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# FINITE ALMOST SIMPLE 5-PRIMARY GROUPS AND THEIR GRUENBERG-KEGEL GRAPHS

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Abstract. The finite almost simple 5-primary groups and their Gruenberg-Kegel graphs are determined. In addition, a list of finite simple 5-primary groups is essentially refined.

**Keywords:** finite almost simple group, prime graph, 5-primary group.

## 1. Introduction

Let $G$ be a finite group. Denote by $\pi(G)$ the set of prime divisors of the order of $G$. If $|\pi(G)| = n$ then the group $G$ is called $n$-primary. The prime graph (Gruenberg-Kegel graph) $\Gamma(G)$ of $G$ is defined as a graph with the vertex set $\pi(G)$, in which two different vertices $p$ and $q$ are adjacent if and only if there exists an element of order $pq$ in $G$.

Many investigators in the finite group theory are interested by various particular cases of the general problem of the study of finite groups by the properties of their Gruenberg-Kegel graphs. In the frame of this general problem, our attention draw first of all a more detailed study of the class of finite groups with disconnected prime graph. In fact, this class generalizes widely the class of finite Frobenius groups as is obvious from the well-known stuctural Gruenberg-Kegel theorem on finite groups with disconnected prime graph (see [27]). And Frobenius groups occupy

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an absolutely exceptional place in the finite group theory. Note also that the class of finite groups with disconnected prime graph coincides with the class of finite groups having an isolated subgroup (i. e., a proper subgroup containing the centralizer of any its nontrivial element) which have studied by many known algebraists (Frobenius, Suzuki, Feit, Thompson, G. Higman, Arad, Chillag, Busarkin, Gorchakov, Podufalov and others).

Finite simple groups with disconnected prime graph are determined in the papers of Williams [27] and the author [14]. They compose sufficiently restricted class of all finite simple groups, but include many "small" in various senses groups which arise often in the investigations. For example, all finite simple groups of exceptional Lie type, besides the groups $E_7(q)$ for $q > 3$, and also all finite simple groups from well-known "Atlas of finite groups" [7], besides the group $A_{10}$, have disconnected prime graph. The classification of connected components of prime graph for finite simple groups, obtained in [27, 14], were applied by Lucido [23] for the obtaining an analogous classification for all finite almost simple groups, i. e., groups with nonabelian simple socle. The problem of the study of finite groups with disconnected prime graph, which are not almost simple, is solved for several particular cases only, because here some nontrivial problems related with modular representations of finite almost simple groups arise (see the author's survey [15]).

In the frame of above-mentioned problem, the first author and Khramtsov in [16, 17, 18, 19] studied finite groups having disconnected prime graph whose the number of vertices is at the most 4. We continue these investigations with the purpose of the study of finite 5-primary groups with disconnected prime graph. In the given work, we make a necessary preliminary step for this by the determining the finite almost simple 5-primary groups and their Gruenberg-Kegel graphs. The following theorem is proved.

**Theorem 1.** *Gruenberg-Kegel graphs of finite almost simple* 5*-primary groups are given in Table.*

Table is given in Section 3.

As an immediate corollary of Theorem 1, the following theorem having an independent interest is obtained.

**Theorem 2.** *A finite almost simple group $G$ is* 5*-primary if and only if one of the following statements holds:*

(1) *the socle of $G$ is isomorphic to one of the groups $A_{11}$, $A_{12}$, $L_2(q)$ for $q \in \{2^6, 2^8, 2^9, 5^3, 5^4, 7^3, 7^4, 7^7, 17^2, 17^3\}$, $L_3(9)$, $L_3(27)$, $L_4(q)$ for $q \in \{4, 5, 7\}$, $L_5(2)$, $L_5(3)$, $L_6(2)$, $U_3(q)$ for $q \in \{16, 17, 25, 81\}$, $U_4(q)$ for $q \in \{4, 5, 7, 9\}$, $U_5(3)$, $U_6(2)$, $S_4(q)$ for $q \in \{8, 16, 17, 25, 49\}$, $S_6(3)$, $S_8(2)$, $O_7(3)$, $O_8^+(3)$, $O_8^-(2)$, $G_2(q)$ for $q \in \{4, 5, 7, 8\}$, $M_{22}$, $J_3$, $HS$, $He$ or $M^cL$;*

(2) *$G \cong L_2(2^p)$, where $p$ is a prime, $p \geq 11$ and $|\pi(2^{2p} - 1)| = 4$;*

(3) *$G \cong Aut(L_2(2^p))$, where $p$, $2^p - 1$ and $(2^p + 1)/3$ are some different primes and $p \geq 7$;*

(4) *$G \cong Aut(L_2(3^p))$ or $O^2(Aut(L_2(3^p)))$, where $p$ and $(3^p - 1)/2$ are primes, $p \geq 5$, $|\pi((3^p + 1)/4)| = 1$;*

(5) *$G \cong L_2(p)$ or $PGL_2(p)$, where $p$ is a prime, $p \geq 41$ and $|\pi(p^2 - 1)| = 4$;*

(6) *the socle of $G$ is isomorphic to $L_2(p^2)$, where $p$ is a prime, $p \geq 11$, $|\pi(p^2 - 1)| = 3$ and $p^2 + 1 = 2t$ or $2t^2$ for an odd prime $t$;*

(7) *$G$ is isomorphic to $L_2(p^r)$ or $PGL_2(p^r)$, where $p \in \{3,5,7,17\}$, $r$ is a prime, $3 < r \neq p$ and $|\pi(p^{2r}-1)| = 4$;*

(8) *$U_3(2^p) \leq G \leq PGU_3(2^p):2$, where $p \geq 5$ and $2^p-1$ are odd primes, $|\pi((2^p+1)/3)| = |\pi((2^{2p}-2^p+1)/3)| = 1$;*

(9) *the socle of $G$ is isomorphic to $L_3^{\epsilon}(p)$, where $\epsilon \in \{+,-\}$, $p$ is a prime, $17 \neq p \geq 11$, $|\pi(p^2-1)| = 3$, and $|\pi(\frac{p^2+\epsilon p+1}{(3,p-\epsilon 1)})| = 1$;*

(10) *$G \cong S_4(p)$ or $PGSp_4(p)$, where $p$ is a prime, $p \geq 11$, $|\pi(p^2-1)| = 3$ and $p^2+1) = 2r$ or $2r^2$ for an odd prime $r$;*

(11) *$G \cong Sz(2^p)$, where $p$ and $2^p-1$ are primes, $p \geq 7$ and $|\pi(2^{2p}+1)| = 3$;*

(12) *$G \cong Aut(Sz(8))$.*

As the first corollary of Theorem 2, a list of finite simple 5-primary groups obtained in [12, 28] is essentially refined. The results of this paper show that finite simple 5-primary groups besides the groups $L_4(q)$ for $q \in \{4,7\}$ and $U_4(q)$ for $q \in \{4,5,7,9\}$ have disconnected prime graph.

In [12] the following Problem 3.12 was posed : *For which power primes $q$ does $q^2-1$ have at most five different prime divisirs?* The case of this problem when $|\pi(q^2-1)| \leq 2$ is very known (see Lemma 1). The case when $|\pi(q^2-1)| = 3$ is considered in [17] (see Lemma 2). As the second corollary of Theorem 2, we obtain the following result for the case $|\pi(q^2-1)| = 4$.

**Theorem 3.** *Let $q$ be a natural power of a prime $p$ and $|\pi(q^2-1)| = 4$. Then one of the following statements holds:*

(1) *$q \in \{2^6,\ 2^8,\ 2^9,\ 5^3,\ 5^4,\ 7^3,\ 7^4,\ 7^7,\ 17^2,\ 17^3\}$;*

(2) *$q = 2^r$, $r$ is a prime and $p \geq 11$;*

(3) *$q = p$ and $p \geq 41$;*

(4) *$q = p^2$, $p \geq 11$, $|\pi(p^2-1)| = 3$ and $|\pi(\frac{p^2+1}{2})| = 1$;*

(5) *$q = p^r$, $p \in \{3,\ 5,\ 7,\ 17\}$ and $r$ is a prime coprime to $q(q^2-1)$.*

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## 2. Notation and preliminary results

We use mainly standard notation and terminology (see [2, 5, 7, 10]). In particular, if $A$ and $B$ are groups, $p$ is a prime and $n$ is a natural number then we use the following notation: $\pi(n)$ for the set of all prime divisors of $n$, $n_p$ for the maximal power of $p$ dividing $n$, $Z_n$ (or simply $n$) for a cyclic group of order $n$, $p^n$ for the elementary abelian $p$-group of order $p^n$, $A.B$ for an extension of $A$ by $B$, $A:B$ ($A \leftthreetimes B$) for a split extension (semi-direct product) of $A$ by (with, on) $B$, $Lie(p)$ for the set of finite simple groups of Lie type over fields of characteristic $p$. If $K \in Lie(r)$ then $Inndiag(K)$ is the group generated by all inner and diagonal automorphisms of $K$ and $Outdiag(K) = Inndiag(K)/Inn(K)$. Moreover, $L_n^{\varepsilon}(q)$ denotes $L_n(q) = PSL_n(q) \cong A_{n-1}(q)$ for $\varepsilon = +$ and $U_n(q) = PSU_n(q) \cong {}^2A_{n-1}(q)$ for $\varepsilon = -$; $S_{2n}(q) = PSp_{2n}(q) \cong C_n(q)$, $PGL_n^{\varepsilon}(q) = Inndiag(L_n^{\varepsilon}(q))$ ($\varepsilon \in \{+,-\}$), $PGSp_{2n}(q) = Inndiag(S_{2n}(q))$.

Assume that $\overline{G}$ is a connected simple adjoint linear algebraic group defined over an algebraically closed field of a positive characteristic $p$. Let $\sigma$ be an surjective endomorphism of $\overline{G}$ such that $\overline{G}_\sigma = C_G(\sigma)$ is finite. Then $G = O^{p'}(\overline{G}_\sigma)$ is a finite group of Lie type (which is simple besides 8 known cases) and $\overline{G}_\sigma \cong Inndiag(G)$.

Moreover, all finite adjoint groups of Lie type both split and twisted can be derived in this way. If $\overline{T}$ is a $\sigma$-stable maximal torus of $\overline{G}$ then $\overline{T}\cap\overline{G}_\sigma$ (respectively $\overline{T}\cap G$) is called a maximal torus of $\overline{G}_\sigma$ (respectively $G$).

For a graph $\Gamma$, $E(\Gamma)$ is the set of its edges.

If $q$ is a natural number, $r$ is an odd prime and $(q,r)=1$, then $e(r,q)$ denotes *multiplicative order of $q$ modulo $r$*, i. e., the minimal natural number $m$ with $q^m\equiv 1\ (mod\ r)$. If $q$ is odd, set $e(2,q)=1$ for $q\equiv 1\ (mod\ 4)$ and $e(2,q)=2$ for $q\equiv -1\ (mod\ 4)$.

Define the following function $\nu$ from $\mathbb{N}$ to $\mathbb{N}$:

$$\nu(m)=\begin{cases} m & \text{if}\quad m\equiv 0\ (mod\ 4),\\ m/2 & \text{if}\quad m\equiv 2\ (mod\ 4),\\ 2m & \text{if}\quad m\equiv 1\ (mod\ 4),\end{cases}$$

Consider some results which are used in the proof of Theorem 1.

**Lemma 1** ([11]). *Let $q$ be a prime power. Then $|\pi(q^2-1)|\le 2$ if and only if $q\in\{2,3,4,5,7,8,9,17\}$.*

**Lemma 2** ([17]). *Let $q=p^m$, where $p$ is a prime, $m\in\mathbb{N}$, and $|\pi(q^2-1)|=3$. Then one of the following statements holds:*

*(i) $17\ne q=p\ge 11$;*

*(ii) $q\in\{16,25,27,49,81\}$,*

*(iii) $p\in\{2,3\}$, $m$ and $(q-1)/(2,q-1)$ are odd primes, $|\pi((q+1)/(p+1)|=1$ and $m$ does not divide $q(q^2-1)$.*

**Lemma 3** ([9]). *Let $p$ and $q$ be primes such that $p^a-q^b=1$ for some natural numbers $a$ and $b$. Then $(p^a,q^b)\in\{(3^2,2^3),(2^a,q),(p,2^b)\}$, where $a$ is a prime and $b$ is a power of 2.*

**Lemma 4** ([29]). *Let $q$ and $n$ be natural numbers, $q\ge 2$. There exists a prime that divides $q^n-1$ and does not divide $q^i-1$ for $1\le i<n$, except in the following cases: $q=2$ and $n=6$; $q=2^k-1$ for some prime $k$ and $n=2$.*

**Lemma 5** ([22]). *Diophantine equation $x^2+1=2y^n$ in integers $x>0,\ n>2,\ y>1$ has unique solution $(x,y,n)=(239,13,4)$.*

**Lemma 6** ([25], Propositions 2.1, 3.1, 4.1). *Let $G=L_n(q)$, $n\ge 2$, $q=p^m$ for a prime $p$ and $r,\ s\in\pi(G)$. Then $r$ and $s$ are non-adjacent in $\Gamma(G)$ if and only if one of the following conditions holds:*

(1) *$r,s\notin\{2,p\}$, $1<e(r,q)\le e(s,q)$, $e(r,q)+e(s,q)>n$ and $e(r,q)$ does not divide $e(s,q)$;*

(2) *$s=p$, $r$ is odd and $e(r,q)>n-2$;*

(3) *$s=p$, $n=2$ and $r=2$;*

(4) *$s=p$, $n=3$, $r=3$ and $(q-1)_3=3$;*

(5) *$r$ divides $q-1$, $s\notin\{2,p\}$, $e(s,q)=n$ and $n_r<(q-1)_r$;*

(6) *$r$ divides $q-1$, $s\notin\{2,p\}$, $e(s,q)=n$ and $n_r=(q-1)_r>2$;*

(7) *$r$ divides $q-1$, $s\notin\{2,p\}$, $e(s,q)=n-1$ and $(q-1)_r\le n_r$.*

**Lemma 7** ([25], Propositions 2.2, 3.1, 4.2). *Let $G=PSU_n(q)$, $n\ge 3$, $q=p^m$ for a prime $p$ and $r,\ s\in\pi(G)$. Then $r$ and $s$ are non-adjacent in $\Gamma(G)$ if and only if one of the following conditions holds:*

(1) *$r,s\notin\{2,p\}$, $1<\nu(e(r,q))\le\nu(e(s,q))$, $\nu(e(r,q))+\nu(e(s,q))>n$ and $\nu(e(r,q))$ does not divide $\nu(e(s,q))$;*

(2) *$s = p$, $r$ is odd and $\nu(e(r,q)) > n-2$;*
(3) *$s = p$, $n = 3$, $r = 3$ and $(q+1)_3 = 3$;*
(4) *$r$ divides $q+1$, $s \notin \{2,p\}$, $\nu(e(s,q)) = n$ and $n_r < (q+1)_r$;*
(5) *$r$ divides $q+1$, $s \notin \{2,p\}$, $\nu(e(s,q)) = n$ and $n_r = (q+1)_r > 2$;*
(6) *$r$ divides $q+1$, $s \notin \{2,p\}$, $\nu(e(s,q)) = n-1$ and $(q+1)_r \le n_r$.*

**Lemma 8** ([3] and [4], Propositions 7-9,12)**.** (1) *Let $G = L_n^{\varepsilon}(q)$. Every maximal torus of $G$ has the order*

$$\frac{1}{(n, q-(\varepsilon 1))(q-(\varepsilon 1))}(q^{n_1}-(\varepsilon 1)^{n_1})(q^{n_2}-(\varepsilon 1)^{n_2})\cdot\ldots\cdot(q^{n_k}-(\varepsilon 1)^{n_k})$$

*for appropriate partition $n_1+n_2+\ldots+n_k = n$ of $n$. Moreover, for any partition of $n$ there exists a maximal torus of $G$ of corresponding order. Every maximal torus of $Inndiag(G)$ has the order $(n, q-\varepsilon 1)|T|$, where $T$ is some maximal torus of $G$.*

(2) *Let $G = S_{2n}(q)$. Every maximal torus of $G$ has the order*

$$\frac{1}{(2,q-1)}(q^{n_1}-1)(q^{n_2}-1)\cdot\ldots\cdot(q^{n_k}-1)(q^{l_1}+1)(q^{l_2}+1)\cdot\ldots\cdot(q^{l_m}+1))$$

*for appropriate partition $n_1+n_2+\ldots+n_k+l_1+l_2+\ldots+l_m = n$ of $n$. Moreover, for any such partition of $n$ there exists a maximal torus of $G$ of corresponding order. Every maximal torus of $Inndiag(G)$ has the order $(2,q-1)|T|$, where $T$ is some maximal torus of $G$.*

**Lemma 9** ([10], Theorem 4.5.1, Propositions 2.5.12, 4.9.1 and 4.9.2)**.** *Let $G = L_n(q)$, $n \ge 2$, $q = p^m$, $p$ be a prime, $m \in \mathbb{N}$, $x$ be an element of a prime order $r$ in $Aut(G) \setminus Inndiag(G)$, and $G_x = O^{p'}(C_G(x))$. Then the following statements hold:*

(1) *$Aut(G) = Inndiag(G) \leftthreetimes (\Phi \times \langle g \rangle)$, where $Outdiag(G) \cong Z_{(n,q-1)}$, $\Phi = \langle f \rangle \cong Aut(GF(q)) \cong Z_m$ is the group of field automorphisms of $G$, $g = 1$ for $n = 2$ and $g$ is a graph automorphism of order 2 of $G$ for $n \ge 3$, $\Phi$ acts on $Outdiag(G)$ as $Aut(GF(q))$ does on the multiplicative subgroup of $GF(q)$ of the same order as $Outdiag(G)$ and $g$ inverts the group $Outdiag(G)$;*

(2) *If $x \in Inndiag(G)\Phi$, then $r$ divides $m$, $G_x \cong L_n(q^{\frac{1}{r}})$, and $C_{Inndiag(G)}(x) \cong Inndiag(G_x)$;*

(3) *If $n \ge 3$, 2 divides $m$ and $x \in Inndiag(G)f^{m/2}g$, then $r = 2$, $G_x \cong L_n^-(q)$, and $C_{Inndiag(G)}(x) \cong Inndiag(G_x)$;*

(4) *If $x \in Inndiag(G)g$ and $n$ is odd, then $G_x \cong B_{\frac{n-1}{2}}(q)$;*

(5) *If $x \in Inndiag(G)g$, $n \ge 4$ is even and $p = 2$, then $C_G(x)$ is isomorphic either to $C_{\frac{n}{2}}(q)$, or to the centralizer of some involution in $C_{\frac{n}{2}}(q)$;*

(6) *If $x \in Inndiag(G)g$, $n \ge 4$ is even and $p > 2$, then $G_x$ is isomorphic one of the groups $C_{\frac{n}{2}}(q)$, $B_{\frac{n-1}{2}}(q)$, $D_{\frac{n}{2}}(q)$, ${}^2D_{\frac{n}{2}}(q)$.*

**Lemma 10** ([10], Theorem 4.5.1, Propositions 2.5.12, 4.9.1 and 4.9.2)**.** *Let $G = L_n^-(q)$, $n \ge 3$, $q = p^m$, $p$ be a prime, $m \in \mathbb{N}$, $x$ be an element of a prime order $r$ in $Aut(G) \setminus Inndiag(G)$, and $G_x = O^{p'}(C_G(x))$. Then the following statements hold:*

(1) *$Aut(G) = Inndiag(G) \leftthreetimes \Phi$, where $Outdiag(G) \cong Z_{(n,q+1)}$, $\Phi = \langle f \rangle \cong Aut(GF(q^2)) \cong Z_{2m}$ is the group of field automorphisms of $G$, $\Phi$ acts on $Outdiag(G)$ as $Aut(GF(q^2))$ does on the multiplicative subgroup of $GF(q^2)$ of the same order as $Outdiag(G)$;*

(1) *If $r > 2$, then $G_x \cong L_n^-(q^{\frac{1}{p}})$, and $C_{Inndiag(G)}(x) \cong Inndiag(G_x)$;*

(2) *If $r = 2$ and $n$ is odd, then $G_x \cong B_{\frac{n-1}{2}}(q)$;*

(3) *If* $r = p = 2$ *and* $n$ *is even, then* $C_G(x)$ *is isomorphic either to* $C_{\frac{n}{2}}(q)$, *or to the centralizer of some involution in* $C_{\frac{n}{2}}(q)$;

(6) *If* $r = 2$, $n$ *is even and* $p > 2$, *then* $G_x$ *is isomorphic one of the groups* $C_{\frac{n}{2}}(q)$, $D_{\frac{n}{2}}(q)$, ${}^2D_{\frac{n}{2}}(q)$.

**Lemma 11** ([10], Propositions 2.5.12, 4.9.1; [13], § 2.2.4). *Let* $G = S_4(q)$, $q = p^m > 2$, $p$ *be a prime,* $m \in \mathbb{N}$ *and* $x$ *be an element of a prime order* $r$ *in* $Aut(G) \setminus Inndiag(G)$. *Then the following statements hold:*

(1) *if* $p = 2$ *then* $Aut(G) \cong G \leftthreetimes Z_{2m}$;

(2) *if* $p > 2$ *then* $Aut(G) = Inndiag(G) \leftthreetimes \Phi$, *where* $Outdiag(G) \cong Z_2$, $\Phi = \langle f \rangle \cong Aut(GF(q)) \cong Z_m$ *is the group of field automorphisms of* $G$;

(3) *if* $m$ *is odd and* $pr = 4$ *then* $C_G(x) \cong Sz(q)$;

(4) *if* $m$ *is even or* $pr \neq 4$ *then,* $C_G(x) \cong S_4(q^{\frac{1}{r}})$.

**Lemma 12** ([10], Propositions 2.5.12, 4.9.1). *Let* $G = Sz(q)$, $q = 2^{2m+1}$ *and* $m \in \mathbb{N}$. *Then the following statements hold:*

(1) $Out(G) \cong Z_{2m+1}$;

(2) *any element* $x$ *of a prime order* $r$ *(dividing* $2m+1$*) in* $Aut(G) \setminus Inn(G)$ *is* $Inn(G)$*-conjugated to a field automorphism of* $G$ *and* $C_G(x) \cong Sz(q^{\frac{1}{r}})$.

**Lemma 13** ([12, 28]). *Let* $G$ *be a finite simple* 5*-primary group. Then* $G$ *is isomorphic to one of the following groups*:

(1) $A_{11}$, $A_{12}$, $M_{22}$, $J_3$, $HS$, $He$, $M^cL$, $L_5(2)$, $U_6(2)$, $O_7(3)$, $S_4(3)$, $S_6(3)$, $O_8^+(3)$, $O_8^-(2)$, $G_2(4)$, $G_2(5)$, $S_8(2)$;

(2) $L_3(9)$, $L_4(4)$, $L_4(5)$, $L_4(7)$, $L_5(3)$, $L_6(2)$, $U_3(17)$, $U_4(4)$, $U_4(5)$, $U_4(7)$, $U_4(9)$, $U_5(3)$, ${}^3D_4(3)$, $G_2(7)$, $G_2(8)$, $S_4(8)$, $S_4(17)$;

(3) $L_2(q)$, *where* $|\pi(q^2-1)| = 4$;

(4) $L_3^{\epsilon}(q)$, *where* $\epsilon \in \{+,-\}$, $|\pi((q^2-1)(q^3+1))| = 4$ *and* $|\pi(\frac{q^2+\epsilon q+1}{(3,q-\epsilon 1)})| = 1$;

(5) $S_4(q)$, *where* $|\pi(q^4-1)| = 4$ *and* $|\pi(\frac{q^2+1}{(2,q-1)})| = 1$;

(6) $Sz(q)$, *where* $q = 2^{2m+1} \geq 2^7$ *and* $|\pi((q-1)(q^2+1))| = 4$.

## 3. Proof of Theorems 1 and 2

Let $G$ be a finite almost simple 5-primary group with the socle $P$. Then $P$ is a nonabelian simple group and $P \leq G \leq Aut(P)$ (we identify $P$ and $Inn(P)$). Therefore, $3 \leq |\pi(P)| \leq 5$. If $|\pi(P)| = 3$, then by [11, 7] $|\pi(Aut(P))| = 3$, a contradiction. If $|\pi(P)| = 4$, then by [17] for $G$ one of the items (3), (4), (12) of Theorem 2 holds and $\Gamma(G)$ is as in Table. Further we will suppose that $|\pi(P)| = 5$. Then $P$ is isomorphic to one of the groups from the items (1)-(6) of Lemma 13.

1. Let $P$ be from the item (1) of Lemma 13. Then, by [7], $G$ is isomorphic to one of the groups from the item (1) of Theorem 2 and $\Gamma(G)$ is as in Table.

2. Let $P$ be from the item (2) of Lemma 13.

Let $P \cong L_3(9)$. By [27], $\pi_1(P) = \{2,3,5\}$ and $\pi_2(P) = \{7,13\}$. By Lemma 6, the graph $\Gamma(P)$ is as in Table, i. e., $E(\Gamma(P)) = \{\{2,3\},\{2,5\},\{7,13\}\}$. Let $P < G$. By Lemma 9, we have $Aut(P) = Inn(P) \leftthreetimes (\langle f \rangle \times \langle g \rangle)$, where $f$ and $g$ are involutive field and graph automorphism of $P$, respectively. By Lemma 9 and [1, Tables 8.3, 8.4], $C_P(x) \cong L_3(3)$ for any involution $x \in Pf$, $C_P(x)) \cong U_3(3)$ for any involution $x \in Pfg$ and $F^*(C_P(x)) \cong B_1(9) \cong L_2(9)$ for any involution $x \in Pg$. Hence, $E(\Gamma(P\langle f \rangle)) = \{\{2,3\},\{2,5\},\{2,13\},\{7,13\}\}$, $E(\Gamma(P\langle g \rangle)) = E(\Gamma(P))$,

$E(\Gamma(P\langle fg\rangle)) = \{\{2,3\},\{2,5\},\{2,7\},\{7,13\}\}$, and

$$E(\Gamma(P\langle f,g\rangle)) = \{\{2,3\},\{2,5\},\{2,7\},\{2,13\},\{7,13\}\}.$$

Therefore, $\Gamma(G)$ is as in Table.

Let $P \cong L_4(4)$. By Lemma 6, the graph $\Gamma(P)$ is as in Table, i. e., $E(\Gamma(P)) = \{\{2,3\},\{2,5\},\{3,5\},\{3,7\},\{5,17\}\}$. Let $P < G$. By Lemma 9, we have $Aut(P) = Inn(P) \leftthreetimes (\langle f\rangle \times \langle g\rangle$, where $f$ and $g$ are involutive field and graph automorphism of $P$, respectively. By Lemma 9 and [1, Tables 8.8, 8.9], $C_P(x) \cong L_4(2) \cong A_8$ for any involution $x \in Pf$, $C_P(x)) \cong {}^2A_3(2) \cong U_4(2)$ for any involution $x \in Pfg$ and $C_P(x) \cong C_2(4) \cong S_4(4)$ for any involution $x \in Pg$. Hence, $E(\Gamma(P\langle f\rangle)) = \{\{2,3\},\{2,5\},\{2,7\},\{3,5\},\{3,7\},\{5,17\}\}$, $E(\Gamma(P\langle fg\rangle)) = E(\Gamma(P))$, $E(\Gamma(P\langle g\rangle)) = \{\{2,3\},\{2,5\},\{2,17\},\{3,5\},\{3,7\},\{5,17\}\}$,

$$E(\Gamma(P\langle f,g\rangle)) = \{\{2,3\},\{2,5\},\{2,7\},\{2,17\},\{3,5\},\{3,7\},\{5,17\}\}.$$

Therefore, $\Gamma(G)$ is as in Table.

Let $P \cong L_4(5)$. By Lemma 6, the graph $\Gamma(P)$ is as in Table, i. e., $E(\Gamma(P)) = \{\{2,3\},\{2,5\},\{3,5\},\{3,13\}\}$. Let $P < G$. By Lemma 9, we have $Aut(P) = Inndiag(P) \leftthreetimes (\langle g\rangle$, where $g$ is the involutive graph automorphism of $P$, $Outdiag(P) \cong Z_4$ and $Out(P) \cong D_8$. If $T \in Syl_{13}(P)$ then $|T| = 13$ and $N_P(T)/C_P(T) \cong Z_4$, hence $G = PC_G(T)$. Therefore, $\{2,13\} \in E(\Gamma(G))$. By Lemma 9 and [1, Tables 8.8, 8.9], $F^*(C_P(x)) \cong C_2(5)$ or $\Omega_4^{\pm}(5)$ for any involution $x \in Inndiag(P)g$. Hence, if $G \cap Inndiag(P) = P$ then $E(\Gamma(G)) = E(\Gamma(P)) \cup \{2,13\}$. By Lemma 8, if $G \cap Inndiag(P) > P$ then $\{2,31\} \in E(\Gamma(G))$. Therefore, $\Gamma(G)$ is as in Table.

Let $P \cong L_4(7)$. By Lemma 6, the graph $\Gamma(P)$ is as in Table, i. e., $E(\Gamma(P)) = \{\{2,3\},\{2,5\},\{2,7\},\{2,19\},\{3,5\},\{3,7\},\{3,19\}\}$. By Lemma 9, $Out(P) \cong 2^2$. Therefore, if $P < G$ then $\Gamma(G) = \Gamma(P)$.

Let $P \cong L_5(3)$. By [27], $\pi_1(P) = \{2,3,5,13\}$ and $\pi_2(P) = \{11\}$.By Lemma 6, the graph $\Gamma(P)$ is as in Table, i. e., $E(\Gamma(P)) = \{\{2,3\},\{2,5\},\{2,13\},\{3,13\}\}$. By Lemma 9, we have $Aut(P) = Inn(P) \leftthreetimes \langle g\rangle$, where $g$ is involutive graph automorphism of $P$. By Lemma 9 and [1, Tables 8.18, 8.19], $C_P(x) \cong B_2(3) \cong S_4(3)$ for any involution $x \in Pg$, Therefore, if $P < G$ then $G = Aut(P)$ and $\Gamma(G) = \Gamma(P)$.

Let $P \cong L_6(2)$. By [14], $\pi_1(P) = \{2,3,5,7\}$ and $\pi_2(P) = \{31\}$. By Lemma 6, the graph $\Gamma(P)$ is as in Table, i. e., $E(\Gamma(P)) = \{\{2,3\},\{2,5\},\{3,5\},\{2,7\},\{3,7\}\}$. By Lemma 9, we have $Aut(P) = Inn(P) \leftthreetimes \langle g\rangle$, where $g$ is involutive graph automorphism of $P$. By Lemma 9 and [1, Tables 8.18, 8.19], $C_P(x) \cong C_3(2) \cong S_6(2)$ for any involution $x \in Pg$, Therefore, if $P < G$ then $G = Aut(P)$ and $\Gamma(G) = \Gamma(P)$.

Let $P \cong U_3(17)$. By [27], $\pi_1(P) = \{2,3,17\}$, $\pi_2(P) = \{7,13\}$. By Lemma 7, the graph $\Gamma(P)$ is as in Table, i. e., $E(\Gamma(P)) = \{\{2,3\},\{2,17\},\{3,17\},\{7,13\}\}$. By Lemma 10, we have $Aut(P) = Inndiag(P) \leftthreetimes \langle f\rangle$, where $f$ is involutive field automorphism of $P$ and $f$ inverts $Inndiag(P) \cong Z_3$. By Lemma 10 and [1, Tables 8.5, 8.6], $F^*(C_P(x)) \cong L_2(17)$ for any involution $x \in Pf$, Let $P < G$. Suppose that $|G : P| = 2$. Then we can assume that $G = P\langle f\rangle$. By Lemma 10 and [1, Tables 8.5, 8.6], $F^*(C_P(x)) \cong L_2(17)$ for any involution $x \in Pf$. Hence, $E(\Gamma(P\langle f\rangle)) = E(\Gamma(P))$, i. e., $\Gamma(G)$ is as in Table. Suppose that $|G : P| = 3$. Then $G = Inndiag(P)$. By Lemma 8, $G$ has a maximal torus of order $3 \cdot 7 \cdot 13$. Hence, $E(\Gamma(G)) = \{\{2,3\},\{2,17\},\{3,7\},\{3,13\},\{3,17\},\{7,13\}\}$, i. e., $\Gamma(G)$ is as in Table. It is clear that $E(\Gamma(Aut(P))) = E(\Gamma(Inndiag(P)))$.

Let $P \cong U_4(4)$. By Lemma 7, the graph $\Gamma(P)$ is as in Table, i. e., $E(\Gamma(P)) = \{\{2,3\},\{2,5\},\{3,5\},\{3,17\},\{5,13\}\}$. By Lemma 10, we have $Aut(P) = Inn(P) \leftthreetimes \langle f\rangle$, where $f$ is field automorphism of order 4 of $P$, and if $x$ is an involution in $Aut(P) \setminus Inndiag(P)$ then $C_P(x)$ is isomorphic either to $S_4(4)$ or the centralizer of some involution in $S_4(4)$. Therefore, if $P < G$ then

$$E(\Gamma(G)) = \{\{2,3\},\{2,5\},\{2,17\},\{3,5\},\{3,17\},\{5,13\}\},$$

i. e., $\Gamma(G)$ is as in Table.

Let $P \cong U_4(5)$. By Lemma 7, the graph $\Gamma(P)$ is as in Table, i. e.,

$$E(\Gamma(P)) = \{\{2,3\},\{2,5\},\{2,13\},\{3,5\},\{3,7\}\}.$$

By Lemma 10, we have $Aut(P) = Inndiag(P) \leftthreetimes \langle f\rangle$, where $f$ are involutive field automorphism of $P$ and $f$ centralizes $Inndiag(P) \cong Z_2$. By Lemma 8, $Inndiag(P)$ has a maximal torus of order $7 \cdot 2 \cdot 3^2$, hence

$$E(\Gamma(Inndiag(P))) = \{\{2,3\},\{2,5\},\{2,7\},\{2,13\},\{3,5\},\{3,7\}\}$$

as in Table. By Lemma 10 and [1, Tables 8.10, 8.11], if $x$ is an involution in $Aut(P) \setminus Inndiag(P)$ then $F^*(C_P(x))$ is isomorphic to one of the groups $S_4(5)$, $SO_4^+(5) \cong L_2(5) \times L_2(5)$ or $SO_4^-(5) \cong L_2(25)$, hence $x$ does not centralize an element of order 7 in $P$. Therefore, if $G \cap Inndiag(P) = P$, then $E(\Gamma(G)) = E(\Gamma(P))$, i. e., $\Gamma(G)$ is as in Table. It is clear that $E(\Gamma(Aut(P))) = E(\Gamma(Inndiag(P)))$.

Let $P \cong U_4(7)$. By Lemma 7, the graph $\Gamma(P)$ is as in Table, i. e.,

$$E(\Gamma(P)) = \{\{2,3\},\{2,7\},\{2,43\},\{3,5\},\{3,7\}\}.$$

By Lemma 10, we have $Aut(P) = Inndiag(P) \leftthreetimes \langle f\rangle$, where $f$ are involutive field automorphism of $P$ and $f$ inverts $Inndiag(P) \cong Z_4$. By Lemma 8, $Inndiag(P)$ has a maximal torus of order $2^2 \cdot 3 \cdot 5^2$, hence if $P < G \leq Inndiag(P)$ then $E(\Gamma(G)) = \{\{2,3\},\{2,5\},\{2,7\},\{3,7\},\{3,5\},\{2,43\}\}$. By Lemma 10 and [1, Tables 8.10, 8.11], $Aut(P) \setminus Inndiag(P)$ contains an involution $x$ such that $F^*(C_P(x)) \cong S_4(7)$, hence $x$ centralizes an element $y$ of order 5 in $P$. Since Sylow 5-subgroup of $P$ is cyclic, $Aut(P) = Inn(P) \leftthreetimes D$, where $D$ is a Sylow 2-subgroup of $C_{Aut(P)}(y)$. Therefore, if $P < G$ then

$$E(\Gamma(G)) = \{\{2,3\},\{2,5\},\{2,7\},\{3,7\},\{3,5\},\{2,43\}\},$$

i. e., $\Gamma(G)$ is as in Table.

Let $P \cong U_4(9)$. By Lemma 7, the graph $\Gamma(P)$ is as in Table, i. e.,

$$E(\Gamma(P)) = \{\{2,3\},\{2,5\},\{2,41\},\{3,5\},\{5,73\}\}.$$

By Lemma 10, we have

$$Aut(P) = Inndiag(P) \leftthreetimes \langle f\rangle,$$

where $f$ are involutive field automorphism of $P$ and $f$ centralizes $Inndiag(P) \cong Z_2$. By Lemma 8, $Inndiag(P)$ has a maximal torus of order $2 \cdot 5 \cdot 73$, hence $E(\Gamma(Inndiag(P))) = \{\{2,3\},\{2,5\},\{2,41\},\{2,73\},\{3,5\},\{5,73\}\}$ By Lemma 10 and [1, Tables 8.10, 8.11], if $x$ is an involution in $Aut(P) \setminus Inndiag(P)$ then $F^*(C_P(x))$ is isomorphic to one of the groups $S_4(9)$, $SO_4^+(9) \cong L_2(9) \times L_2(9)$ or $SO_4^-(5) \cong L_2(81)$, hence $x$ does not centralize an element of order 73 in $P$. Therefore, $\Gamma(G)$ is as in Table.

Let $P \cong U_5(3)$. By [27], $\pi_1(P) = \{2,3,5,7\}$, $\pi_2(P) = \{61\}$. By Lemma 7, the graph $\Gamma(P)$ is as in Table, i. e., $E(\Gamma(P)) = \{\{2,3\},\{2,5\},\{2,7\},\{3,7\}\}$. By Lemma

10, we have $Aut(P) = Inn(P) \leftthreetimes \langle f \rangle$, where $f$ is involutive field automorphism of $P$. By Lemma 10 and [1, Tables 8.20, 8.21], if $x$ is an involution in $Aut(P) \setminus Inn(P)$ then $F^*(C_P(x)) \cong S_4(3)$, hence $x$ does not centralize an element of order 61 in $P$. Therefore, $E(\Gamma(Aut(P))) = E(\Gamma(P))$.

Let $P \cong {}^3D_4(3)$. By [27], $\pi_1(P) = \{2, 3, 7, 13\}$, $\pi_2(P) = \{73\}$. By [25, Proposition 2.5], the graph $\Gamma(P)$ is as in Table, i. e.,

$$E(\Gamma(P)) = \{\{2, 3\}, \{2, 7\}, \{2, 13\}, \{3, 7\}, \{3, 13\}\}.$$

Let $P < G$. Then, by [10, Propositions 2.5.12 and 4.9.2], $Aut(P) \cong P \leftthreetimes Z_3$ and if $x$ is an element of order 3 in $Aut(P) \setminus P$ then $C_P(x)$ is isomorphic either $G_2(3)$ or the centralizer of some element of order 3 in $G_2(3)$, hence $x$ does not centralize an element of order 73 in $P$. Therefore, $E(\Gamma(Aut(P)) = E(\Gamma(P))$.

Let $P \cong S_4(8)$. By [14], $\pi_1(P) = \{2, 3, 7\}$, $\pi_2(P) = \{5, 13\}$. By [1, Tables 8.14], $P$ contains some subgroups isomorphic to $\Omega_4^+(8) \cong L_2(8) \times L_2(8)$ and $\Omega_4^-(8) \cong L_2(64)$. Hence, the graph $\Gamma(P)$ is as in Table, i. e., $E(\Gamma(P)) = \{\{2, 3\}, \{2, 7\}, \{3, 7\}, \{5, 13\}\}$. By Lemma 11, $Aut(P) \cong P \leftthreetimes Z_6$. Furthermore, if $x$ is an element of order 3 in $G \setminus P$ then $C_P(x) \cong S_4(2) \cong S_6$, hence $E(\Gamma(P\langle x \rangle)) = \{\{2, 3\}, \{2, 7\}, \{3, 5\}, \{3, 7\}, \{5, 13\}\}$; if $t$ is an element of order 2 in $G \setminus P$ then $C_P(x) \cong Sz(8)$, hence $E(\Gamma(P\langle t \rangle)) = \{\{2, 3\}, \{2, 5\}, \{2, 7\}, \{2, 13\}, \{3, 7\}, \{5, 13\}\}$. Therefore,

$$E(\Gamma(Aut(P))) = \{\{2, 3\}, \{2, 5\}, \{2, 7\}, \{2, 13\}, \{3, 5\}, \{3, 7\}, \{5, 13\}\}.$$

Therefore, $\Gamma(G)$ is as in Table.

Let $P \cong S_4(17)$. By [27], $\pi_1(P) = \{2, 3, 17\}$, $\pi_2(P) = \{5, 29\}$. By [1, Tables 8.12], $P$ contains some subgroups isomorphic to $\Omega_4^+(17) \cong SL_2(17) \circ SL_2(17)$ and $\Omega_4^-(17) \cong L_2(17^2)$. Hence, the graph $\Gamma(P)$ is as in Table, i. e., $E(\Gamma(P)) = \{\{2, 3\}, \{2, 17\}, \{3, 17\}, \{5, 29\}\}$. Let $P < G$. Then $G = Aut(P) \cong PGSp_4(17)$. By Lemma 8, maximal tori in $G$ have orders $2^8$, $2^5 \cdot 3^2$, $2^2 \cdot 3^4$, $2 \cdot 5 \cdot 29$, hence $E(\Gamma(G)) = \{\{2, 3\}, \{2, 5\}, \{2, 17\}, \{2, 29\}, \{3, 17\}, \{5, 29\}\}$. Therefore, $\Gamma(G)$ is as in Table.

3. Let $P$ be from the item (3) of Lemma 13. Then $P \cong L_2(q)$, where $|\pi(q^2 - 1)| = 4$.

Let $q = 2^m$. Since $P$ is 5-primary, $m > 5$. By [14], $s(P) = 3$, $\pi_1(G) = \{2\}$, $\pi_2(G) = \pi(q - 1)$, $\pi_3(G) = \pi(q + 1)$.

Suppose that $|\pi(q - \epsilon 1)| = 1$ for $\epsilon \in \{+, \}$, i. e., $q - \epsilon 1 = r^k$ for a prime $r$ and $k \in \mathbb{N}$. Then $\pi(q + \epsilon 1) = \{r_1, r_2, r_3\}$ is a clique in the graph $\Gamma(P)$.

Let $\epsilon = +$. Then, by Lemma 3, $k = 1$ and $m > 7$ is prime. We obtain that $q - 1 = r > 3$ and $\pi(q + 1) = \{r_1, r_2, r_3\}$ is a clique in the graph $\Gamma(P)$. Therefore, $\Gamma(P)$ is as in Table. Suppose that $P < G$. Then, by Lemma 9, $G = Aut(P) = P \leftthreetimes \langle f \rangle$, where $f$ is the field automorphism of order $m$ of $P$. By Lemma 10, if $x$ is an element of order $m$ in $G \setminus P$ then $C_P(x) \cong L_2(2) \cong S_3$. This contradicts the condition $3 < m \in \pi(P)$. Therefore, $G = P$.

Let $\epsilon = -$. Then, by Lemma 3, $k = 1$ and $m = 2^l \geq 8$. We obtain that $q + 1 = r > 3$ and $\pi(q - 1) = \{r_1, r_2, r_3\}$ is a clique in the graph $\Gamma(P)$. If $m = 8$ then $r = 257$ and $\pi(q - 1) = \{3, 5, 17\}$. If $m > 8$ then $2^{16} - 1$ divides $q - 1$, which is impossible. Therefore, $m = 8$ and $\Gamma(P)$ is as in Table. Suppose that $P < G$. By Lemma 9, $Aut(P) = P \leftthreetimes \langle f \rangle$, where $f$ is the field automorphism of order 8 of $P$. Hence, $G$ contains the involution $f^4$. Again, by Lemma 9, if $x$ is an involution in $G \setminus P$ then $C_P(x) \cong L_2(16)$. But $\pi(L_2(16)) = \{2, 3, 5, 17\}$, therefore, $\Gamma(G)$ is as in Table.

Further, we can assume that $\pi(q-1)=\{r,s\}$ and $\pi(q+1)=\{r_1,s_1\}$. Let $p$ be a prime divisor of $m$.

Suppose that $p=2$. Then $m\geq 6$ and $q-1={q_0}^2-1$, where $q={q_0}^2$. Hence, by Lemma 1, $q_0=8$, i. e. $q=2^6$. The graph $\Gamma(P)$ is as in Table. In the case $P<G$, we apply Lemma 9 and obtain that $\Gamma(G)$ is as in Table.

Further, we can assume that $m$ is odd and $p>2$. Let $q={q_0}^p$. In view of Lemma 4, $|\pi(q_0^2-1)|\leq 2$. Hence, by Lemma 1, $q_0\in\{2,8\}$.

Suppose that $q_0=8$. Then 3 divides $m$ and we can assume that $p=3$, i. e. $q=2^9$. The graph $\Gamma(P)$ is as in Table. In the case $P<G$, we apply Lemma 9 and obtain that $\Gamma(G)$ is as in Table.

New, we can assume that $m=p>7$. Then for $P$ the item (2) of Theorem 2 holds and $\Gamma(P)$ is as in Table. Suppose that $P<G$. By Lemma 9, $G=Aut(P)=P\leftthreetimes\langle f\rangle$, where $f$ is the field automorphism of order $p$ of $P$. Hence, $p\in\pi(P)$. Again, by Lemma 9, $C_P(f)\cong L_2(2)\cong S_3$ and hence $p\in\{2,3\}$, a contradiction.

Let $q=p^m$, where $p$ is an odd prime, $m\in\mathbb{N}$ and $q\equiv\epsilon 1\ (mod\ 4)$ for $\epsilon\in\{+,-\}$. By [27], $s(P)=3$ and $\pi_1(G)=\pi(q-\epsilon 1)$, $\pi_2(G)=\{p\}$, $\pi_3(G)=\pi(q+\epsilon 1)/2)$ are cliques in the graph $\Gamma(P)$. If $q-\epsilon 1=2^k$ then $\pi(\frac{q+\epsilon 1}{2})=\{r_1,r_2,r_3\}$ and, by Lemma 2, $m=1$. If $\pi(q-\epsilon 1)=\{2,s\}$ then $\pi(\frac{q+\epsilon 1}{2})=\{r_1,r_2\}$. If $\pi(q-\epsilon 1)=\{2,s,t\}$ then $\pi(\frac{q+\epsilon 1}{2})=\{r\}$. Therefore, the graph $\Gamma(P)$ is determined.

By Lemma 9, $G=Aut(P)=Inndiag(P)\leftthreetimes\langle f\rangle$, where $f$ is the field automorphism of order $m$ of $P$, and $f$ centralizes the group $Outdiag(P)\cong Z_2$. If $G=Inndiag(P)$ then, by Lemma 8, maximal tori in $G$ have orders $q-1$ and $q+1$, hence the graph $\Gamma(Inndiag(P))$ is determined.

If $m=1$ then $p\geq 41$ (if $p<41$ then $|\pi(Aut(P))|\leq 4$) and $Aut(P)=Inndiag(P)\cong PGL_2(p)$, hence $G$ satisfies the item (4) of Theorem 2 and the graph $\Gamma(G)$ is as in Table.

Let $m>1$, $r$ is a prime divisor of $m$ and $q_0=p^{m/r}$.

Let $r=2$. Then $q^2-1=q_0^4-1=(q_0-1)(q_0+1)(q_0^2+1)$ and $(q_0-1,q_0+1)=(q_0^2+1,q_0^2-1)=2$.

By Lemma 4, $\pi(q_0^2-1)\subset\pi(q^2-1)$, hence, $|\pi(q_0^2-1)|\leq 3$.

Suppose that $|\pi(q_0^2-1)|\leq 2$. By Lemma 1, $q_0\in\{3,5,7,9,17\}$. If $q\neq 17^2$ then $|\pi(Aut(L_2(q)))|\leq 4$. Hence $q=17^2$ and, cosequently, $\pi(L_2(q))=\{2,3,5,17,29\}$. By Lemma 9, if $x$ is an involution in $Pf$ then $C_P(x)\cong PGL_2(17)$. Thus, for all $G$ with $P\leq G\leq Aut(P)$, $G$ satisfies the item (1) of Theorem 2 and $\Gamma(G)$ is as in Table.

Now, suppose that $|\pi(q_0^2-1)|=3$. Then, by Lemma 2, one of the following cases holds:

(i) $q_0=p$,

(ii) $q_0\in\{25,27,49,81\}$,

(iii) $p=3$ and $q_0=3^n$, where $n$ is an odd prime and $n$ does not divide $q_0(q_0^2-1)$.

Suppose that the case (i) holds. Then $q=p^2$, $p\geq 11$, $p^2\equiv 1\ (mod\ 12)$, $\pi(p^2-1)=\{2,3,s\}$, and $p^2+1=2t^k$, where t is a prime and $k\in\mathbb{N}$. By Lemma 5, either $k\leq 2$, or $p=239$, $t=13$ and $k=4$. The second possibility does not arise, since $\pi(239^2-1)=\{2,3,5,7,13,17\}$. Hence, $k\leq 2$. By Lemma 9, if $x$ is an involution in $Pf$ then $C_P(x)\cong PGL_2(p)$. Thus, for all $G$ with $P\leq G\leq Aut(P)$, $G$ satisfies the item (6) of Theorem 2 and $\Gamma(G)$ is as in Table.

Suppose that the case (ii) holds. Then $q\in\{5^4,3^6,7^4,3^8\}$. If $q=3^6$ then $\pi(q^2-1)=\{2,5,7,13,73\}$. If $q=3^8$ then $\pi(q^2-1)=\{2,5,17,41,193\}$. Hence, $q\in$

$\{5^4, 7^4\}$. If $q = 5^4$ then $\pi(q^2-1) = \{2,3,13,313\}$. If $q = 7^4$ then $\pi(q^2-1) = \{2,3,5,1201\}$. By Lemma 9, if $x$ is an involution in $Pf^2$ then $C_P(x) \cong PGL_2(p^2)$. Thus, for all $G$ with $P \leq G \leq Aut(P)$, $G$ satisfies the item (1) of Theorem 2 and $\Gamma(G)$ is as in Table.

Suppose that the case (iii) holds. Then $q^2-1 = 10(3^n-1)(3^n+1)[(9^n+1)/(9+1)]$ and $(3^n-1,(3^n+1) = (3^{2n}-1,(3^{2n}+1) = 2$. By Lemma 2, $|\pi(3^n \pm 1)| \geq 2$. Let $u \in \pi((3^n-1)/2)$ and $v \in \pi((3^n+1)/4)$, where $u$ and $v$ are odd primes. Then $u$ and $v$ are bigger than 5. By Lemma 3, there exists a prime divisor of the number $(9^n+1)/10$ which does not belong to $\{2,5,u,v\}$. Hence $|\pi(q^2-1)| \geq 5$, a contradiction.

Now, we can assume that $m$ is odd and, in particular, $r > 2$. Then $q^2-1 = (q-1)(q+1) = (q_0^r-1)(q_0^r+1) = (q_0-1)(q_0+1)\frac{q_0^r-1}{q_0-1}\frac{q_0^r+1}{q_0+1}$ and $(q_0-1,q_0+1) = (q_0^r-1,q_0^r+1) = 2$. By Lemma 3, there exist two different prime divisors of the numbers $\frac{q_0^r-1}{q_0-1}$ and $\frac{q_0^r+1}{q_0+1}$ which does not belong to $\pi(q_0^2-1)$. Therefore, $|\pi(q_0^2-1)| \leq 2$. By Lemma 1, $q_0 \in \{3,5,7,9,17\}$. Since $m$ is odd, $q_0 \neq 9$. Hence $q_0 \in \{3,5,7,17\}$, in particular, $q_0 = p$.

Suppose that $r$ divides $|P| = q(q^2-1)/2$. Then $r$ divides the order of the group $C_P(f) \cong PGL_2(p)$ which is equal to $p(p^2-1)$.

If $p = 3$ then $r = 3$ and, cosequently, $|\pi(Aut(P))| = 4$, which is impossible.

Let $p = 5$. Then $r \in \{3,5\}$ and $q \in \{5^3, 5^5\}$. If $q = 5^5$ then $\pi(P) = \pi(Aut(P)) = \{2,3,5,11,71,521\}$, i. e. $|\pi(P)| = 6$. Hence $q = 5^3$. Thus $\pi(P) = \pi(Aut(P)) = \{2,3,11,71,521\}$. By Lemma 9, if $x$ is an element of order 3 in $Aut(p) \setminus Inndiag(P)$ then $C_P(x) \cong PGL_2(5)$. Thus, for all $G$ with $P \leq G \leq Aut(P)$, $G$ satisfies the item (1) of Theorem 2 and $\Gamma(G)$ is as in Table.

Let $p = 7$. Then $r \in \{3,7\}$ and $q \in \{7^3, 7^7\}$. If $q = 7^3$ then $\pi(P) = \pi(Aut(P)) = \{2,3,7,19,43\}$. If $q = 7^7$ then $\pi(P) = \pi(Aut(P)) = \{2,3,5,7,1201\}$. By Lemma 9, if $x$ is an element of order $r$ in $Aut(p) \setminus Inndiag(P)$ then $C_P(x) \cong PGL_2(7)$. Thus, for all $G$ with $P \leq G \leq Aut(P)$, $G$ satisfies the item (1) of Theorem 2 and $\Gamma(G)$ is as in Table.

Let $p = 17$. Then $r \in \{3,17\}$ and $q \in \{17^3, 17^{17}\}$. If $q = 17^{17}$ then

$$\pi(P) = \pi(Aut(P)) = \{2,3,17,10949,1749233,2699538733,4595779232701870912 1\},$$

i. e., $|\pi(P)| = 7$. Hence $q = 17^3$. By Lemma 9, if $x$ is an element of order 3 in $Aut(p) \setminus Inndiag(P)$ then $C_P(x) \cong PGL_2(17)$. Thus, for all $G$ with $P \leq G \leq Aut(P)$, $G$ satisfies the item (1) of Theorem 2 and $\Gamma(G)$ is as in Table.

4. Let $P$ be from the item (4) of Lemma 13. Then $P \cong L_3^{\varepsilon}(q)$, where $q = p^m$, $p$ is a prime, $m \in \mathbb{N}$, $\varepsilon \in \{+,-\}$, $|\pi(\frac{q^2+\varepsilon q+1}{(3,q-\varepsilon 1)})| = 1$, and $|\pi(P)| = 5$.

By [17], $17 \neq q \geq 9$ and $q > 9$ for $\varepsilon = -$. By [27, 14], $s(P) = 2$, $\pi_1(P) = \pi(q(q^2-1))$ and $\pi_2(P) = \{r\}$, where $\pi(\frac{q^2+\varepsilon q+1}{(3,q-\epsilon 1)}) = \{r\}$. Hence $|\pi(q^2-1)| = 3$. By Lemma 2, one of the following cases holds:

(i) $17 \neq q = p \geq 11$;

(ii) $q \in \{16,25,27,49,81\}$,

(iii) $p \in \{2,3\}$, $m$ and $(q-1)/(2,q-1)$ are odd primes, $|\pi((q+1)/(p+1)| = 1$ and $m$ does not divide $q(q^2-1)$.

Let $p = 2$. Then the case (i) is impossible.

Suppose that the case (ii) holds. Then $q = 16$. If $\varepsilon = +$ then $\pi(\frac{q^2+\varepsilon q+1}{(3,q-1)}) = 7 \cdot 13$, a contradiction. Hence, $\varepsilon = -$ and $P = U_3(16)$. Then $\pi_1(P) = \{2,3,5,17\}$,

$\pi_2(P) = \{241\}$. By Lemma 7, the graph $\Gamma(P)$ is as in Table, i. e., $E(\Gamma(P)) = \{\{2,17\},\{3,5\},\{3,17\},\{5,17\}\}$. By Lemma 10, we have $Aut(P) = P \leftthreetimes \langle f \rangle$, where $f$ is field automorphism of order 8 of $P$, and if $x$ is an involution in $Aut(P) \setminus P$ then $C_P(x)$ is isomorphic either to $L_2(16)$ or to the centralizer of some involution in $S_4(4)$. Hence if $P < G$ then $\pi_1(G) = \pi_1(P)$ is a coclique in the graph $\Gamma(G)$. Therefore, for any group $G$ with $P \leq G \leq Aut(G)$, $G$ satisfies the item (1) of Theorem 2 and $\Gamma(G)$ is as in Table.

Suppose that the case (iii) holds. Let $\varepsilon = +$. Then $m$ and $q-1$ are odd primes, and $q-1 > 7$. It is clear that $(q-1,3) = 1$ and 7 divides $q^3 - 1 = 2^{3m} - 1 = 8^m - 1$. Hence $\pi((q^3-1))/(q-1)) = \{7\}$. By Lemma 3, there exists a prime divisor of $2^{3m} - 1$ which does not divide $2^3 - 1 = 7$ and $2^m - 1 = q - 1$, a contradiction. Therefore, $\varepsilon = -$, $m > 3$, $(q+1)_3 = 3$, and $(q^3+1)_3 = 9$.

Let $q-1 = s$ and $t \in \pi((q+1)/3)$. By Lemma 7, $E(\Gamma(P)) = \{\{2,3\},\{3,t\},\{t,s\}\}$, i. e., $\Gamma(P)$ is as in Table. By Lemma 10 and [1, Table 8.5], we have $Aut(P) = Inndiag(P) \leftthreetimes \langle f \rangle$, where $Outdiag(G) \cong Z_3$, $f$ is the field automorphism of order $2m$ of $P$, $f^m$ inverts the group $Outdiag(G)$ and if $x$ is an involution in $Aut(P) \setminus Inndiag(P)$ then $F^*(C_P(x)) \cong L_2(q)$. We have $P \leq G \leq Inndiag(P)\langle f^m \rangle$. If $G = Inndiag(P)$ then maximal tori of $G$ have the orders $q^2-q+1$, $q^2 \pm 1$ or $(q+1)^2$ and, by [1, Table 8.5], $(q+1) \times L_2(q) \cong GU_2(q) < G$, hence, $E(\Gamma(Inndiag(P))) = \{\{2,3\},\{3,t\},\{3,s\},\{3,r\},\{t,s\}\}$, i. e., $\Gamma(G)$ is as in Table. If $G = P\langle f^m \rangle$ then $E(\Gamma(G)) = \{\{2,3\},\{2,t\},\{2,s\},\{3,t\},\{t,s\}\}$, i. e., $\Gamma(P\langle f^m \rangle)$ is as in Table. Thus, $E(\Gamma(Inndiag(P)\langle f^m \rangle) = \{\{2,3\},\{2,t\},\{2,s\},\{3,t\},\{3,s\},\{3,r\},\{t,s\}\}$, i. e., $\Gamma(Inndiag(P)\langle f^m \rangle)$ is as in Table. Therefore, for any group $G$ with $P \leq G \leq Inndiag(P)\langle f^m \rangle$, $G$ satisfies the item (8) of Theorem 2 and $\Gamma(G)$ is as in Table.

Let $p > 2$.

Suppose that the case (i) holds. Then $17 \neq q = p \geq 11$ and $\pi_1(P) = \{2,3,s,p\}$. By [1, Tables 8.3, 8.5], $SL_3^{\varepsilon}(q)$ contains a maximal subgroup isomorphic to $GL_2^{\varepsilon}(q)$. Hence, $P$ contains a maximal subgroup $H$ isomorphic to $Z_a.PGL_2(q)$, where $a = (q-\varepsilon 1)/(3, q-\varepsilon 1)$. Using the information for the case 1 and Lemmas 6-10, we obtain that, for any group $G$ with $P \leq G \leq Aut(P)$, $G$ satisfies the item (9) of Theorem 2 and $\Gamma(G)$ is as in Table.

Suppose that the case (ii) holds. Then $q \in \{25, 27, 49, 81\}$. If $q = 49$ then the set $\pi(\frac{q^2+\varepsilon q+1}{(3,q-1)})$ equals to $\{19, 43\}$ for $\varepsilon = +$ and to $\{13, 181\}$ for $\varepsilon = -$, a contradiction.

Let $q = 25$. Since $(25^2+25+1)/(3,25-1) = 7 \cdot 31$ and $(25^2-25+1)(3,25+1) = 601$, we have $\varepsilon = -$, $r = 601$ and $\pi_1(P) = \{2,3,5,13\}$. By Lemma 7, the graph $\Gamma(P)$ is as in Table, i. e., $E(\Gamma(P)) = \{\{2,3\},\{2,5\},\{2,13\},\{3,13\}\}$. By Lemma 10 and [1, Table 8.5], $Aut(P) = P \leftthreetimes \langle f \rangle$, where $f$ is the field automorphism of order 4 of $P$, and $F^*(C_P(x)) \cong L_2(25)$ for any involution $x$ in $Pf^2$. Therefore, if $P < G$ then $\Gamma(G) = \Gamma(P)$.

Let $q = 27$. Since $27^2 - 27 + 1 = 19 \cdot 37$ and $27^2 + 27 + 1 = 757$, we have $\varepsilon = +$, $r = 757$ and $\pi_1(P) = \{2,3,7,13\}$. By Lemma 6, the graph $\Gamma(P)$ is as in Table, i. e.,

$$E(\Gamma(P)) = \{\{2,3\},\{2,7\},\{2,13\},\{7,13\}\}.$$

By Lemma 9 and [1, Table 8.3], $Aut(P) = P \leftthreetimes \langle f \rangle$, where $f$ is the field automorphism of order 6 of $P$, $F^*(C_P(x)) \cong L_2(27)$ for any involution $x$ in $Pf^3$ and $F^*(C_P(y)) \cong L_3(3)$ for any element $y$ of order 3 in $Pf^2$. Therefore, $\Gamma(G\langle f^3 \rangle) = \Gamma(P)$,

$$E(\Gamma(P\langle f^2 \rangle)) = \{\{2,3\},\{2,7\},\{2,13\},\{3,13\}\{7,13\}\}$$

and $\Gamma(Aut(P)) = \Gamma(P\langle f^2\rangle)$. Therefore, if $P < G$ then $\Gamma(G) = \Gamma(P)$.

Let $q = 81$. Since $81^2 + 81 + 1 = 7 \cdot 13 \cdot 73$ and $81^2 - 81 + 1 = 6481$, we have $\varepsilon = -$, $r = 6481$ and $\pi_1(P) = \{2, 3, 5, 41\}$. By Lemma 7, the graph $\Gamma(P)$ is as in Table, i. e., $E(\Gamma(P)) = \{\{2, 3\}, \{2, 5\}, \{2, 41\}, \{3, 41\}, \{5, 41\}\}$. By Lemma 10 and [1, Table 8.5], $Aut(P) = P \leftthreetimes \langle f\rangle$, where $f$ is the field automorphism of order 8 of $P$, and $F^*(C_P(x)) \cong L_2(81)$ for any involution $x$ in $Pf^2$. Therefore, if $P < G$ then $\Gamma(G) = \Gamma(P)$.

Hence, if the case (ii) holds then $G$ satisfies the item (1) of Theorem 2 and $\Gamma(G)$ is as in Table.

Suppose that the case (iii) holds. Then $p = 3$, $m$ and $u = (q-1)/2$ are odd primes, $m > 3$, $\pi((q+1)/4) = \{s\}$, and $\pi(q^2+\varepsilon q+1)) = \{r\}$. It is clear that $3^3-\varepsilon 1$ divides $q^3 - \varepsilon 1 = 3^{3m} - \varepsilon 1$. Therefore, $r = 13$ for $\varepsilon = +$ and $r = 7$ for $\varepsilon = -$. Hence $r = 2^a 3^b + 1$ for some positive integers $a$ and $b$. But $P$ is a $Crr$-group, a contradiction with [21].

5. Let $P$ be from the item (5) of Lemma 13. Then $P \cong S_4(q)$, where $q = p^m > 2$, $p$ is a prime, $m \in \mathbb{N}$, $|\pi(G)| = |\pi(P)| = 5$, and $|\pi((q^2 + 1)/(2, q-1))| = 1$. If $q \le 9$ then $|\pi(P)| \le 4$. Hence $q \ge 11$.

By the condition, $q^2 + 1 = (2, q-1)r^k$, where $r$ is a prime, $r \ne p$, and $k \in \mathbb{N}$.

Let $p = 2$. Then $r^k - 2^{2m} = 1$. By Lemma 3, $k = 1$ and $m = 2^{m_1} \ge 4$ for $m_1 \in \mathbb{N}$. Hence, $\pi(P) = \pi(2r(q^2-1)) = \pi(2r(2^{2^{m_1+1}} - 1))$. If $m_1 \ge 3$ then $2^{16}-1$ divides $2^{2^{m_1+1}} - 1$ and, consequently, $|\pi(P)| > 5$. Therefore, $m_1 = 2$, $q = 16$ and $\pi(P) = \{2, 3, 5, 17, 257\}$. By [14], $s(P) = 2$, $\pi_1(P) = \{2, 3, 5, 17, \}$, $\pi_2(P) = \{257\}$. By [1, Table 8.22], $L_2(q) \times L_2(q) \cong \Omega_4^+(q) < P$, hence $\pi_1(P)$ is a clique in the graph $\Gamma(P)$. Therefore, $P$ satisfies the item (1) of Theorem 2 and $\Gamma(P)$ is as in Table. If $P < G$ then, by Lemma 11, $Aut(P) \cong P : Z_4$ and $C_G(x) \cong S_4(4)$ for any involution $x$ in $G \setminus P$; therefore, $\Gamma(G) = \Gamma(P)$.

Let $p > 2$. Then $q^2 + 1 = 2r^k$. By [27], $s(P) = 2$, $\pi_1(P) = \pi(q(q^2-1))$, $\pi_2(P) = \{r\}$. By Lemma 5, either $k \le 2$, or $p = 239$, $r = 13$ and $k = 4$. The second possibility does not arise, since $\pi(239^2 - 1) = \{2, 3, 5, 7, 13, 17\}$. Hence, $k \le 2$. By [1, Table 8.22], $SL_2(q) \circ SL_2(q) \cong \Omega_4^+(q) < P$, hence $\pi_1(P)$ is a clique in the graph $\Gamma(P)$. If $G = Inndiag(P)$ then, by Lemma 8, the maximal tori in $G$ have orders $(q\pm 1)^2$ or $q^2 \pm 1$, hence the vertices 2 and $r$ in the graph $\Gamma(G)$ are adjancent and this graph is determined.

Since $|\pi(q^2 - 1)| = 3$, by Lemma 2, one of the following cases holds:

(i) $17 \ne q = p \ge 11$ and $\pi(q^2 - 1) = \{2,\ 3,\ s\}$;

(ii) $q \in \{25, 27, 49, 81\}$,

(iii) $p = 3$, $m$ is an odd prime and $m$ does not divide $q(q^2 - 1)$.

If the case (i) holds then $G$ satisfies the item (11) of Theorem 2 and $\Gamma(G)$ is as in Table.

Suppose that the case (ii) holds. If $q = 27$ then $\pi(q^2 + 1) = \{2, 5, 73\}$, a contradiction. If $q = 81$ then $\pi(q^2 + 1) = \{2, 17, 193\}$, a contradiction. Hence $q = p^2$, where $p \in \{5, 7\}$. By Lemma 11, $Aut(P) = Inndiag(P) \leftthreetimes \langle f\rangle$, where $f$ is the involutive field automorphism of $P$, and $C_P(x) \cong S_4(p)$ for any involution $x$ in $Pf$. Therefore, for any group $G$ with $P \le G \le Aut(G)$, $G$ satisfies the item (1) of Theorem 2 and $\Gamma(G)$ is as in Table.

Suppose that the case (iii) holds. Then $q^2 + 1 = 9^m + 1$ and, consequently, 10 divides $q^2 + 1$. Thus, $r = 5$ and $2r^k \in \{10, 50\}$, i. e. $q \in \{3, 7\}$, a contradiction.

6. Let $P$ be from the item (6) of Lemma 13. Then $P \cong L_3^{\varepsilon}(q)$, $q = p^m$, $p$ is a prime, $m \in \mathbb{N}$, $\varepsilon \in \{+, -\}$, $|\pi(\frac{q^2+\varepsilon q+1}{(3,q-\varepsilon 1)})| = 1$, $|\pi(P)| = 5$.

By [17], $17 \neq q \geq 9$ and $q > 9$ for $\varepsilon = -$. By [27, 14], $s(P) = 2$, $\pi_1(P) = \pi(q(q^2-1))$ and $\pi_2(P) = \{r\}$, where $\pi(\frac{q^2+\varepsilon q+1}{(3,q-\epsilon 1)}) = \{r\}$. Hence $|\pi(q^2-1)| = 3$. By Lemma 2, one of the following cases holds:

(i) $17 \neq q = p \geq 11$;

(ii) $q \in \{16, 25, 27, 49, 81\}$,

(iii) $p \in \{2, 3\}$, $m$ and $(q-1)/(2, q-1)$ are odd primes, $|\pi((q+1)/(p+1)| = 1$ and $m$ does not divide $q(q^2-1)$.

Let $p = 2$. Then the case (i) is impossible.

Suppose that the case (ii) holds. Then $q = 16$. If $\varepsilon = +$ then $\pi(\frac{q^2+\varepsilon q+1}{(3,q-1)}) = 7 \cdot 13$, a contradiction. Hence, $\varepsilon = -$ and $P = U_3(16)$. Then $\pi_1(P) = \{2, 3, 5, 17\}$, $\pi_2(P) = \{257\}$. By Lemma 7, the graph $\Gamma(P)$ is as in Table, i. e., $E(\Gamma(P)) = \{\{2, 17\}, \{3, 5\}, \{3, 17\}, \{5, 17\}\}$. By Lemma 10, we have $Aut(P) = P \leftthreetimes \langle f \rangle$, where $f$ is field automorphism of order 8 of $P$, and if $x$ is an involution in $Aut(P) \setminus P$ then $C_P(x)$ is isomorphic either to $L_2(16)$ or to the centralizer of some involution in $S_4(4)$. Hence if $P < G$ then $\pi_1(G) = \pi_1(P)$ is a coclique in the graph $\Gamma(G)$. Therefore, for any group $G$ with $P \leq G \leq Aut(G)$, $G$ satisfies the item (1) of Theorem 2 and $\Gamma(G)$ is as in Table.

Suppose that the case (iii) holds. Let $\varepsilon = +$. Then $m$ and $q-1$ are odd primes, and $q-1 > 7$. It is clear that $(q-1, 3) = 1$ and 7 divides $q^3 - 1 = 2^{3m} - 1 = 8^m - 1$. Hence $\pi((q^3-1))/(q-1)) = \{7\}$. By Lemma 3, there exists a prime divisor of $2^{3m} - 1$ which does not divide $2^3 - 1 = 7$ and $2^m - 1 = q - 1$, a contradiction. Therefore, $\varepsilon = -$, $m > 3$, $(q+1)_3 = 3$, and $(q^3+1)_3 = 9$.

Let $q-1 = s$ and $t \in \pi((q+1)/3)$. By Lemma 7, $E(\Gamma(P)) = \{\{2, t\}, \{3, t\}, \{t, s\}\}$, i. e., $\Gamma(P)$ is as in Table. By Lemma 10 and [1, Table 8.5], we have $Aut(P) = Inndiag(P) \leftthreetimes \langle f \rangle$, where $Outdiag(G) \cong Z_3$, $f$ is the field automorphism of order $2m$ of $P$, $f^m$ inverts the group $Outdiag(G)$ and if $x$ is an involution in $Aut(P) \setminus Inndiag(P)$ then $F^*(C_P(x)) \cong L_2(q)$. We have $P \leq G \leq Inndiag(P)\langle f^m \rangle$. If $G = Inndiag(P)$ then maximal tori of $G$ have the orders $q^2 - q + 1$, $q^2 \pm 1$ or $(q+1)^2$ and, by [1, Table 8.5], $(q+1) \times L_2(q) \cong GU_2(q) < G$, hence, $E(\Gamma(Inndiag(P))) = \{\{2, 3\}, \{2, t\}, \{3, t\}, \{3, s\}, \{3, r\}, \{t, s\}\}$, i. e., $\Gamma(G)$ is as in Table. If $G = P\langle f^m \rangle$ then $E(\Gamma(G)) = \{\{2, 3\}, \{2, t\}, \{2, s\}, \{3, t\}, \{t, s\}\}$, i. e., $\Gamma(P\langle f^m \rangle)$ is as in Table. Thus, $E(\Gamma(Inndiag(P)\langle f^m \rangle)) = \{\{2, 3\}, \{2, t\}, \{2, s\}, \{3, t\}, \{3, s\}, \{3, r\}, \{t, s\}\}$, i. e., $\Gamma(Inndiag(P)\langle f^m \rangle)$ is as in Table. Therefore, for any group $G$ with $P \leq G \leq Inndiag(P)\langle f^m \rangle$, $G$ satisfies the item (8) of Theorem 2 and $\Gamma(G)$ is as in Table.

Let $p > 2$.

Suppose that the case (i) holds. Then $17 \neq q = p \geq 11$ and $\pi_1(P) = \{2, 3, s, p\}$. By [1, Tables 8.3, 8.5], $SL_3^{\varepsilon}(q)$ contains a maximal subgroup isomorphic to $GL_2^{\varepsilon}(q)$. Hence, $P$ contains a maximal subgroup $H$ isomorphic to $Z_a.PGL_2(q)$, where $a = (q - \varepsilon 1)/(3, q - \varepsilon 1)$. Using the information for the case 1 and Lemmas 6-10, we obtain that, for any group $G$ with $P \leq G \leq Aut(P)$, $G$ satisfies the item (9) of Theorem 2 and $\Gamma(G)$ is as in Table.

Suppose that the case (ii) holds. Then $q \in \{25, 27, 49, 81\}$. If $q = 49$ then the set $\pi(\frac{q^2+\varepsilon q+1}{(3,q-1)})$ equals to $\{19, 43\}$ for $\varepsilon = +$ and to $\{13, 181\}$ for $\varepsilon = -$, a contradiction.

Let $q = 25$. Since $(25^2 + 25 + 1)/(3, 25 - 1) = 7 \cdot 31$ and $(25^2 - 25 + 1)(3, 25 + 1) = 601$, we have $\varepsilon = -$, $r = 601$ and $\pi_1(P) = \{2, 3, 5, 13\}$. By Lemma 7, the graph

$\Gamma(P)$ is as in Table, i. e., $E(\Gamma(P)) = \{\{2,3\},\{2,5\},\{2,13\},\{3,13\}\}$. By Lemma 10 and [1, Table 8.5], $Aut(P) = P \leftthreetimes \langle f\rangle$, where $f$ is an automorphism of order 4 of $P$, and $F^*(C_P(x)) \cong L_2(25)$ for any involution $x$ in $Pf^2$. Therefore, if $P < G$ then $\Gamma(G) = \Gamma(P)$.

Let $q = 27$. Since $27^2 - 27 + 1 = 19 \cdot 37$ and $27^2 + 27 + 1 = 757$, we have $\varepsilon = +$, $r = 757$ and $\pi_1(P) = \{2,3,7,13\}$. By Lemma 6, the graph $\Gamma(P)$ is as in Table, i. e.,

$$E(\Gamma(P)) = \{\{2,3\},\{2,7\},\{2,13\},\{3,13\},\{7,13\}\}.$$

By Lemma 9 and [1, Table 8.3], $Aut(P) = P \leftthreetimes \langle f\rangle$, where $f$ is an automorphism of order 6 of $P$, $F^*(C_P(x)) \cong L_2(27)$ for any involution $x$ in $Pf^3$ and $F^*(C_P(y)) \cong L_3(3)$ for any element $y$ of order 3 in $Pf^2$. Therefore, if $P < G$ then $\Gamma(G) = \Gamma(P)$.

Let $q = 81$. Since $81^2 + 81 + 1 = 7 \cdot 13 \cdot 73$ and $81^2 - 81 + 1 = 6481$, we have $\varepsilon = -$, $r = 6481$ and $\pi_1(P) = \{2,3,5,41\}$. By Lemma 7, the graph $\Gamma(P)$ is as in Table, i. e., $E(\Gamma(P)) = \{\{2,3\},\{2,5\},\{2,41\},\{5,41\}\}$. By Lemma 10 and [1, Table 8.5], $Aut(P) = P \leftthreetimes \langle f\rangle$, where $f$ is the field automorphism of order 8 of $P$, and $F^*(C_P(x)) \cong L_2(81)$ for any involution $x$ in $Pf^2$. Therefore, if $P < G$ then $\Gamma(G) = \Gamma(P)$.

Hence, if the case (ii) holds then $G$ satisfies the item (1) of Theorem 2 and $\Gamma(G)$ is as in Table.

Suppose that the case (iii) holds. Then $p = 3$, $m$ and $u = (q-1)/2$ are odd primes, $m > 3$, $\pi((q+1)/4) = \{s\}$, and $\pi(q^2+\varepsilon q+1)) = \{r\}$. It is clear that $3^3 - \varepsilon 1$ divides $q^3 - \varepsilon 1 = 3^{3m} - \varepsilon 1$. Therefore, $r = 13$ for $\varepsilon = +$ and $r = 7$ for $\varepsilon = -$. Hence $r = 2^a 3^b + 1$ for some positive integers $a$ and $b$. But $P$ is a $Crr$-group, a contradiction with citeli (see also Table 1 in the Shi's survey [24]).

6. Let $P$ be from the item (6) of Lemma 13. Then $P \cong Sz(q)$, where $q =\geq 2^7$ and $|\pi(G)| = |\pi(P)| = 5$. By [14], $s(P) = 4$, $\pi_1(P) = \{2\}$, $\pi_2(P) = \pi(q-1)$, $\pi_3(P) = \pi(q - \sqrt{2q} + 1)$, $\pi_4(P) = \pi(q + \sqrt{2q} + 1)$. By [20], the maximal subgroups in $P$ are conjugated to one of Frobenius groups of the form $q^2 : (q-1)$, $D_{2(q-1)}$, or $(q \pm \sqrt{2q} + 1) : 4$. By Lemma 12, $Aut(G) = P \leftthreetimes \langle f\rangle$, where $\langle f\rangle \cong Aut(GF(q)) \cong Z_{2m+1}$ is the group of field automorphisms of $P$. Since 5 divides $q^2+1 = 4^{2m+1}+1$, 5 divides $q - \varepsilon\sqrt{2q} + 1$ for some $\varepsilon \in \{+,-\}$.

If $\pi(q - \varepsilon\sqrt{2q} + 1) = \{5\}$ then $P$ is a $C55$-group and consequently, by [8], $q \in \{8,\ 32\}$, which is impossible. Hence, $|\pi(q - \varepsilon\sqrt{2q} + 1)| \geq 2$. The equality $|\pi(P)| = 5$ implies that $|\pi(q-1)| = |\pi(q+\varepsilon\sqrt{2q}+1)| = 1$ and $|\pi(q-\varepsilon\sqrt{2q}+1)| = 2$. By Lemma 2, $q-1 = r$ and $2m+1 = p$ are odd primes. Let $\pi(q-\varepsilon\sqrt{2q}+1) = \{5,\ s\}$ and $\pi(q+\varepsilon\sqrt{2q}+1) = \{t\}$. Then $P$ satisfies the item (11) of Theorem 2 and $\Gamma(P)$ is as in Table.

Let $P) < G$. Then $G = Aut(Sz(q)) \cong Sz(q) : p$. Since $|\pi(G)| = 5$, we have $p \in \pi(Sz(q))$. It is clear that $p \notin \{2,\ 3,\ r\}$. If $p = s$ or $p = t$ then $f$ centralizes an element of order $s$ or $t$ in some cyclic subgroup of order $q-\varepsilon\sqrt{2q}+1$ or $q-\varepsilon\sqrt{2q}+1$, respectively, in $P$. But, this contradicts the isomorphism $C_P(f) \cong 5 : 4$. Therefore, this case is impossible.

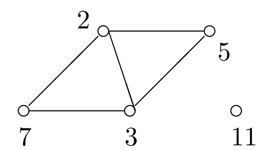
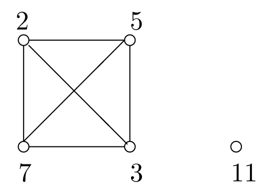
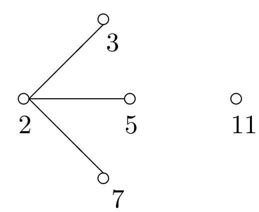
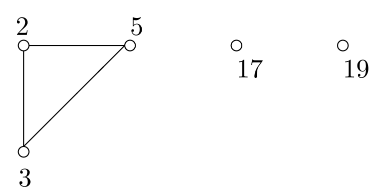
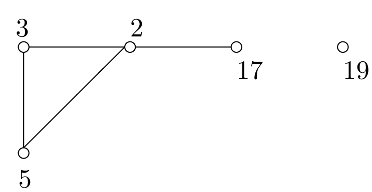
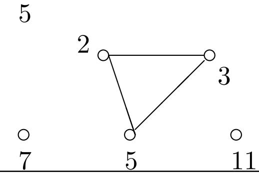
The necessary condition of Theorem 2 is proved. The sufficient condition of Theorem 2 is obvious.

Theorems 1 and 2 are completely proved.

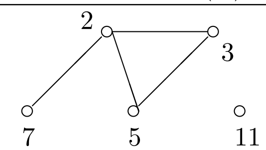
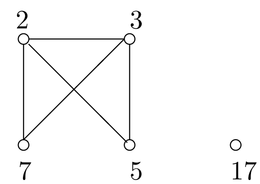
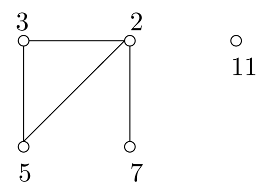
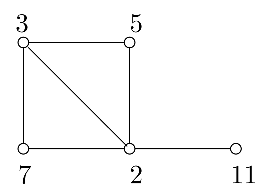
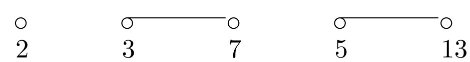
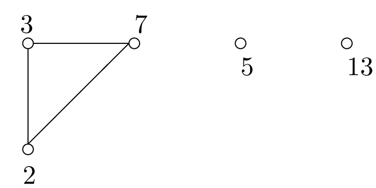
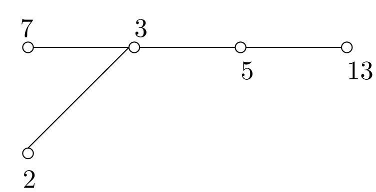
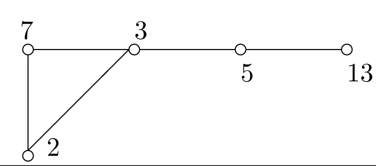
## 4. The table

In Table, we list all finite almost simple 5-primary groups $G$ and their prime graphs $\Gamma(G)$. In addition, $d$, $f$ and $g$ denote outer diagonal, field and graph automorphisms of a group $Soc(G)$ of Lie type, respectively.

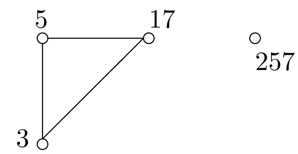
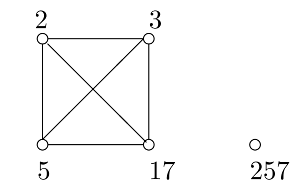
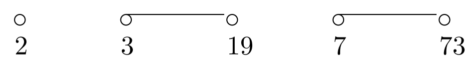
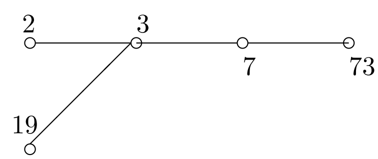
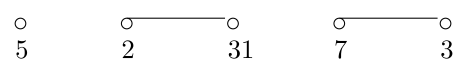
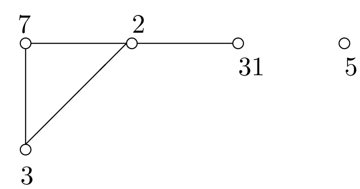
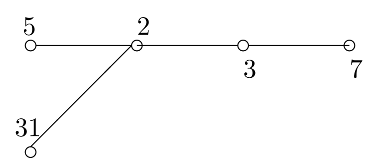
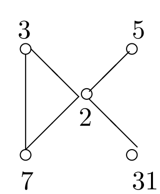
**Table. Finite almost simple 5-primary groups $G$ and their prime graphs $\Gamma(G)$**

| Group $G$ | Graph $\Gamma(G)$ |
|---|---|
| $A_{11}, S_{11}$ |   |
| $A_{12}, S_{12}$ |   |
| $M_{22}$ |   |
| $Aut(M_{22})$ |   |
| $J_3$ |   |
| $Aut(J_3)$ |   |
| $HS$ |   |

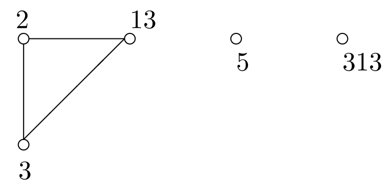
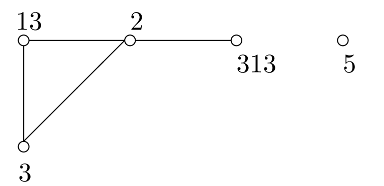
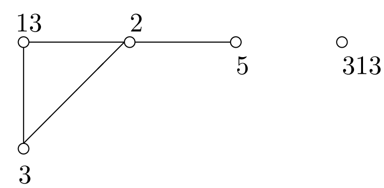
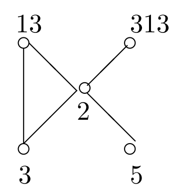
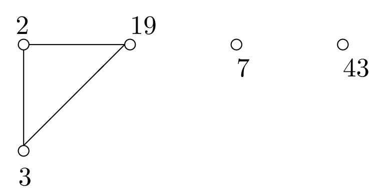
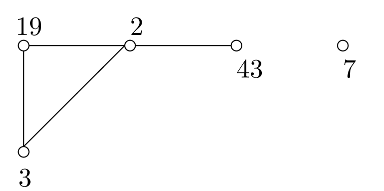
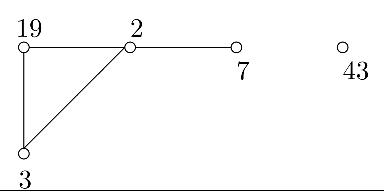
Continue Table

| Group $G$ | Graph $\Gamma(G)$ |
|---|---|
| $Aut(HS)$ |  |
| $He, Aut(He)$ |  |
| $M^cL$ |  |
| $Aut(M^cL)$ |  |
| $L_2(2^6)$ |  |
| $L_2(2^6):2$ |  |
| $L_2(2^6):3$ |  |
| $Aut(L_2(2^6)) \cong L_2(2^6):6$ |  |

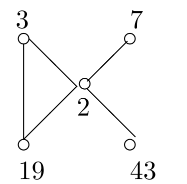
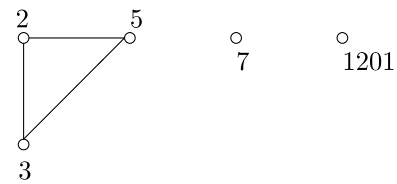
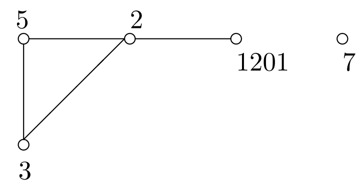
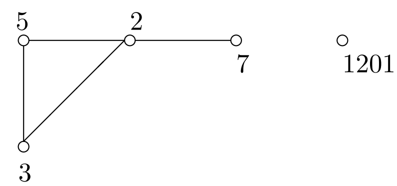
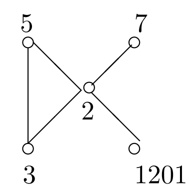
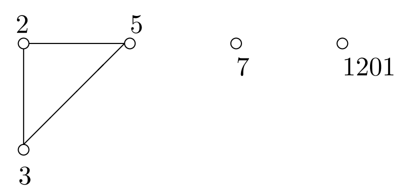
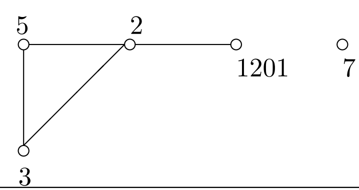
Continue Table

| Group $G$ | Graph $\Gamma(G)$ |
|---|---|
| $L_2(2^8)$ |   |
| $L_2(2^8) < G \leq Aut(L_2(2^8)) \cong L_2(2^8):8$ |   |
| $L_2(2^9)$ |   |
| $L_2(2^9) < G \leq Aut(L_2(2^9)) \cong L_2(2^9):9$ |   |
| $L_2(5^3)$ |   |
| $PGL_2(5^3)$ |   |
| $L_2(5^3):3$ |   |
| $Aut(L_2(5^3)) \cong L_2(5^3).6$ |   |

Continue Table

| Group $G$ | Graph $\Gamma(G)$ |
|---|---|
| $L_2(5^4), L_2(5^4)\,\langle fd\rangle$ |   |
| $PGL_2(5^4) = L_2(5^4)\langle d\rangle$ |   |
| $L_2(5^4)\,\langle f\rangle$ |   |
| $L_2(5^4).2^2 \leq G \leq Aut(L_2(5^4)) \cong$ $\cong L_2(5^4).(2\times 4)$ |   |
| $L_2(7^3)$ |   |
| $PGL_2(7^3)$ |   |
| $L_2(7^3):3$ |   |

Continue Table

| Group $G$ | Graph $\Gamma(G)$ |
|---|---|
| $Aut(L_2(7^3))$ |  |
| $L_2(7^4), L_2(7^4)\left\langle f^2d\right\rangle, \lvert f\rvert = 4$ |  |
| $PGL_2(7^4) = L_2(7^4)\left\langle d\right\rangle$ |  |
| $L_2(7^4)\left\langle f^2\right\rangle, L_2(7^4)\left\langle f\right\rangle, L_2(7^4)\left\langle fd\right\rangle$ |  |
| $Aut(L_2(7^4))$ |  |
| $L_2(7^7)$ |  |
| $PGL_2(7^7)$ |  |

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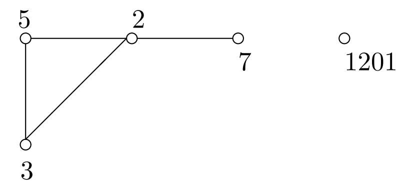
<tr><td>$L_2(7^7):7$</td><td>
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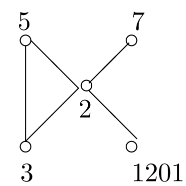
<tr><td>$Aut(L_2(7^7)) \cong L_2(7^7).14$</td><td>
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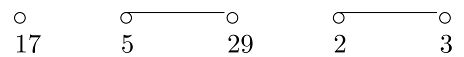
<tr><td>$L_2(17^2), L_2(17^2)\langle fd\rangle$</td><td>
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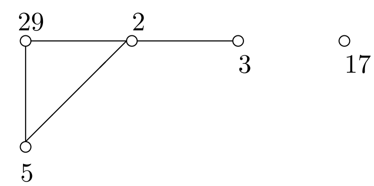
<tr><td>$PGL_2(17^2) = L_2(17^2)\langle d\rangle$</td><td>
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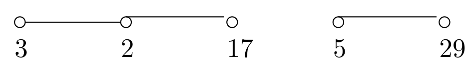
<tr><td>$L_2(17^2)\langle f\rangle$</td><td>
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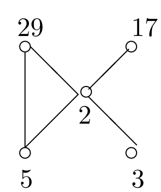
<tr><td>$Aut(L_2(17^2)) \cong L_2(17^2).2^2$</td><td>
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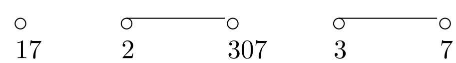
<tr><td>$L_2(17^3)$</td><td>
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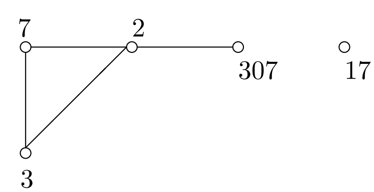
<tr><td>$PGL_2(17^3)$</td><td>
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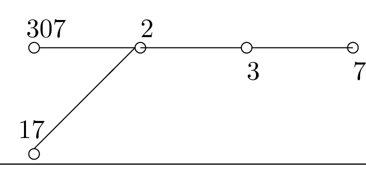
<tr><td>$L_2(17^3):3$</td><td>
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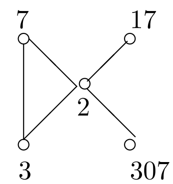
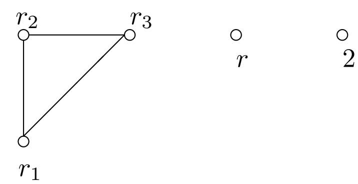
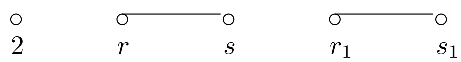
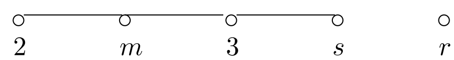
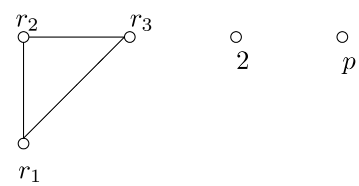
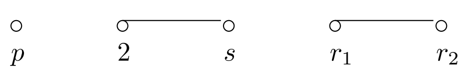
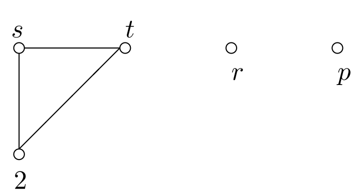
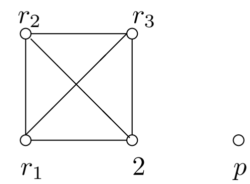
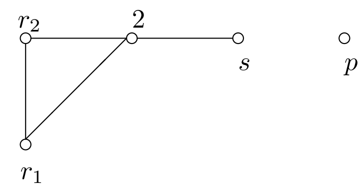
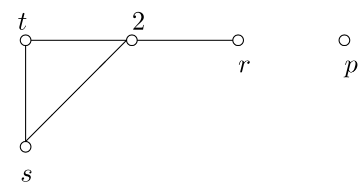
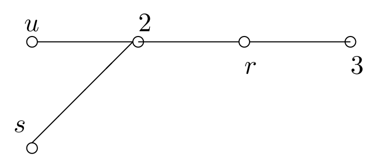
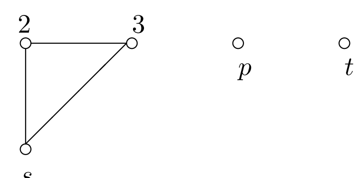
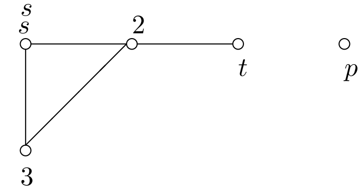
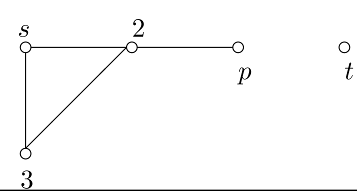
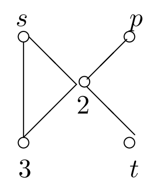
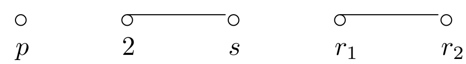
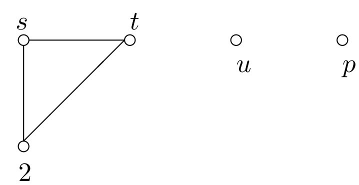
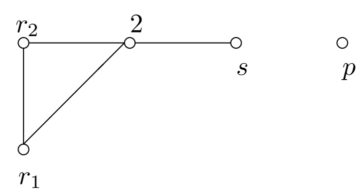
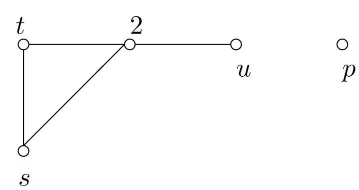
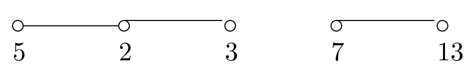
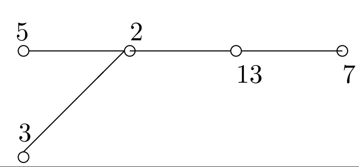
| Group $G$ | Graph $\Gamma(G)$ |
| --- | --- |
| $Aut(L_2(17^3)) \cong L_2(17^3).6$ |  |
| $L_2(q)$, $q = 2^m$, $m$ is a prime, $m \geq 11$ | |
| $q - \varepsilon 1 = r$ is a prime, $\varepsilon \in \{+, -\}$, $\pi(q + \epsilon 1) = \{r_1, r_2, r_3\}$ |  |
| $\pi(q-1) = \{r, s\}$, $\pi(q+1) = \{r_1, s_1\}$ |  |
| $G = Aut(L_2(2^m)) \cong L_2(2^m) : m$, $m, r = 2^m - 1, s = \frac{2^m+1}{3}$ are different primes, bigger 5 |  |
| $L_2(p)$, $p$ is a prime, $p \geq 41$, $\lvert\pi(p^2-1)\rvert = 4$ | |
| $\pi(p-1) = \{2\}$, $\pi(\frac{p+1}{2}) = \{r_1, r_2, r_3\}$ |  |
| $\pi(p-1) = \{2, s\}$, $\pi(\frac{p+1}{2}) = \{r_1, r_2\}$ |  |
| $\pi(p-1) = \{2, s, t\}$, $\pi(\frac{p+1}{2}) = \{r\}$ |  |
| $PGL_2(p)$, $p$ is a prime, $p \geq 41$, $\lvert\pi(p^2-1)\rvert = 4$ | |

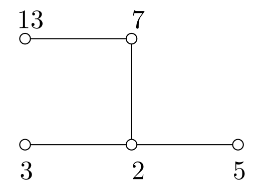
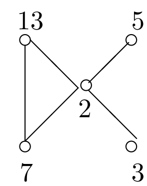
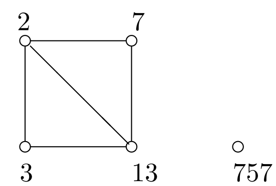
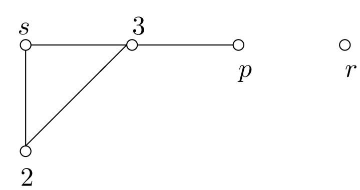
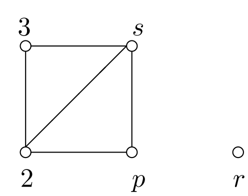
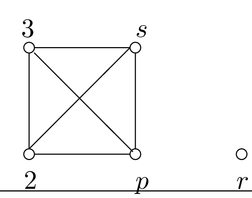
Continue Table

| Group $G$ | Graph $\Gamma(G)$ |
|---|---|
| $\pi(p-1)=\{2\}$, $\pi(\frac{p+1}{2})=\{r_1,r_2,r_3\}$ |   |
| $\pi(p-1)=\{2,s\}$, $\pi(\frac{p+1}{2})=\{r_1,r_2\}$ |   |
| $\pi(p-1)=\{2,s,t\}$, $\pi(\frac{p+1}{2})=\{r\}$ |   |
| $Aut(L_2(3^r))$, $O^2(Aut(L_2(3^r)))$, $r$ and $u=\frac{3^r-1}{2}$ are primes, bigger 3, $\pi(3^r+1)=\{2,s\}$ |   |
| $L_2(p^2)$, $L_2(p^2)\langle fd\rangle$, $p$ is a prime, $p\geq 11$, $\lvert\pi(p^2-1)\rvert=\{2,3,s\}$, $p^2+1=2t^k$, $k\leq 2$, $s$ and $t$ are primes |   |
| $PGL_2(p^2)=L_2(p^2)\langle d\rangle$ $p$ is a prime, $p\geq 11$, $\lvert\pi(p^2-1)\rvert=\{2,3,s\}$, $p^2+1=2t^k$, $k\leq 2$, $s$ and $t$ are primes |   |
| $L_2(p^2)\langle f\rangle$ $p$ is a prime, $p\geq 11$, $\lvert\pi(p^2-1)\rvert=\{2,3,s\}$, $p^2+1=2t^k$, $k\leq 2$, $s$ and $t$ are primes |   |

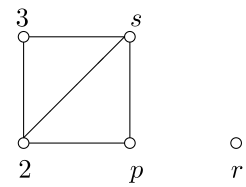
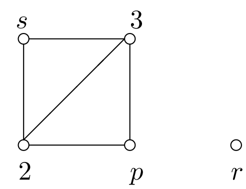
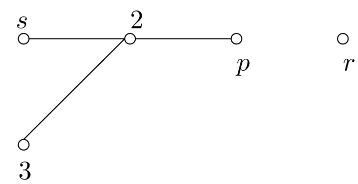
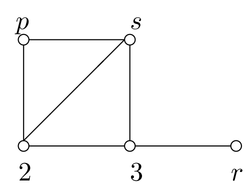
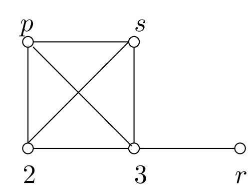
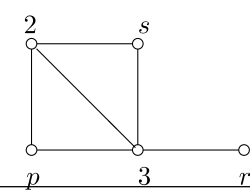
Continue Table

| Group $G$ | Graph $\Gamma(G)$ |
|---|---|
| $Aut(L_2(p^2))$, $p$ is a prime, $p \geq 11$, $\lvert\pi(p^2-1)\rvert = \{2,3,s\}$, $p^2+1 = 2t^k$, $k \leq 2$, $s$ and $t$ are primes |  |
| $L_2(p^r)$, $p$ and $r$ are odd primes, $p \in \{3,5,7,17\}$, $r$ does not divide $p(p^{2r}-1)$, $\varepsilon \in \{+,-\}$ | |
| $\pi(p^r - \varepsilon 1) = \{2,s\}$, $\pi(\frac{p^r+\varepsilon 1}{2}) = \{r_1, r_2\}$ |  |
| $\pi(p^r - \varepsilon 1) = \{2,s,t\}$, $\pi(\frac{p^r+\varepsilon 1}{2}) = \{u\}$ |  |
| $PGL_2(p^r)$, $p$ and $r$ are odd primes, $p \in \{3,5,7,17\}$, $r$ does not divide $p(p^{2r}-1)$, $\varepsilon \in \{+,-\}$ | |
| $\pi(p^r - \varepsilon 1) = \{2,s\}$, $\pi(\frac{p^r+\varepsilon 1}{2}) = \{r_1, r_2\}$ |  |
| $\pi(p^r - \varepsilon 1) = \{2,s,t\}$, $\pi(\frac{p^r+\varepsilon 1}{2}) = \{u\}$ |  |
| $L_3(9), L_3(9)\langle g\rangle$ |  |
| $L_3(9)\langle f\rangle$ |  |

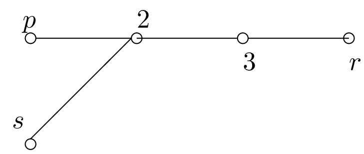
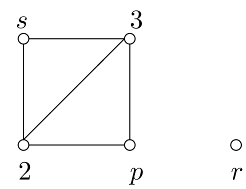
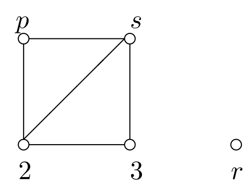
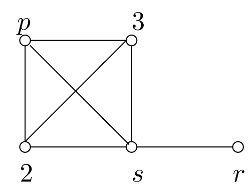
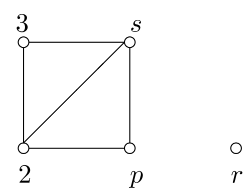
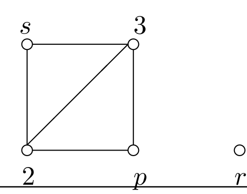
Continue Table

| Group $G$ | Graph $\Gamma(G)$ |
|---|---|
| $L_3(9)\langle fg\rangle$ |   |
| $Aut(L_3(9)) \cong L_3(9):2^2$ |   |
| $L_3(27) \leq G \leq Aut(L_3(27)) \cong L_3(27).6$ |   |
| $L_3(p)$, $p$ is a prime, $17 \neq p \geq 11$, $\pi(p^2-1)=\{2,3,s\}$, $\pi(\frac{p^2p+p+1}{(3,p-1)})=\{r\}$ | |
| $p-1=2^m$ |   |
| $p+1=2^m$, $(p-1)_3=3$ |   |
| $p+1=2^m$, $(p-1)_3>3$ |   |

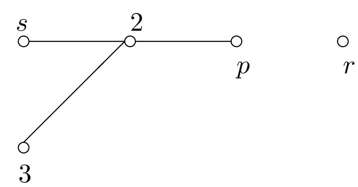
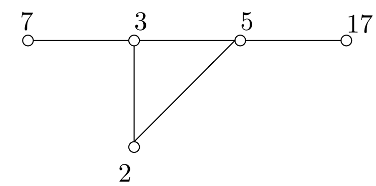
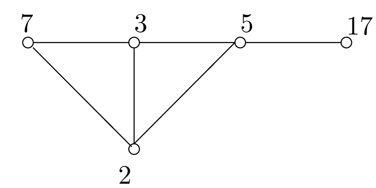
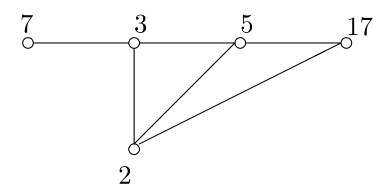
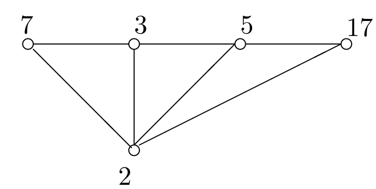
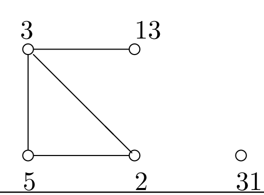
Continue Table

| Group $G$ | Graph $\Gamma(G)$ |
|---|---|
| $p \pm 1 \neq 2^m$, $(p-1)_3 = 1$ |  |
| $p \pm 1 \neq 2^m$, $(p-1)_3 > 3$ |  |
| $p \pm 1 \neq 2^m$, $(p-1)_3 = 3$ |  |
| $PGL_3(p) \cong L_3(p):3$, $Aut(L_3(p))$, $17 \neq p \geq 11$, $\pi(p^2-1) = \{2,3,s\}$, $\pi(\frac{p^2p+p+1}{(3,p-1)}) = \{r\}$ | |
| $p + 1 = 2^m$, $(p-1)_3 = 3$ |  |
| $p + 1 = 2^m$, $(p-1)_3 > 3$ |  |
| $p \pm 1 \neq 2^m$, $(p-1)_3 > 3$ |  |

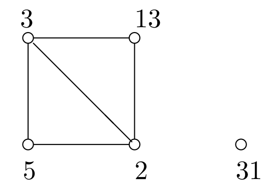
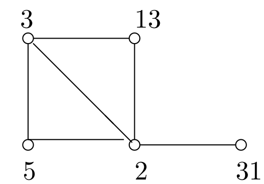
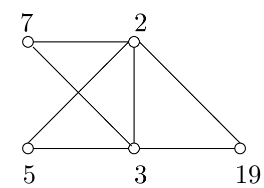
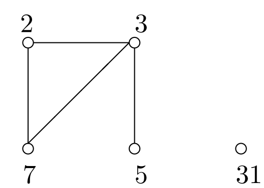
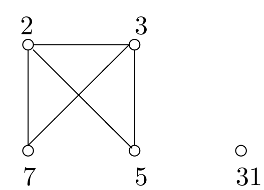
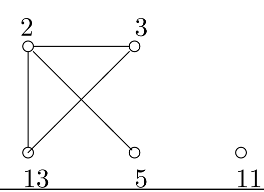
Continue Table

| Group $G$ | Graph $\Gamma(G)$ |
|---|---|
| $p \pm 1 \neq 2^m$, $(p-1)_3 = 3$ |   |
| $L_3(p) : 2, 11 \neq p \geq 17$, $\pi(p^2-1) = \{2, 3, s\}$, $\pi(\frac{p^2p+p+1}{(3,p-1)}) = \{r\}$ | |
| $p - 1 = 2^m$ |   |
| $p + 1 = 2^m$, $(p-1)_3 = 3$ |   |
| $p + 1 = 2^m$,$(p-1)_3 > 3$ |   |
| $p \pm 1 \neq 2^m$,$(p-1)_3 = 1$ |   |
| $p \pm 1 \neq 2^m$,$(p-1)_3 > 3$ |   |

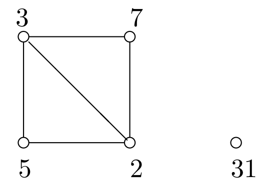
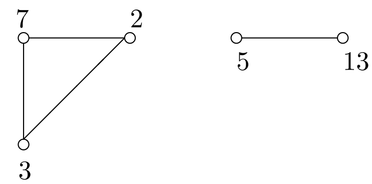
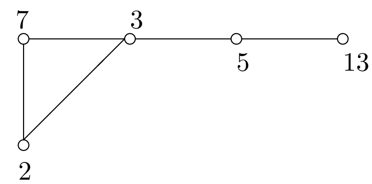
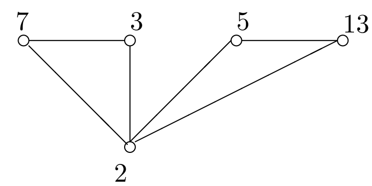
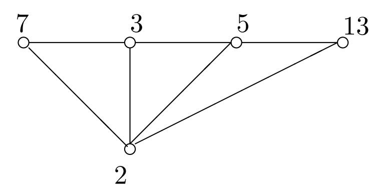
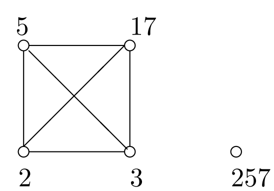
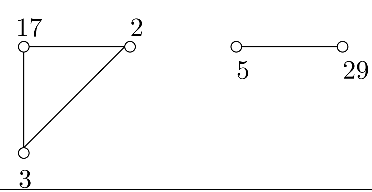
Continue Table

| Group $G$ | Graph $\Gamma(G)$ |
|---|---|
| $p \pm 1 \neq 2^m$, $(p-1)_3 = 3$ |  |
| $L_4(4)$, $L_4(4)\langle fg\rangle$ |  |
| $L_4(4)\langle f\rangle$ |  |
| $L_4(4)\langle g\rangle$ |  |
| $Aut(L_4(4)) \cong L_4(4) : 2^2$ |  |
| $L_4(5)$ |  |

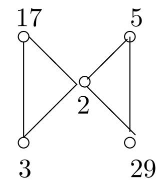
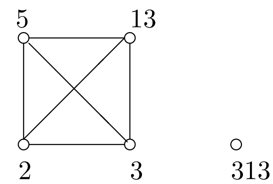
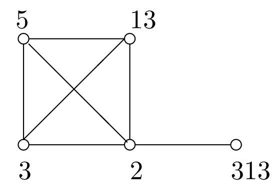
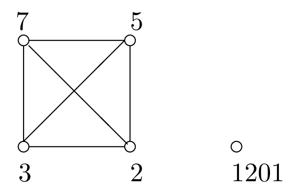
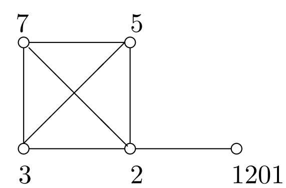
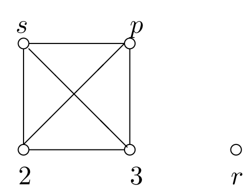
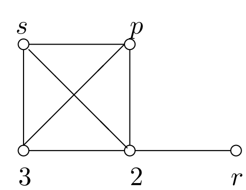
Continue Table

| Group $G$ | Graph $\Gamma(G)$ |
|---|---|
| $L_4(5) < G \leq Aut(L_4(5)) \cong L_4(5).D_8$ | |
| $G \cap PGL_4(5) = L_4(5)$ |   |
| $G \cap PGL_4(5) > L_4(5)$ |   |
| $L_4(7) \leq G \leq Aut(L_4(7)) \cong L_4(7).2^2$ |   |
| $L_5(2)$ |   |
| $Aut(L_5(2)) \cong L_5(2):2$ |   |
| $L_5(3), Aut(L_5(3)) \cong L_5(3):2$ |   |

Continue Table

| Group $G$ | Graph $\Gamma(G)$ |
|---|---|
| $L_6(2), Aut(L_6(2)) \cong L_6(2):2$ |  |
| $S_4(8)$ |  |
| $S_4(8):3$ |  |
| $S_4(8):2$ |  |
| $Aut(S_4(8)) \cong S_4(8):6$ |  |
| $S_4(16) \leq G \leq Aut(S_4(16)) \cong S_4(16):4$ |  |
| $S_4(17)$ |  |

Continue Table

| Group $G$ | Graph $\Gamma(G)$ |
|---|---|
| $Aut(S_4(17)) \cong PGSp_4(17) \cong S_4(17).2$ |   |
| $S_4(25)$, $S_4(25)\langle f\rangle$, $S_4(25)\langle fd\rangle$ |   |
| $S_4(25)\langle d\rangle$, $Aut(S_4(25)) \cong S_4(25).2^2$ |   |
| $S_4(49)$, $S_4(49)\langle f\rangle$, $S_4(49)\langle fd\rangle$ |   |
| $S_4(49)\langle d\rangle$, $Aut(S_4(49)) \cong S_4(49).2^2$ |   |
| $S_4(p)$, $p$ is a prime, $17 \neq p \geq 11$, $\pi(p^2-1) = \{2,3,s\}$, $\pi(\frac{p^2+1}{2}) = \{r\}$ |   |
| $PGSp_4(p)$, $p$ is a prime, $17 \neq p \geq 11$, $\pi(p^2-1) = \{2,3,s\}$, $\pi(\frac{p^2+1}{2}) = \{r\}$ |   |

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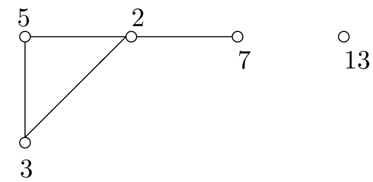
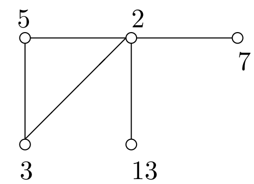
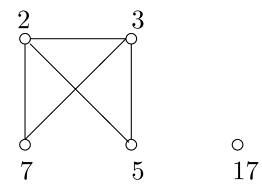
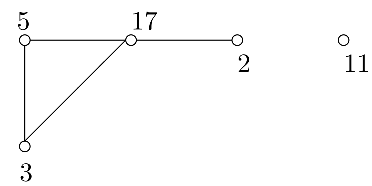
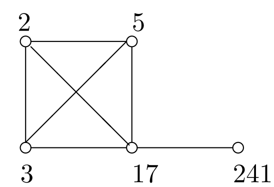
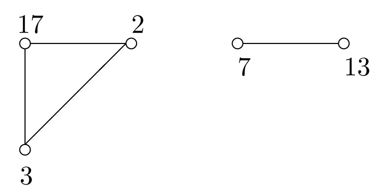
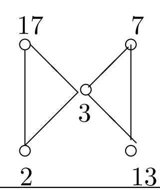
| Group $G$ | Graph $\Gamma(G)$ |
|---|---|
| $S_6(3)$ |  |
| $Aut(S_6(3)) \cong S_6(3).2$ |  |
| $S_8(2)$ |  |
| $U_3(16)$ |  |
| $U_3(16) < G \leq Aut(U_3(16)) \cong U_3(16):8$ |  |
| $U_3(17), U_3(17).2$ |  |
| $U_3(17).3 \cong PGU_3(17) \leq G \leq$ $\leq Aut(U_3(17)) \cong U_3(17).S_3$ |  |

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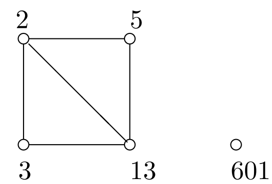
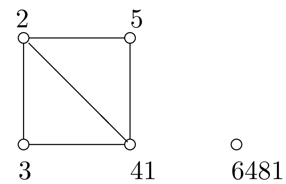
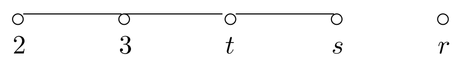
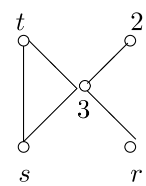
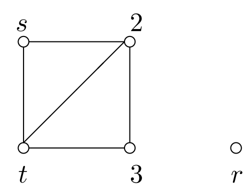
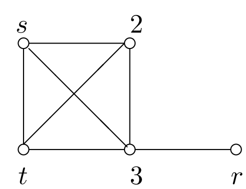
| Group $G$ | Graph $\Gamma(G)$ |
|---|---|
| $U_3(25) \leq G \leq Aut(U_3(25)) \cong U_3(25):4$ |   |
| $U_3(81) \leq G \leq Aut(U_3(81)) \cong U_3(81).8$ |   |
| $U_3(2^p)$, $p$, $s = 2^p - 1$ and $t = (2^p + 1)/3$ are primes, $p \geq 5$, $\pi(2^{2p} - 2^p +) = \{r\}$ |   |
| $PGU_3(2^p)$, $p$, $s = 2^p - 1$ and $t = (2^p + 1)/3$ are primes, $p \geq 5$, $\pi(2^{2p} - 2^p +) = \{r\}$ |   |
| $U_3(2^p) : 2$, $p$, $s = 2^p - 1$ and $t = (2^p + 1)/3$ are primes, $p \geq 5$, $\pi(2^{2p} - 2^p +) = \{r\}$ |   |
| $PGU_3(2^p) : 2$, $p$, $s = 2^p - 1$ and $t = (2^p + 1)/3$ are primes, $p \geq 5$, $\pi(2^{2p} - 2^p +) = \{r\}$ |   |
| $U_3(p)$, $p$ is a prime, $17 \neq p \geq 11$, $\pi(p^2 - 1) = \{2, 3, s\}$, $\pi(\frac{p^2 p - p + 1}{(3, p+1)}) = \{r\}$ | |

Continue Table

<table>
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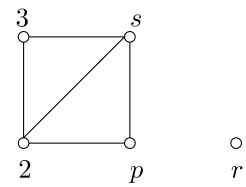
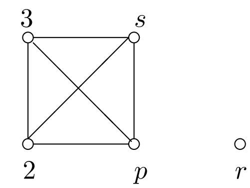
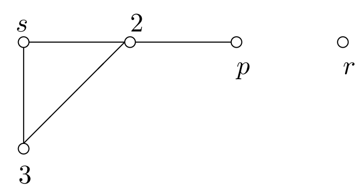
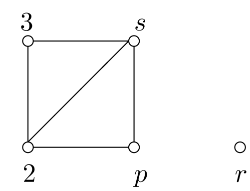
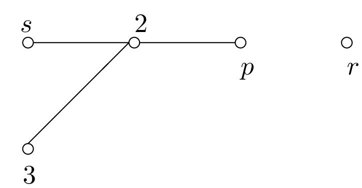
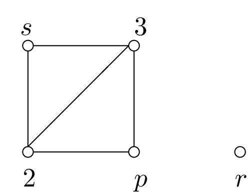
<tr><td>$p-1=2^m$, $(p+1)_3=3$</td><td></td></tr>
<tr><td>$p-1=2^m$, $(p+1)_3>3$</td><td></td></tr>
<tr><td>$p+1=2^m$</td><td></td></tr>
<tr><td>$p\pm 1\neq 2^m$, $(p+1)_3=1$</td><td></td></tr>
<tr><td>$p\pm 1\neq 2^m$, $(p+1)_3=3$</td><td></td></tr>
<tr><td>$p\pm 1\neq 2^m$, $(p+1)_3>3$</td><td></td></tr>
<tr><td>$PGU_3(p)=U_3(p):3$, $U_3(p):S_3$, $p$ is a prime, $17\neq p\geq 11$, $\pi(p^2-1)=\{2,3,s\}$, $\pi(\frac{p^2p-p+1}{(3,p+1)})=\{r\}$</td><td></td></tr>
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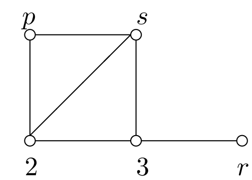
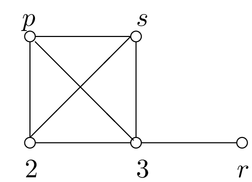
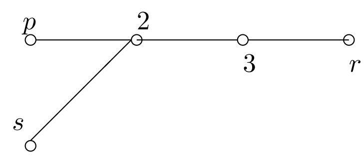
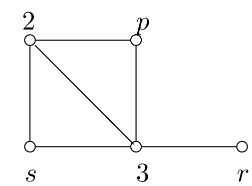
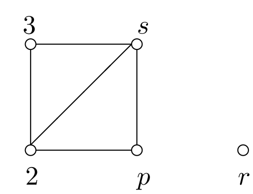
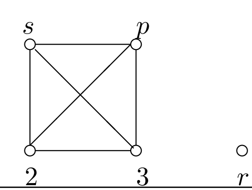
Continue Table

| Group $G$ | Graph $\Gamma(G)$ |
|---|---|
| $p-1=2^m$, $(p+1)_3=3$ |   |
| $p-1=2^m$, $(p+1)_3>3$ |   |
| $p\pm 1\neq 2^m$, $(p+1)_3=3$ |   |
| $p\pm 1\neq 2^m$, $(p+1)_3>3$ |   |
| $U_3(p)$ : 2, $p$ is a prime, $17\neq p\geq 11$, $\pi(p^2-1)=\{2,3,s\}$, $\pi(\frac{p^2p-p+1}{(3,p+1)})=\{r\}$ | |
| $p-1=2^m$, $(p+1)_3=3$ |   |
| $p-1=2^m$, $(p+1)_3>3$ |   |

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| Group $G$ | Graph $\Gamma(G)$ |
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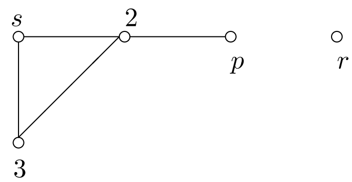
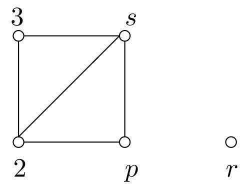
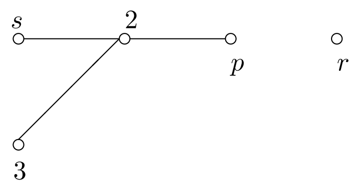
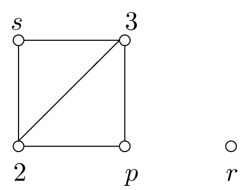
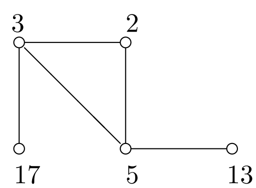
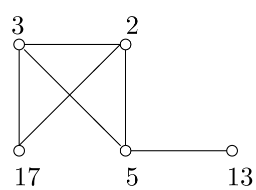
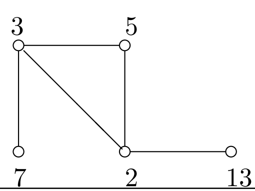
| $p+1=2^m$ |   |
| $p \pm 1 \neq 2^m$, $(p+1)_3 = 1$ |   |
| $p \pm 1 \neq 2^m$, $(p+1)_3 = 3$ |   |
| $p \pm 1 \neq 2^m$, $(p+1)_3 > 3$ |   |
| $U_4(4)$ |   |
| $U_4(4) < G \leq Aut(U_4(4)) \cong U_4(4):4$ |   |
| $U_4(5) \leq G \leq Aut(U_4(5)) \cong U_4(5).2^2$<br>$G \cap PGU_4(5) = U_4(5)$ |   |

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| Group $G$ | Graph $\Gamma(G)$ |
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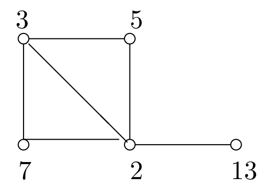
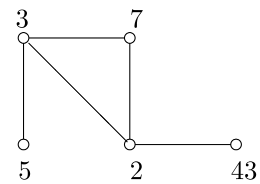
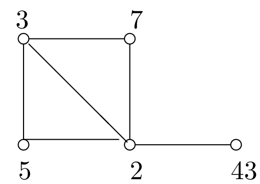
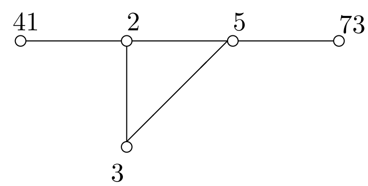
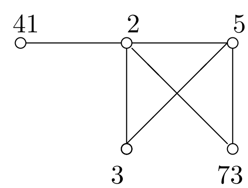
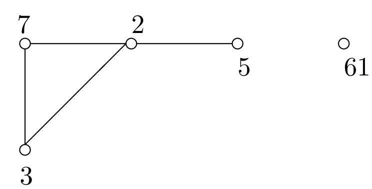
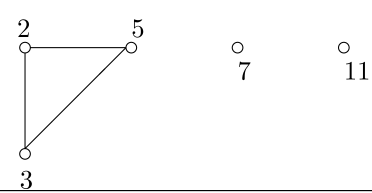
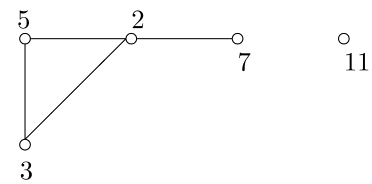
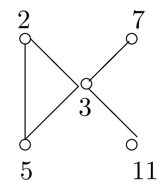
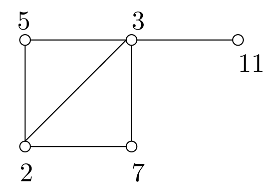
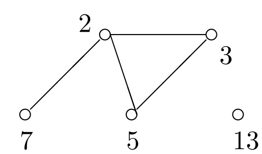
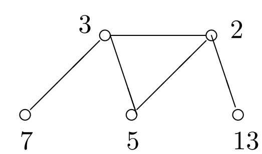
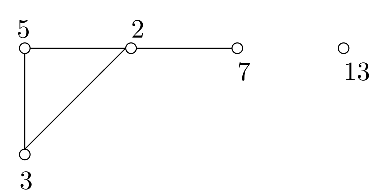
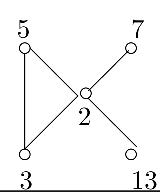
| $G \cap PGU_4(5) > U_4(5)$ |  |
| $U_4(7)$ |  |
| $U_4(7) < G \leq Aut(U_4(7)) \cong U_4(7).D_8$ |  |
| $U_4(9) \leq G \leq Aut(U_4(9)) \cong U_4(9).(2 \times 4)$ | |
| $G \cap PGU_4(9) = 1$ |  |
| $G \cap PGU_4(9) > U_4(9)$ |  |
| $U_5(3),\ Aut(U_5(3)) \cong U_5(3):2$ |  |
| $U_6(2)$ |  |

Continue Table

| Group $G$ | Graph $\Gamma(G)$ |
|---|---|
| $U_6(2).2$ |  |
| $U_6(2).3$ |  |
| $Aut(U_6(2)) \cong U_6(2).S_3$ |  |
| $O_7(3)$ |  |
| $Aut(O_7(3)) \cong O_7(3):2$ |  |
| $O_8^+(3)$ |  |
| $O_8^+(3) < G \leq Aut(O_8^+(3)) \cong O_8^+(3).S_4$ $\lvert G : O_8^+(3)\rvert = 2^k \geq 2$ |  |

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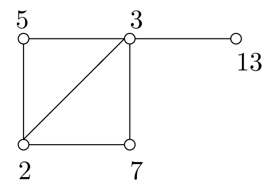
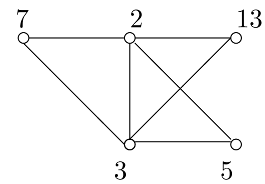
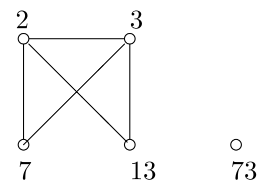
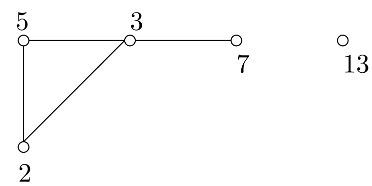
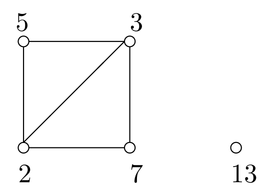
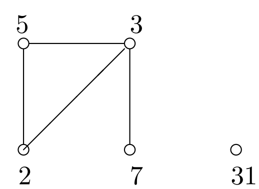
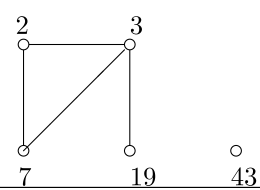
| Group $G$ | Graph $\Gamma(G)$ |
|---|---|
| $\|G : O_8^+(3)\| = 3$ |   |
| 6 divides $\|G : O_8^+(3)\|$ |   |
| ${}^3D_4(3),\ Aut({}^3D_4(3))$ |   |
| $G_2(4)$ |   |
| $Aut(G_2(4)) \cong G_2(4) : 2$ |   |
| $G_2(5)$ |   |
| $G_2(7)$ |   |

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| Group $G$ | Graph $\Gamma(G)$ |
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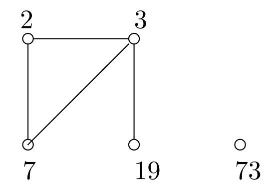
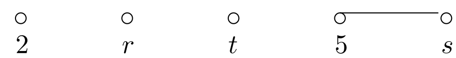
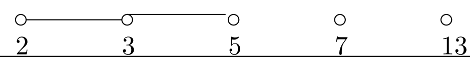
| $G_2(8),\ Aut(G_2(8)) \cong G_2(8):3$ |  |
| $Sz(q)$, $q=2^p \geq 2^7$, $p$, $q-1=r$ are primes, $\varepsilon \in \{+,-\}$, $\pi(q-\varepsilon\sqrt{2q}+1)=\{5,s\}$, $\pi(q+\varepsilon\sqrt{2q}+1)=\{t\}$ |  |
| $Aut(Sz(8)) \cong Sz(8):3$ |  |

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