	22	34	3	4	6

In the case $p = 3$, we cannot hope to find all irreducibles for $G/\mathrm{O}_3(G_0)$ because of its sheer order $2^{66} \cdot 3^3$ and so need another strategy. For $K = \mathbb{F}_3$ and $B = B(2, 3;\, 12)$, define

$$W = KB/(x^8 + x^4 + 1 \mid x \in B). \tag{6}$$

The polynomial $x^8 + x^4 + 1$ here "encodes" the quadratic action. Namely, if $|x|$ divides 4 then $x^8 + x^4 + 1 = 0$, otherwise, $y = x^4$ has order 3 and $x^8 + x^4 + 1 = y^2 + y + 1$. Thus, we may view W as the free cyclic KB-module with quadratic action of the elements of B of order 3. An analog of the following lemma was proved by A. S. Mamontov without computer help.

Lemma 2. *The KB-module W is finite-dimensional. Namely,* $\dim W = 16$.

Proof. Define the 2-generator associative K-algebra with 1

$$A = \langle\, m, n \;\mid\; 0 = m^2 - 1 = n^2 + n + 1 = (mn)^8 + (mn)^4 + 1 \,\rangle.$$

Clearly, as a K-algebra, W is a homomorphic image of A under the homomorphism that extends the map $\varphi : (m, n) \mapsto (\overline{a}, \overline{b})$, where $(\overline{a}, \overline{b})$ is the image in W of the generating 2,3-pair (a, b) of B. We prove that φ is in fact an isomorphism. Using the Magma method "FPAlgebra" for constructing finitely presented algebras it is readily verified that $\dim A = 16$. Now, the elements $m, n \in A$ are invertible and the group $A_0 = \langle m, n\rangle$ can be checked to have order 432, exponent 12, and quadratic action on A of the elements of order 3. This implies that A_0 is a homomorphic image of B and A is a (cyclic) KB-module with quadratic action. Since W is a free KB-module with these properties, there exists a homomorphism $W \to A$ which is clearly an inverse of φ. □

As every irreducible module is cyclic, we conclude that V is an irreducible homomorphic image of the free module W. It can be checked that up to isomorphism W has only three irreducible factors $V_i^{(3)}$, $i = 1, 2, 3$, which are listed in the second part of Table 1.

We remark that there is no known analog of Lemma 2 that could be used to find the modules with cubic action in characteristic 2.

4. 2,3-extensions

Now that we have restricted the possibilities for V in (5), we need to check, for every possible extension E with a given V, if E is 2,3-generated of period 12. Since E can be constructed as a polycyclic group and Magma is efficient in calculating the exponent of a polycyclic group, the verification of whether E has period 12 presents no problem (in fact, if E is a *split* extension, it will automatically have period 12 due to the restrictions on V). Therefore, we will concentrate on finding sufficient conditions for E to be a 2,3-extension of G_0.

Lemma 3. *Let the extension* (5) *be nonsplit, where* $G_0 = \langle a, b\rangle$ *and* V *is irreducible. Then* E *is a* $|a|, |b|$*-extension if and only if there are preimages under* π *of* a *and* b *of orders* $|a|$ *and* $|b|$*, respectively.*

Proof. Let $\hat{a}$, $\hat{b}$ be such preimages and let $\hat{E} = \langle \hat{a}, \hat{b}\rangle$. We have $E = \hat{E}V$ and $\hat{E} \cap V \neq 0$, since E is nonsplit. But V is irreducible, which yields $\hat{E} \cap V = V$ and so $\hat{E} = E$. The converse holds by definition. □

There are different ways to check whether $\pi^{-1}(a)$ contains an element of order $|a|$. Since $|\pi^{-1}(a)| = |V|$, for small modules V, one could use exhaustive search through all preimages. However, the following is a more conceptual approach which works for larger modules, too.

Let $\tau : G_0 \to E$ be a transversal in (5), i.e., $\pi \circ \tau = \mathrm{id}_{G_0}$. If we choose τ such that $\tau(1) = 1$ then the corresponding 2-cocycle $\gamma : G_0 \times G_0 \to V$, which is defined by

$$\gamma(g_1, g_2) = \tau(g_1g_2)^{-1}\tau(g_1)\tau(g_2), \quad g_1, g_2 \in G_0, \tag{7}$$

will be *normalized*, i.e. $\gamma(1, 1) = 0$. (We use additive notation in V.) The set $Z^2_N(G_0, V)$ of all normalized 2-cocycles is a subspace in $Z^2(G_0, V)$. We also define $B^2_N(G_0, V) = Z^2_N(G_0, V) \cap B^2(G_0, V)$ to be the set of normalized 2-coboundaries.

Lemma 4. *Let* G_0 *be a finite group, let* V *be a* KG_0*-module over a field* K*, and let an extension* (5) *be defined by a 2-cocycle* $\gamma \in Z^2_N(G_0, V)$*. For* $g \in G_0$*, define*

$$\psi_g(\gamma) = \gamma(g, g) + \gamma(g, g^2) + \ldots + \gamma(g, g^{|g|-1}),$$
$$g_0 = 1 + g + \ldots + g^{|g|-1}.$$

Then

(*i*) $\exists\ \hat{g} \in \pi^{-1}(g) : \ |\hat{g}| = |g| \quad \Leftrightarrow \quad \psi_g(\gamma) \in \mathrm{Im}(g_0)$.
(*ii*) $\forall\ \hat{g} \in \pi^{-1}(g) \quad |\hat{g}| = |g| \quad \Leftrightarrow \quad \psi_g(\gamma) = 0$ *and* $\mathrm{Im}(g_0) = 0$.
(*iii*) *If* $\mathrm{Im}(g_0) = 0$ *then* $B^2_N(G_0, V) \subseteq \mathrm{Ker}\,\psi_g$*. In particular,* ψ_g *induces a* K*-linear map* $\overline{\psi}_g : H^2(G_0, V) \to V$*.*

Proof. Let $\tau : G_0 \to E$ be a transversal in (5) such that $\tau(1) = 1$. Then $\pi^{-1}(g) = \{\tau(g)v \mid v \in V\}$. We see that $(\tau(g)v)^{|g|} = \tau(g)^{|g|} + vg_0$ is the zero of V if and only if $\tau(g)^{|g|} = -vg_0$. An inductive application of (7) yields $\tau(g)^{|g|} = \tau(1)\psi_g(\gamma) = \psi_g(\gamma)$. These remarks imply (i) and (ii).

We now prove (iii). Let $\gamma_f \in B^2_N(G_0, V)$, where $f : G \to V$ satisfies $f(1) = 0$. This means that $\gamma_f(g_1, g_2) = f(g_1g_2) - f(g_1) \cdot g_2 - f(g_2)$ for all $g_1, g_2 \in G$. We have

$$\begin{aligned} \gamma_f(g, g) &= f(g^2) - f(g) \cdot g - f(g), \\ \gamma_f(g, g^2) &= f(g^3) - f(g) \cdot g^2 - f(g^2), \\ &\dots \\ \gamma_f(g, g^{|g|-1}) &= f(1) - f(g) \cdot g^{|g|-1} - f(g^{|g|-1}). \end{aligned}$$

Summing up gives $\psi_g(\gamma_f) = -f(g) \cdot g_0 = 0$, because $\mathrm{Im}(g_0) = 0$. The claim follows. □

The meaning of Lemmas 3 and 4(i) is that they reduce checking the $2,3$-generation of nonsplit extensions to a linear calculation in the module V. Here is an analogous result for split extensions.

Lemma 5. *Let $G_0 = \langle a, b\rangle$ be a finite group and let V be an irreducible finite-dimensional KG_0-module over a field K. Denote by E the natural semidirect product of G_0 and V, and set $a_0 = 1 + a + \dots a^{|a|-1}$, $b_0 = 1 + b + \dots b^{|b|-1}$. Then E is an $|a|, |b|$-extension of G_0 if and only if*

$$\dim \mathrm{Ker}\, a_0 + \dim \mathrm{Ker}\, b_0 - \dim V > \dim H^1(G_0, V) - \dim H^0(G_0, V).$$

Proof. For $v \in V$, we have $(av)^{|a|} = va_0$, which is the zero of V if and only if $v \in \mathrm{Ker}\, a_0$, and similarly for b. Hence, the number of $|a|, |b|$-pairs of elements of E that cover (a, b) is $|\mathrm{Ker}\, a_0| \cdot |\mathrm{Ker}\, b_0|$. If (a_1, b_1) is such a pair then $\langle a_1, b_1\rangle V = E$ and, due to the irreducibility of E, we see that $\langle a_1, b_1\rangle$ either equals E or is a complement to V. Conversely, every complement G_1 uniquely determines an $|a|, |b|$-pairs above (a, b). Hence, the number of generating $|a|, |b|$-pairs is the difference between the number of all pairs and the number of complements. Observe that the number of complements that are conjugate to a fixed one, G_1, equals

$$|E : N_E(G_1)| = |V : C_V(G_1)| = |V : C_V(G_0)| = |V|/|H^0(G_0, V)|$$

and does not depend on the complement. Also, it is well known that there are $|H^1(G_0, V)|$ conjugacy classes of complements to V in E. Hence, there are a total of

$$|V| \cdot |H^1(G_0, V)|/|H^0(G_0, V)|$$

complements. Passing to dimensions, we obtain the required inequality. □

5. The search

The results in the previous section can be used to check the $2,3$-generation of all extensions (5) with a given irreducible module V by running through the elements of $H^2(G_0, V)$, unless the order $|H^2(G_0, V)|$ is too big. This was done for all modules V from Table 1 except for $V_2^{(2)}$ and $V_5^{(2)}$, and no $2,3$-extensions of G_0 of period 12 were found. For the excluded two modules, this method would require searching through as many as 2^{24} and 2^{36} extensions, respectively. In these cases, we do the elimination in a different way, which we briefly explain.

Let $\mathcal{X}$ be the representation corresponding to one of the excluded modules V. Observe that the element $g_1 = ab \in G_0$ has order 12. Hence, a necessary condition for E to be of period 12 is that all elements in $\pi^{-1}(g_1)$ have order 12. Since $\mathcal{X}(g_0) = 0$, where $g_0 = 1 + g_1 + \ldots + g_1^{|g_1|-1}$, Lemma 4$(ii), (iii)$ implies that the extensions E satisfying this condition are defined by the elements of $\operatorname{Ker} \overline{\psi}_{g_1}$. This kernel may turn out to be a proper subspace of $H^2(G_0, V)$, in which case the dimension of the extension space to search in is reduced. We may repeat this procedure by taking a new element $g_2 \in G_0$ of order divisible by 4 which should be, in a sense, "independent" of g_1 and attempt to reduce the dimension further. It turns out that the (images in G_0 of) elements of L in (3), which we used as candidates for relators, are also good candidates for such independent elements of G_0.

A more formalized version of this procedure is given below.

```
Input:
  V; // A G0-module in characteristic 2
  L0; // The list of first 20 elements of L whose image in G0 has order 4 or 12
Start:
  H2 := H^2(G0,V); // The second cohomology group
  dim_e := Dimension(H2); // Initial dimension of the extension space
  bas_m := MatrixBasis(H2); // Basis in matrix form
Cycle:
   For g in L0
   { psi_g := [ Sum( <g,g^i> @ TwoCocycle(v) : i in [1 .. Order(g)-1] )
                | v in bas_m
               ];              // psi_g in matrix form
     If psi_g = 0
     Then print "No extensions eliminated for g = ", g;
     Else { bas_m := Kernel(psi_g) * bas_m; // Reducing the extension space
            dim_e := NumberOfRows(bas_m);
            print "Dimension of extension space reduced to ",dim_e; }
   }
```

As a result, we found that, as g runs through the first few ($\leqslant 20$) such elements of L, the dimension of $\cap_g \operatorname{Ker} \overline{\psi}_g$ is 2 for $V = V_2^{(2)}$ and 0 for $V = V_5^{(2)}$. This left us to consider only the split extensions for both modules and three inequivalent (but isomorphic) nonsplit extensions for $V = V_2^{(2)}$. All these, despite having period 12, were found not to be 2, 3-generated. This final elimination proves that G_0 is in fact B_0 as claimed in Theorem 1.

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References

[1] D.V. Lytknia, V.D. Mazurov, A.S. Mamontov, *On local finiteness of some groups of period* 12, Sib. Math. J., **53**:6 (2012), 1105–1109. MR3074446

[2] V.D. Mazurov, A.S. Mamontov, *Involutions in groups of exponent* 12, Algebra and Logic, **52**:1 (2013), 67–71. MR3113800

[3] A.S. Mamontov, *Groups of period* 12 *without elements of order* 12, Sib. Math. J., **54**, N1 (2013), 114–118. MR3089334

[4] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.6.4; 2013. (see `http://www.gap-system.org`)

[5] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput., 24 (1997), 235–265. MR1484478

[6] `http://www.ime.usp.br/~zavarn/progs/b0.magma`

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