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THE COMPLEXITY OF THE EDGE 3-COLORABILITY PROBLEM FOR GRAPHS WITHOUT TWO INDUCED FRAGMENTS EACH ON AT MOST SIX VERTICES

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ABSTRACT. We obtain a complete complexity dichotomy for the EDGE 3-COLORABILITY within the family of hereditary classes defined by forbidden induced subgraphs on at most 6 vertices and having at most two 6-vertex forbidden induced structures.

Keywords: computational complexity, edge 3-colorability, hereditary class, efficient algorithm

1. INTRODUCTION

A *coloring* is an arbitrary mapping from the set of vertices or edges of a graph into a set of colors of the graph such that any adjacent vertices (or edges) are colored by different colors. The VERTEX k-COLORABILITY is to verify whether vertices of a given graph can be colored by at most k colors. The EDGE k-COLORABILITY is the edge analogue of the VERTEX k-COLORABILITY. The VERTEX COLORABILITY (resp. the EDGE COLORABILITY) is to find the minimum number of colors necessary for coloring vertices (resp. edges) of a given graph. All four problems defined above are NP-complete (see [10, 14, 19]) in the class of all graphs and the same remains true under its substantial restrictions. At the same time, some areas of tractability

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are known i.e., graph classes where the problems are polynomially solvable (see, for example, the corresponding surveys and references in [25, 26, 27, 41, 46]).

A class of graphs is called *hereditary* if it is closed under deletion of vertices. It is well known that any hereditary (and only hereditary) graph class $\mathcal{X}$ can be defined by a set of its forbidden induced subgraphs $\mathcal{S}$. We write $\mathcal{X} = Free(\mathcal{S})$ in this case. There is a unique minimal under inclusion set $\mathcal{S}$ with this property denoted by $Forb(\mathcal{X})$. If $Forb(\mathcal{X})$ is finite, then $\mathcal{X}$ is called *finitely defined*.

The complexity of the VERTEX COLORABILITY and the VERTEX k-COLORABILITY restricted to graph classes defined by one or more forbidden induced subgraphs has been studied by many authors [8, 9, 11, 13, 15, 17, 18, 41, 46, 48]. Kral' et al. obtained a complete complexity dichotomy for the VERTEX COLORABILITY within hereditary classes having one forbidden induced subgraph [17]. They also initialized a study for two forbidden induced subgraphs, but there are classes defined by two 4-vertex forbidden induced structures for which the complexity status of this problem remains open [22]. The complexity of the VERTEX 3-COLORABILITY is known for all classes of the form $Free(\{H\})$ with $|V(H)| \leq 6$ [9]. A similar result for H-free graphs with $|V(H)| \leq 5$ was recently obtained for the VERTEX 4-COLORABILITY [11]. On the other hand, the complexity status of the VERTEX k-COLORABILITY is still open for P_7-free graphs ($k = 3$) and for P_6-free graphs ($k = 4$).

For the EDGE COLORABILITY and the EDGE k-COLORABILITY the author does not know any facts similar to the exhaustive classifications above. The first result of this type is presented in this article. Namely, we give a complete complexity dichotomy for the EDGE 3-COLORABILITY in the family of hereditary classes defined by forbidden induced subgraphs on at most 6 vertices and having at most two 6-vertex forbidden induced fragments.

2. Notation

As usual, P_n and C_n stand respectively for the simple path with n vertices and the chordless cycle with n vertices. The graphs *claw*, *diamond*, *bug*, *hantel*, *barbell*, *glider* are drawn on the picture below.

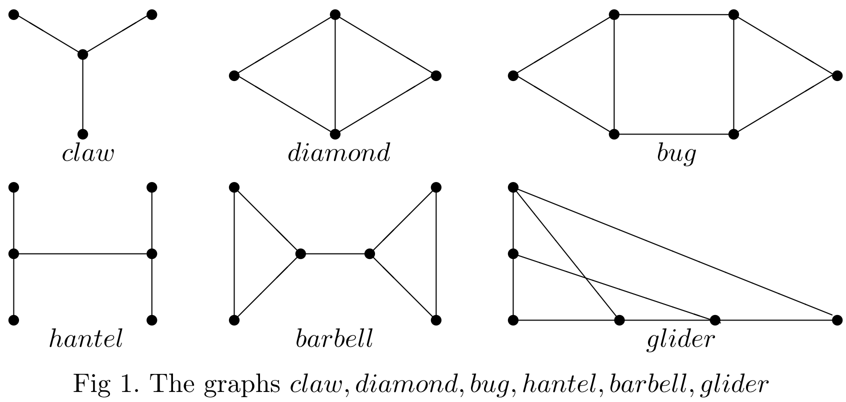

Fig 1. The graphs *claw*, *diamond*, *bug*, *hantel*, *barbell*, *glider*

A graph $T_{i,j,k}$ has vertices $x_0, x_1, x_2, \ldots, x_i, y_1, y_2, \ldots, y_j, z_1, z_2, \ldots, z_k$ and edges $(x_0, x_1), (x_1, x_2), \ldots, (x_{i-1}, x_i), (x_0, y_1), (y_1, y_2), \ldots, (y_{j-1}, y_j), (x_0, z_1), (z_1, z_2), \ldots, (z_{k-1}, z_k)$. A graph $D_{i,j,k}$ has vertices $x_0, y_0, z_0, x_1, x_2, \ldots, x_i, y_1, y_2, \ldots, y_j, z_1, z_2, \ldots,$

z_k and edges $(x_0, y_0), (x_0, z_0), (y_0, z_0), (x_0, x_1), (x_1, x_2), \ldots, (x_{i-1}, x_i), (y_0, y_1), (y_1, y_2), \ldots, (y_{j-1}, y_j), (z_0, z_1), (z_1, z_2), \ldots, (z_{k-1}, z_k)$.

Fig 2. The graphs $T_{i,j,k}$ and $D_{i,j,k}$

A graph kG is the disjoint union of k copies of a graph G. For a graph G and a set $V' \subseteq V(G)$ the formula $G \setminus V'$ denotes the subgraph of G induced by $V \setminus V'$. The *hereditary closure of a class* $\mathcal{X}$ (denoted by $[\mathcal{X}]$) is the set of all induced subgraphs of its members.

We refer to textbooks in graph theory for any graph terminology undefined here.

3. Boundary graph classes

3.1. The notion of a boundary class and its significance

The notion of a boundary graph class is a helpful tool for analyzing the computational complexity of graph problems within the family of hereditary graph classes. This notion was originally introduced by V. Alekseev for the independent set problem [1]. It was applied for the dominating set problem later [3]. A study of boundary graph classes for some graph problems was continued in the paper of Alekseev et al. [2] where the notion was stated in its most general form. Let us give necessary definitions.

Let Π be an NP-complete graph problem. The term «graph problem» is not defined here and it is understood intuitively as a question on the input graph. A hereditary graph class is called Π-*easy* if Π is polynomial-time solvable for its graphs. If the problem Π is NP-complete for graphs in a hereditary class, then this class is called Π-*hard*. A class of graphs is said to be Π-*limit* if this class is the intersection of an infinite monotonically decreasing sequence of Π-hard classes. In other words, $\mathcal{X}$ is Π-limit if there is an infinite sequence $\mathcal{X}_1 \supseteq \mathcal{X}_1 \supseteq \ldots$ of Π-hard classes such that $\mathcal{X} = \bigcap\limits_{k=1}^{\infty} \mathcal{X}_k$. Each Π-hard class is Π-limit. A minimal under inclusion Π-limit class is called Π-*boundary*.

The following theorem certifies the significance of the boundary class notion (see [1]).

Theorem 1. *A finitely defined class $\mathcal{X}$ is Π-hard if it contains some Π-boundary class. If $\mathcal{X}$ does not contain a Π-boundary class, then it is Π-easy (unless $P = NP$).*

The theorem means that finding out all Π-boundary classes implies a complete classification of finitely defined graph classes with respect to the complexity of Π. Two concrete classes of graphs are known to be boundary for several graph problems. The first of them is $\mathcal{T}$. It is constituted by all forests with at most three leaves in each connected component. The second one is $\mathcal{D}$ consisting of line graphs of graphs in $\mathcal{T}$. In other words, $\mathcal{T}$ (resp. $\mathcal{D}$) is the class of graphs whose connected components belong to $[\bigcup_{i=1}^{\infty}\{T_{i,i,i}\}]$ (resp. to $[\bigcup_{i=1}^{\infty}\{D_{i,i,i}\}]$). The papers [2, 4, 29, 35] are good surveys about graph problems for which either $\mathcal{T}$ or $\mathcal{D}$ is boundary. Note that in some papers $\mathcal{T}$ is denoted by $\mathcal{S}$ and $\mathcal{D}$ is denoted by $\mathcal{T}$. We will use the $\{\mathcal{T},\mathcal{D}\}$-notation here.

The most important fields of research in the theory of boundary classes are revealing new boundary classes for some graph problems and attempts to get a comprehensive description of some *boundary systems* (i.e., the sets of boundary classes). Among important advances in the first field one could mention pointing out the first two boundary classes for the hamiltonian cycle problem [16], the first boundary class for the 3-satisfiability problem [23]. Until recently, a complete description of a boundary system was not known for any graph problem. Only partial results of this type existed (see [1, 20, 40]). The first comprehensive description of boundary systems was obtained in [34] where a generalization of the EDGE k-COLORABILITY was considered.

V. Alekseev, R. Boliac, D. Korobitsyn and V. Lozin conjectured in [2] that there is a graph problem with an infinite boundary system. By Theorem 1 the cardinality of a boundary system can be interpreted as a complexity measure of the corresponding graph problem. It was conjectured in [2] that there is a problem with a large value of the measure. This conjecture was proved in [28, 31] by showing that the boundary systems for the VERTEX 3-COLORABILITY and the EDGE 3-COLORABILITY are infinite. The results of [28, 31] were further improved by proving that the systems for the VERTEX COLORABILITY and the EDGE COLORABILITY, for each $k \geq 3$ for the VERTEX k-COLORABILITY and the EDGE k-COLORABILITY have continuum cardinalities [16, 30, 32, 33]. One could consider these results as a Gödel argument in the sense that the boundary systems for the mentioned problems are quite complicated and attempts to get their exhaustive descriptions look hopeless.

So, advances in complete descriptions of boundary systems are small. But, for some graph problems Theorem 1 and known parts of boundary systems (together with other results) help obtaining a complexity dichotomy in a family of hereditary classes with small forbidden induced fragments or with a small number of such forbidden induced subgraphs. For the VERTEX 3-COLORABILITY an exhaustive classification of classes with two forbidden induced structures on at most five vertices was obtained in [39]. Some new polynomial and intractable cases were found in [36, 37, 38] for the VERTEX COLORABILITY. We will apply some known limit and boundary classes for EDGE 3-COLORABILITY to achieve our aims.

3.2. Limit and boundary classes for the edge 3-colorability

Let G be a graph with two chosen vertices such that there is an automorphism of G mapping these vertices to each other. *Replacement of an edge $e = (a, b)$ by the graph G* is deleting e from a graph, identifying a with one of the chosen vertices of G and b with the other chosen vertex of G. Clearly, the resultant graph does not depend on the choice of a vertex identified with a.

For a binary sequence π of length l *π-sheaf* is a graph obtained from P_{2l+2} by replacements of its edges. For each $i \in \{1, 2, \dots, l\}$ the $2i$-th edge of this path is replaced by *diamond* (if $\pi_i = 0$) or by *bug* (if $\pi_i = 1$). By T_π we denote the graph obtained by replacements of all edges of *claw* by π-sheafs. Replacements of all edges of $D_{1,1,1}$ incident to its leaves by π-sheafs yield the graph D_π.

Let π be an infinite binary sequence now and $\pi^{(l)}$ be its subsequence consisting of the first l members. The class $\mathcal{T}_\pi$ (resp. $\mathcal{D}_\pi$) is the set of graphs whose connected components belong to $[\bigcup\limits_{l=1}^{\infty}\{T_{\pi^{(l)}}\}]$ (resp. to $[\bigcup\limits_{l=1}^{\infty}\{D_{\pi^{(l)}}\}]$). Continuum cardinality of the boundary system for the EDGE 3-COLORABILITY immediately follows from the following result (proved in [30]).

Theorem 2. *For any infinite binary sequence π the classes $\mathcal{T}_\pi$ and $\mathcal{D}_\pi$ are boundary for the* EDGE 3-COLORABILITY.

A graph is called *subcubic* if degrees of all its vertices are at most three. Clearly, the EDGE 3-COLORABILITY for a class of graphs is polynomially reduced to the same problem for its subcubic part. The set $\mathcal{F}(3)$ of subcubic forests is a limit class for the problem [21]. We will need three more limit classes for it. They can be obtained by some graph operations called inscribing a triangle, diamond and glider implantations.

Let G be a subcubic graph and the neighborhood of $x \in V(G)$ consist of pairwise nonadjacent vertices $y_1, y_2, \ldots, y_k$. *Inscribing a triangle* for x is to delete x, add new vertices x_1, x_2, x_3 and the edges $(x_1, x_2), (x_1, x_3), (x_2, x_3), (x_1, y_1)$ (if $k \geq 1$), (x_2, y_2) (if $k \geq 2$), (x_3, y_3) (if $k = 3$).

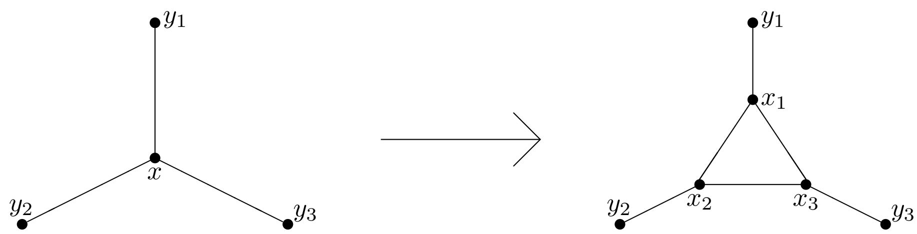

Fig 3. Inscribing a triangle

The graph G is edge 3-colorable iff the resultant graph has this property. By $\mathcal{F}^t(3)$ we denote the hereditary closure of the set of graphs produced by inscribing a triangle for each vertex of any graph in $\mathcal{F}(3)$.

Let H be a subcubic graph and $e = (x, y) \in E(H)$. *Diamond implantation* for e is to delete the edge, add new vertices x', y' and the edges $(x, x'), (x', y'), (y', y)$, replace (x', y') by *diamond*.

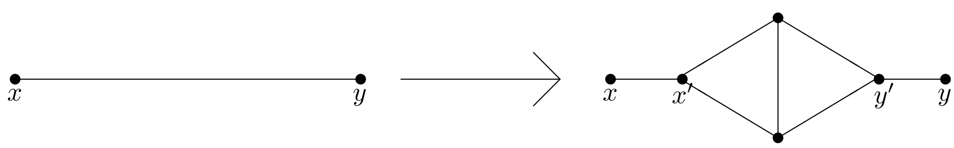

Fig 4. Diamond implantation

Glider implantation is defined in a similar way (see Fig 5).

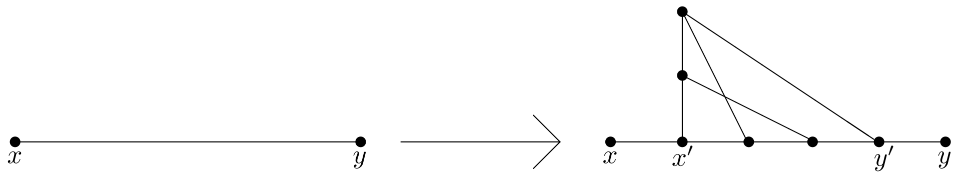

Fig 5. Glider implantation

It is easy to see that both implantations preserve edge 3-colorability. The class $\mathcal{F}^d(3)$ is the hereditary closure of the set of graphs obtained by diamond implantation for every edge of any graph in $\mathcal{F}(3)$. The class $\mathcal{F}^g(3)$ is the hereditary closure of the class of graphs produced by inscribing a triangle for each vertex of any graph in $\mathcal{F}(3)$ and, furthermore, implanting *glider* for any edge outside triangles.

Lemma 1. *The classes* $\mathcal{F}^t(3), \mathcal{F}^d(3), \mathcal{F}^g(3)$ *are limit for the* EDGE 3-COLORABILITY.

Proof. Let $\{\mathcal{X}_i\}$ be a monotonically decreasing sequence of hard classes for the EDGE 3-COLORABILITY converging to $\mathcal{F}(3)$. We can assume that each its member consists of subcubic graphs. Inscribing a triangle for each vertex of any graph in $\mathcal{X}_i$ produces some set of graphs. Let $\mathcal{Y}_i$ be the hereditary closure of this set. The class $\mathcal{Y}_i$ is hard for the EDGE 3-COLORABILITY, since the problem is hard for $\mathcal{X}_i$. We have $\mathcal{Y}_1 \supseteq \mathcal{Y}_2 \supseteq \ldots$ and $\mathcal{F}^t(3) = \bigcap_{i=1}^{\infty} \mathcal{Y}_i$. So, $\mathcal{F}^t(3)$ is limit for the EDGE 3-COLORABILITY. Proofs for $\mathcal{F}^d(3), \mathcal{F}^g(3)$ are similar. □

The classes $\mathcal{F}(3), \mathcal{F}^t(3), \mathcal{F}^d(3), \mathcal{F}^g(3)$ are likely to be boundary for the EDGE 3-COLORABILITY. Proving (or disproving) this fact is a challenging problem.

4. Treewidth of graphs and its significance

It is well known that many NP-complete graph problems are polynomial-time solvable for trees. It is also true for sets of graphs that are close to trees with respect to some qualitative or quantitative measure. In other words, if for graphs in a class this measure grows slowly, then one can expect that a considered graph problem is solvable in polynomial time for them. Treewidth is such a measure defined as follows. A *k-tree* is a graph that can be obtained from the $(k+1)$-clique by iterative implementation of the rule: add a new vertex to a k-tree G and k edges incident to the new vertex and all vertices of a k-clique of G. A *partial k-tree* is a subgraph (not necessarily induced) of a k-tree. *Treewidth* of a graph is the smallest value of k for which the graph is a partial k-tree.

The notion of treewidth has an equivalent definition in terms of tree decompositions. The concept of tree decompositions was originally introduced by R. Halin [12]. Later it was rediscovered by N. Robertson and P. Seymour [42] and afterwards it was studied by many other authors.

Many graph problems are efficiently (polynomially) solved for classes with treewidth uniformly bounded by a constant [5, 6, 7, 45]. It is also true for the VERTEX k-COLORABILITY and the EDGE k-COLORABILITY for any k (see [6]).

The significance of treewidth inspires determining sufficient conditions that a given graph class has bounded treewidth. N. Robertson and P. Seymour showed that for any planar graph G the set of G-minor-free graphs (i.e., graphs where repeating removal of vertices and edges, and contraction of edges can not produce G) is a family with uniformly bounded treewidth [43]. V. Lozin and D. Rautenbach proved in [24] the following result.

Lemma 2. *Let $\mathcal{X}$ be a hereditary class, $\mathcal{T} \nsubseteq \mathcal{X}$, $\mathcal{D} \nsubseteq \mathcal{X}$ and for some constant d degrees of vertices of all its graphs are bounded by d. Then there is a constant $C = C(\mathcal{X}, d)$ such that treewidth of each graph in $\mathcal{X}$ is at most C.*

5. Auxiliary results

5.1. Bounding graph diameter or forbidding an induced fragment

Lemma 3. *Let G be a connected graph in $Free(\{D_{0,0,k}\})$ $(k \geq 2)$. Then G is triangle-free or its diameter is at most $2k+2$.*

Proof. Let G contains a triangle with vertices x, y, z. We will show that the eccentricity of x is at most $k+1$. This fact and the triangle inequality implies the bound for the diameter. Assume the opposite, i.e. that G contains the shortest induced path P on at least $k+3$ vertices connecting x with some other vertex of G. We enumerate all vertices of P starting from x. No one of the vertices x, y, z can be adjacent to a vertex of P with an index greater than three (P is not shortest otherwise). Let $n \leq 3$ be the largest index of a vertex in P that is adjacent to x or to y or to z. It is easy to verify that two or three vertices in $\{x, y, z\}$ and the vertices of P with indices in $\{n, n+1, \ldots, k+n\}$ induce a subgraph isomorphic to $D_{0,0,k}$. We have a contradiction. □

Similarly, one can show the validity of Lemma 4.

Lemma 4. *Let G be a connected graph in $Free(\{T_{1,1,k}\})$ $(k \geq 2)$. Then G is claw-free or its diameter is at most $2k+2$.*

5.2. Some polynomial-time reductions for the edge 3-colorability

The idea of a compression of a given graph is frequently used as a part of an algorithm to solve a given graph problem. For example, such an idea is dropping vertices of degrees at most $k-1$ for the *clique problem* (to find the maximum subset of pairwise adjacent vertices in a graph) assuming that the current feasible solution has at least $k+1$ vertices. Sometimes, a data reduction itself produces an efficient algorithm. For example, deleting any neighbor of a *simplicial vertex* (i.e., a vertex whose neighborhood induces a clique) for solving the *independent set problem* (to find the maximum subset of pairwise nonadjacent vertices in a given graph) for chordal graphs. A graph is *chordal* if it has no induced cycles with four and more vertices. A chordal graph always has a simplicial vertex.

The following assertion by A. Schrijver (see his monography [44]) is a data compression for the EDGE k-COLORABILITY.

Lemma 5. *Let a vertex v of some graph G and all its neighbors have degrees at most k and at most one vertex of the neighbourhood has exactly k adjacent vertices. Then G is edge k-colorable iff $G \setminus \{v\}$ is edge k-colorable.*

It was noticed in [44] that Lemma 5 implies a new proof of the famous Vizing's theorem [47] claiming that a graph with maximum degree of vertices at most k is edge $(k+1)$-colorable. The Schrijver's result is useful for proving the next two lemmas.

Lemma 6. *The* 3-EDGE COLORABILITY *for* $\{barbell, claw\}$*-free graphs is polynomially reduced to the same problem for subcubic* $\{barbell, claw, D_{2,2,2}\}$*-free graphs.*

Proof. Let $G \in Free(\{barbell\})$ be a *claw*-free subcubic graph that contains an induced copy of $D_{2,2,2}$. We will show that $deg(x_1) = deg(y_1) = deg(z_1) = 2$. Assume that among them at least one vertex (say x_1) has three neighbors. If x is the neighbor of x_1 different from x_0 and x_2, then $(x, x_2) \notin E(G)$ (since G is a subcubic *barbell*-free graph) and $(x, x_2) \in E(G)$ (as G is *claw*-free). We have a contradiction.

By G' we denote the graph obtained by deleting the vertices x_0, y_0, z_0, adding a vertex v and edges $(v, x_1), (v, y_1), (v, z_1)$. The graph G is edge 3-colorable iff it is so for G'. By Lemma 5 the graphs G' and $G' \setminus \{v\}$ are simultaneously edge 3-colorable or each of them has no such a coloring. Hence G is edge 3-colorable iff $G \setminus \{x_0, y_0, z_0\}$ has this property. Doing the same for all copies of $D_{2,2,2}$ we obtain a polynomial-time reduction to subcubic $\{barbell, claw, D_{2,2,2}\}$-free graphs. ☐

Lemma 7. *The* 3-EDGE COLORABILITY *for* $\{hantel, C_3\}$*-free graphs is polynomially reduced to the same problem for subcubic* $\{hantel, C_3, T_{4,4,4}\}$*-free graphs.*

Proof. Let $G \in Free(\{hantel\})$ be a triangle-free subcubic graph that contains an induced copy of $T_{4,4,4}$. By Lemma 5 one may assume that among the vertices x_1, y_1, z_1 of $T_{4,4,4}$ at least two have degrees equal to three. Let $(x_1, x) \in E(G), (y_1, y) \in E(G)$ and $x, y \notin V(T_{4,4,4})$. Assume that $x \neq y$. Since G is triangle-free, then $(x, x_2) \notin E(G)$ and $(y, y_2) \notin E(G)$. Either $y, y_1, y_2, x_1, x_0, z_1$ or $x, x_1, x_2, y_1, x_0, z_1$ induce a copy of *hantel*. We have a contradiction. Hence $x = y$. If x and z_2 are adjacent, then $(x, z_3) \notin E(G), (x, z_1) \notin E(G)$ and $x_1, x, y_1, z_1, z_2, z_3$ induce *hantel*. Therefore $(x, z_2) \notin E(G)$. If $(x, z_1) \in E(G)$, then x_2, y_2 and z_2 have degrees equal to two (as G is $\{hantel, C_3\}$-free). Deleting x and contracting the edges $(x_0, x_1), (x_0, y_1), (x_0, z_1)$ yield a graph G'. It is easy to verify that G is edge 3 colorable iff G' is edge 3-colorable. This fact and Lemma 5 imply that G is edge 3-colorable iff it is so for $G \setminus \{x_0, x, x_1, y_1, z_1\}$. Hence x and z_1 are not adjacent. The vertex z_1 can not have a neighbor outside $V(T_{4,4,4})$, since G is not *hantel*-free otherwise.

Let $deg(x_2) = deg(y_2) = 2$. If there is a neighbor $z \notin \{x_1, y_1\}$ of x, then $deg(z) \leq 2$ (otherwise, the neighborhood of z, z, x, x_1, y_1 induce *hantel*). Let us fix some edge 3-coloring of $H = G \setminus \{x_0, x, x_1, y_1\}$ (if one exists). As $deg(y_2) = deg(z_1) = 2$ (in G), then there is a color c_1 that differs from the colors of $(y_2, y_3), (z_1, z_2)$. By the same reason, there is a color c_2 (perhaps $c_1 = c_2$) different from the colors of the edges incident to z or x_2 in H. If $c_1 = c_2$, then the coloring of H can be extended to an edge 3-coloring of G by coloring $(y_2, y_1), (z, x), (x_2, x_1), (z_1, x_0)$ in c_1 and $(y_1, x), (x, x_1), (x_1, x_0), (x_0, y_1)$ in the remaining two colors. If $c_1 \neq c_2$, then an edge 3-coloring of G is obtained by coloring (y_1, y_2),
$(z_1, x_0), (x, x_1)$ in c_1, $(x, z), (x_1, x_2), (y_1, x_0)$ in c_2 and $(y_1, x), (x_0, x_1)$ in the last third color. Hence G is edge 3-colorable iff H is edge 3-colorable.

Let both vertices x_2, y_2 have neighbors outside $V(T_{4,4,4})$. They must be coinciding, nonadjacent to x_3 and y_3, adjacent to x. Hence $deg(x_3) = deg(y_3) = 2$. The edges $(x_2, x_3), (y_2, y_3), (x_0, z_1)$ have different colors in any edge 3-coloring of G (by reductio ad absurdum). Deleting the common neighbor z' of x_2 and y_2, deleting x and contracting $(x_0, x_1), (x_1, x_2), (x_0, y_1), (y_1, y_2)$ produce an edge 3-colorable graph iff G is edge 3-colorable. This observation and Lemma 5 imply that G is edge 3-colorable iff it is so for $G \setminus \{x_0, x, z', x_1, y_1, x_2, y_2\}$.

Let among x_2 and y_2 only one vertex (say x_2) has a neighbor. Let $\{a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k\}$ be a set with the maximal cardinality such that $(a_1, a_2), (a_2, a_3), \ldots, (a_{k-1}, a_k), (b_1, b_2), (b_2, b_3), \ldots, (b_{k-1}, b_k), (a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$ are edges of G and $x_0 = a_1, y_1 = b_1, x_1 = a_2, x = b_2, x_2 = a_3, z'' = b_3$. It is computed in polynomial time. As $deg(z_1) = 2$, then $z_1 \notin \{a_1, b_1, a_2, b_2, \ldots, a_{k-1}, b_{k-1}\}$. If $k \in \{2, 3\}$, then $z_1 \notin \{a_k, b_k\}$ (otherwise, $x_0, x_1, x_2, y_1, y_2, z_1, z_2$ do not induce $T_{2,2,2}$). The same is true for $k > 3$ (G contains an induced copy of *hantel* otherwise). Similarly, $y_2 \notin \{a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k\}$. Without loss of generality, one may assume that a_k is adjacent to $a' \notin \{a_{k-1}, b_k\}$ and b_k is adjacent to $b' \notin \{a_k, b_{k-1}\}$ (obviously, $a' \neq b'$). Due to the maximality above, $(a', b') \notin E(G)$. As G is $\{hantel, C_3\}$-free, then $deg(a') \leq 2$, $deg(b') \leq 2$. If $a' = z_1$ or $b' = z_1$, then $\{a_k, b_k\} = \{z_2, z_3\}$ and $\{a', b'\} = \{z_1, z_4\}$. If $a' = y_2$ or $b' = y_2$, then $\{a_k, b_k\} = \{y_3, y_4\}$. Denote the graph $G \setminus \{a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k\}$ by H. We will show that any edge 3-coloring of H can be extended to an edge 3-coloring of G.

Consider an arbitrary edge 3-coloring of H (if one exists). As z_1, y_2, a', b' have degrees in G at most two, then $\{a', b'\} \cap \{z_1, y_2\} = \emptyset$ and there are colors c_1 and c_2 such that c_1 differs from the colors of the edges incident to y_2 or z_1 and c_2 differs from the colors of the edges incident to a' or b'. If $c_1 = c_2$, then $(y_2, b_1), (z_1, a_1), (a_2, b_2), (a_3, b_3), \ldots, (a_{k-1}, b_{k-1}), (a', a_k), (b', b_k)$ are colored in c_1, the remaining uncolored edges of G are colored in two other colors (as they constitute an even cycle). If $c_1 \neq c_2$ and k is even, then $(y_2, y_1), (z_1, x_0)$ and all elements of $\{(a_i, b_i) : 1 < i \leq k\}$ are colored in c_1, $(a', a_k), (b', b_k), (a_1, b_1)$ and all edges from $\{(a_{2i}, a_{2i+1}) : 1 \leq i \leq \frac{k}{2} - 1\} \cup \{(b_{2i}, b_{2i+1}) : 1 \leq i \leq \frac{k}{2} - 1\}$ are colored in c_2. If $c_1 \neq c_2$ and k is odd, then $(y_2, y_1), (z_1, x_0), (a_2, a_3), (b_2, b_3), (a_4, a_5), (b_4, b_5), \ldots, (a_{k-1}, a_k), (b_{k-1}, b_k)$ are colored in c_1, $(a', a_k), (b', b_k), (a_1, a_2), (b_1, b_2), (a_3, a_4), (b_3, b_4) \ldots, (a_{k-2}, a_{k-1}), (b_{k-2}, b_{k-1})$ in c_2. In both cases the uncolored edges are colored in the third color.

The elimination process described in the previous paragraphs finishes the reduction. □

Lemma 8. *The* 3-EDGE COLORABILITY *for* $\{hantel, 2C_3\}$*-free graphs is polynomially reduced to the same problem for graphs of bounded treewidth.*

Proof. Let G be a $\{hantel, 2C_3\}$-free graph. By the previous lemma and Lemma 2 one can consider that G has a triangle T. Deleting its vertices with their neighborhoods produces a triangle-free graph. Hence G has at most 4 triangles. Let G' be the subgraph induced by vertices of G lying at distance at least 7 from each vertex of T. Any possible induced copy of $T_{4,4,4}$ in G with the central vertex in $V(G')$ has no joint vertices with any triangle of G. Such a copy of $T_{4,4,4}$ will be called separated. For a separated copy of $T_{4,4,4}$ one can apply the elimination process from the previous lemma if $a' \neq b'$. Therefore we can assume that any separated copy of $T_{4,4,4}$ has a ladder where a' or b' belongs to a triangle. These ladders for different copies of $T_{4,4,4}$ are not intersected. Each triangle of G interrupts at most three ladders. Hence G' has at most 12 separated copies of $T_{4,4,4}$ i.e., G' is $13T_{4,4,4}$-free. The absolute difference between treewidths of G and G' is at most $|V(G)| - |V(G')|$ (see the definition of treewidth) that is no more than $3(1+1+2+4+8+16+32) = 192$. By Lemma 2 and Lemma 7 we have the validity of the claim. □

6. Main result

Theorem 3. *Let $\mathfrak{F}$ be the family of hereditary classes defined by forbidden induced subgraphs on at most 6 vertices and having at most two 6-vertex forbidden fragments. The* EDGE 3-COLORABILITY *is NP-complete for $\mathcal{X} \in \mathfrak{F}$ if $\mathcal{D} \cup \mathcal{T} \cap Free(\{T_{1,2,2}\}) \subseteq \mathcal{X}$ or $\mathcal{T} \cup \mathcal{D} \cap Free(\{D_{0,1,1}\}) \subseteq \mathcal{X}$. It is polynomial-time solvable for all other classes from $\mathfrak{F}$.*

Proof. It is easy to see that *hantel* is the unique graph in $\mathcal{F}(3) \setminus \mathcal{T}$ on at most six vertices. The same is true for *barbell* and $\mathcal{F}^t(3) \setminus \mathcal{D}$. Since $\mathcal{F}(3)$ and $\mathcal{F}^t(3)$ are limit classes for the EDGE 3-COLORABILITY, then one may assume $\mathcal{F}(3) \nsubseteq \mathcal{X}$ and $\mathcal{F}^t(3) \nsubseteq \mathcal{X}$. Otherwise $\mathcal{X}$ is a hard case for the problem.

Let $\mathcal{D} \subseteq \mathcal{X}$, then $barbell \in Forb(\mathcal{X})$ (as $\mathcal{F}^t(3) \nsubseteq \mathcal{X}$). If $\mathcal{X} = Free(\{barbell\})$ or $\mathcal{X} = Free(\{barbell, hantel\})$ or $\mathcal{X} = Free(\{barbell, T_{1,2,2}\})$, then $\mathcal{X}$ is a hard class for the EDGE 3-COLORABILITY (by Theorem 1, Lemma 1 and $\mathcal{F}^g(3) \subseteq \mathcal{X}$). In all other cases $Forb(\mathcal{X})$ contains a graph in $(\mathcal{T} \setminus \mathcal{D}) \cap Free(\{T_{1,2,2}\})$ on at most 6 vertices. Hence it is an induced subgraph of $T_{1,1,5}$. Since one may consider only subcubic graphs in $\mathcal{X}$, then by Lemma 4 and Lemma 6 the problem is polynomially reduced to $Free(\{barbell, claw, D_{2,2,2}\})$. This class is easy by Lemma 2.

Let $\mathcal{T} \subseteq \mathcal{X}$ now, then $hantel \in Forb(\mathcal{X})$ (as $\mathcal{F}(3) \nsubseteq \mathcal{X}$). If $\mathcal{X} \neq Free(\{hantel\})$ and $\mathcal{X} \neq Free(\{hantel, barbell\})$, then $Forb(\mathcal{X})$ have an element of $\mathcal{D} \setminus \mathcal{T}$. If $Free(\{hantel, D_{0,1,1}\}) \subseteq \mathcal{X}$, then it is hard for the problem (by Theorem 1, Lemma 1 and $\mathcal{F}^d(3) \subseteq Free(\{hantel, D_{0,1,1}\})$). If $Free(\{hantel, D_{0,1,1}\}) \nsubseteq \mathcal{X}$, then $Forb(\mathcal{X})$ contains a graph $G \in (\mathcal{D} \setminus \mathcal{T}) \cap Free(\{D_{0,1,1}\})$. As $|V(G)| \leq 6$, then $G = 2C_3$ or G is an induced subgraph of $D_{0,0,6}$. By Lemma 8 the first case is polynomial, by Lemmas 3 and 7 the second one is reduced to $Free(\{hantel, C_3, T_{4,4,4}\})$ which is easy by Lemma 2.

If $\mathcal{T} \nsubseteq \mathcal{X}$ and $\mathcal{D} \nsubseteq \mathcal{X}$, then by Lemma 2 the class $\mathcal{X}$ is easy for the problem. □

Extending the complexity dichotomy above for larger forbidden structures is a challenging research problem.

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