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HOMOLOGY AND BISIMULATION OF ASYNCHRONOUS TRANSITION SYSTEMS AND PETRI NETS

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Abstract. Homology groups of labelled asynchronous transition systems and Petri nets are introduced. Examples of computing the homology groups are given. It is proved that if labelled asynchronous transition systems are bisimulation equivalent, then they have isomorphic homology groups. A method of constructing a Petri net with given homology groups is presented.

Keywords: bisimulation, homology groups, simplicial complex, trace monoid, partial action, asynchronous system, Petri net.

1. Introduction

The paper is devoted to the application of algebraic topological methods for classification and studying the mathematical models of concurrency. We consider asynchronous transition systems with label functions on events. Our purpose is to construct a homology theory of labelled asynchronous transition systems for which any bisimulation equivalent asynchronous transition systems have isomorphic homology groups.

We consider a categorical notion of the bisimulation defined by open maps [1]. It was proved in [1], that in the case of labelled transition systems this definition coincides with a strong bisimulation of R. Milner [2]. A characterization of the bisimilation equivalence for asynchronous transition systems was given in [3].

Homology groups have no less than important for the classification and studying the properties of concurrent systems. In particular, they have been applied in the

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work [4] to characterize the condition of solvability for some classes of problems in parallel distribution systems.

In [5], E. Goubault and T. P. Jensen applied homology groups for studying higher dimensional automata. There were obtained some signs of bisimulation equivalence for the higher dimensional automata in terms of the homology groups [5, Prop. 10]. The results were developed in the [6]. In a survey [7], open questions were marked on the relationship of the Goubault homology [6] with directed homotopy. This homology was also applied to prove the homotopy properties for higher dimensional automata in the [8]. Communications between homotopy and bisimilarity of higher dimensional automata was researched in [9].

Homology groups for asynchronous systems and Petri nets were introduced in [10]. These groups were used to find signs of parallelizable asynchronous systems in [11] and were regarded as the homology groups of a topological space of intermediate states for an asynchronous system in [12]. An algorithm for computing the homology groups was developed in [13].

In this paper, we study the homology of the *labelled* asynchronous transition systems and Petri nets.

We work in the category of asynchronous transition systems considered in [14]. But we call them simply *asynchronous systems*. Note that M.A. Bednarczyk [15] studied the broader category of asynchronous systems. Using results of M. Nielsen and G. Winskel [3], we study open morphisms. We introduce homology groups for labelled asynchronous transition systems and Petri nets and prove that Pom_L-bisimilar labelled asynchronous transition systems have isomorphic homology groups (Theorem 1 and Corollary 3). We give some examples of computing the homology groups of asynchronous transition systems and Petri nets. We prove that for an arbitrary finite sequence of finitely generated Abelian groups A_0, A_1, A_2, ... where A_0 is free and not equal 0 there exists a labelled Petri net the ith homology groups of which are isomorphic to A_i for all $i \geqslant 0$.

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2. Asynchronous systems and trace monoid actions

Let us recall some facts on the mathematical models of concurrency [3], [14], [15]. We study asynchronous systems as trace monoids with partial action on sets.

2.1. State spaces and asynchronous systems.

Definition 1. A state space (S, E, I, Tran) *consists of a set* S *of* states, *a set* E *of* events *with a symmetric irreflexive relation* $I \subseteq E \times E$ *of* independence, *and a* transition relation $\mathrm{Tran} \subseteq S \times E \times S$. *The following axioms must be satisfied:*

(i) *If* $(s, a, s') \in \mathrm{Tran}$ & $(s, a, s'') \in \mathrm{Tran}$, *then* $s' = s''$.
(ii) *If* $(a, b) \in I$ & $(s, a, s') \in \mathrm{Tran}$ & $(s', b, s'') \in \mathrm{Tran}$, *then there exists* $s_1 \in S$ *such that* $(s, b, s_1) \in \mathrm{Tran}$ & $(s_1, a, s'') \in \mathrm{Tran}$. *(See Fig. 1)*

Triples $(s, e, s') \in \mathrm{Tran}$ are denoted by $s \xrightarrow{e} s'$ and called *transitions* .

Definition 2. *An* asynchronous system $(S, s_0, E, I, \mathrm{Tran})$ *consists of a state space* (S, E, I, Tran) *with an arbitrary* $s_0 \in S$ *called the* initial state. *Moreover, for every* $a \in E$*, there must be* $s_1, s_2 \in S$ *satisfying* $(s_1, a, s_2) \in Tran$.

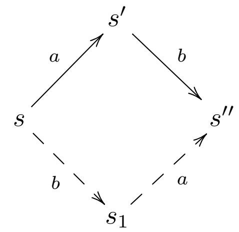

FIGURE 1. To Axiom (ii)

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Definition 3. *A* morphism *between state spaces*

$$(\sigma, \eta) : (S, E, I, \mathrm{Tran}) \to (S', E', I', \mathrm{Tran}')$$

is a pair consisting of a partial map $\eta : E \rightharpoonup E'$ *and a map* $\sigma : S \to S'$ *satisfying the following conditions*

(i) *for any triple* $(s_1, e, s_2) \in \mathrm{Tran}$, *there is the following alternative*

$$\begin{cases} (\sigma(s_1), \eta(e), \sigma(s_2)) \in \mathrm{Tran}', & \text{if the value } \eta(e) \text{ is defined,} \\ \sigma(s_1) = \sigma(s_2), & \text{if } \eta(e) \text{ is not defined;} \end{cases}$$

(ii) *for all* $(e_1, e_2) \in I$, *if* $\eta(e_1)$ *and* $\eta(e_2)$ *both are defined, then* $(\eta(e_1), \eta(e_2)) \in I'$.

Let $\mathcal{A} = (S, s_0, E, I, \mathrm{Tran})$ and $\mathcal{A}' = (S', s'_0, E', I', \mathrm{Tran}')$ be asynchronous systems. A morphism of asynchronous systems $(\sigma, \eta) : \mathcal{A} \to \mathcal{A}'$ is a mophism $(\sigma, \eta) : (S, E, I, \mathrm{Tran}) \to (S', E', I', \mathrm{Tran}')$ between the state spaces such that $\sigma(s_0) = s'_0$.

2.2. **Asynchronous systems and partial actions of trace monoids.** Throughout the paper, we let $\mathcal{A} = (S, s_0, E, I, \mathrm{Tran})$ and $\mathcal{A}' = (S', s'_0, E', I', \mathrm{Tran}')$ be asynchronous systems.

For an arbitrary category $\mathcal{C}$, let $\mathcal{C}^{op}$ be the opposite category.

Denote by $PSet$ the category of sets and partial maps. Let M be a monoid considered as the category with a single object. A *partial right action of a monoid* M *on a set* S is a functor $M^{op} \to PSet$, the value of which on the single object is equal to S. The functor assigns to each morphism $\mu \in M$ a partial map $S \rightharpoonup S$ the values of which defined on $s \in S$ are denoted by $s \cdot \mu$. The category $PSet$ is equivalent to the category of pointed sets and pointed maps [14]. Let Set_* be a category of pointed sets, whose distinguished points are equal to a fixed common point $*$, and let morphisms in Set_* are maps preserving the point $*$. The category Set_* is isomorphic to $PSet$. The isomorphism allows us to consider a partial right action of M on S as a functor $M^{op} \to \mathrm{Set}_*$. We denote this functor by (M, S_*). For each $\mu \in M$, its value $(M, S_*)(\mu)$ is the map denoted by $s \mapsto s \cdot \mu$ for all $s \in S_*$.

In particular, the state space can be considered as a set with a partial action of a *trace monoid*. Let us recall the definition of a trace monoid [16].

Let E be a set with a symmetric irreflexive relation $I \subseteq E \times E$. Denote by E^* a free monoid of words with the letters of E. Elements $a, b \in E$ are *independent* if $(a, b) \in I$. We define an equivalence relation on E^* assuming $w_1 \equiv w_2$ if the word w_2 can be obtained from w_1 by a finite sequence permutations of adjacent independent elements. Let $[w]$ be the equivalence class of $w \in E^*$. It is easy to see that the operation $[w_1][w_2] = [w_1 w_2]$ transforms the set of equivalence classes $E^*/\equiv$ in a monoid. This monoid is called a trace monoid $M(E, I)$.

Let (S, E, I, Tran) be a state space. For any $s \in S$ and $e \in E$, there exists at most one $s' \in S$ for which $(s, e, s') \in \mathrm{Tran}$. In this case, we set $s \cdot e = s'$. If Tran does not contain such a triple, then let $s \cdot e = *$. Now we can assign to each state space (S, E, I, Tran) the partial action $(M(E, I), S_*)$ defined as $(s, [e_1 \cdots e_n]) \mapsto (\ldots((s \cdot e_1) \cdot e_2) \ldots \cdot e_n)$. Any asynchronous system can be considered as a partial action $(M(E, I), S_*)$ of the trace monoid on S with initial element $s_0 \in S$. It follows from the definition of action that the formula $s \cdot e \in S$ is equivalent to $(\exists t \in S)(s, e, t) \in \mathrm{Tran}$. This formula means that the value $s \cdot e$ is defined, but $s \cdot e = *$ means that this value is not defined. The morphism between asynchronous systems $\mathcal{A} \to \mathcal{A}'$ can be defined as a pair of maps $\sigma : S \to S'$, $\eta : E \to E' \cup \{1\}$ for which

- the map η can be extended to a homomorphism of monoids $M(E, I) \to M(E', I')$;
- for every $s \in S$ and $e \in E$ satisfying $s \cdot e \in S$, we have $\sigma(s) \cdot \eta(e) \in S \;\&\; \sigma(s) \cdot \eta(e) = \sigma(s \cdot e)$;
- $\sigma(s_0) = s'_0$.

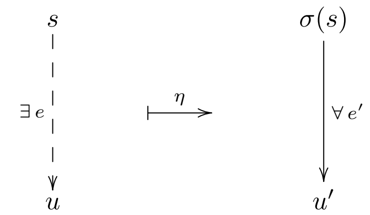
2.3. **Open morphisms.** A state $s \in S$ of asynchronous system $\mathcal{A}$ is *reachable* if there exists a finite sequence of transitions $s_0 \stackrel{e_1}{\to} s_1 \stackrel{e_2}{\to} s_2 \to \cdots \to s_{n-1} \stackrel{e_n}{\to} s$.

If we want to emphasize that the map $f : X \to Y$ is defined on all elements of X, then we call it *total*.

Definition 4. *A morphism of asynchronous systems $(\sigma, \eta) : \mathcal{A} \to \mathcal{A}'$ is* open*, if it has the following properties:*

(i) $\eta : E \to E'$ *is total;*
(ii) *for every reachable state $s \in S$ and transition $(\sigma(s), e', u') \in \mathrm{Tran}'$, there exists $(s, e, u) \in \mathrm{Tran}$ for which $\eta(e) = e'$ and $\sigma(u) = u'$;*
(iii) *for any reachable $s \in S$, if $(s, e_1, u) \in \mathrm{Tran}$ and $(u, e_2, v) \in \mathrm{Tran}$ and $(\eta(e_1), \eta(e_2)) \in I'$, then $(e_1, e_2) \in I$.*

The property (ii) can be shown visually by drawing



For any asynchronous system $\mathcal{A} = (S, s_0, E, I, \mathrm{Tran})$ and a reachable $s \in S$, we let $\mathcal{A}(s) = (S, s, E, I, \mathrm{Tran})$. In particular, $\mathcal{A}(s_0) = \mathcal{A}$.

Proposition 1. *For any open morphism $(\sigma, \eta) : \mathcal{A} \to \mathcal{A}'$ of asynchronous systems and a reachable state $s \in S$, the morphism $(\sigma, \eta) : \mathcal{A}(s) \to \mathcal{A}'(\sigma(s))$ is open.*

3. Bisimulation equivalence of labelled asynchronous systems

In this section, we recall the definitions of $\mathcal{P}$-open morphisms and $\mathcal{P}$-bisimilar objects [1] and we consider Pom_L-bisimilar labelled asynchronous systems [3].

3.1. **$\mathcal{P}$-open morphisms.** Let $\mathcal{M}$ be an arbitrary category and $\mathcal{P} \subseteq \mathcal{M}$ be a subcategory. A morphism $f : X \to Y$ of the category $\mathcal{M}$ is called to be $\mathcal{P}$-*open* [1] if for any commutative diagram in $\mathcal{M}$

$$\begin{array}{ccc} P & \xrightarrow{p} & X \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle f} \\ Q & \xrightarrow[q]{} & Y \end{array}$$

with $m \in Mor\,\mathcal{P}$, there is a morphism $p' : Q \to X$ such that in the diagram

$$\begin{array}{ccc} P & \xrightarrow{p} & X \\ {\scriptstyle m}\downarrow & \overset{p'}{\nearrow} & \downarrow{\scriptstyle f} \\ Q & \xrightarrow[q]{} & Y \end{array}$$

the two triangles commute in the sense of $p' \circ m = p$ and $f \circ p' = q$.

Let X and Y be objects of $\mathcal{M}$. If there exists an object Z of $\mathcal{M}$ with $\mathcal{P}$-open morphisms

$$\begin{array}{ccc} & Z & \\ \swarrow & & \searrow \\ X & & Y \end{array}$$

in the category $\mathcal{M}$, then X and Y are called $\mathcal{P}$-*bisimilar* [1].

3.2. **Labelled asynchronous systems.** Consider the case when the category $\mathcal{M}$ consists of labelled asynchronous systems.

A *labelled asynchronous system* $(\mathcal{A}, \lambda, L)$ consists of an asynchronous system $\mathcal{A}$ with an arbitrary set L of *labels* and a map $\lambda : E \to L$ called *label function.* Each asynchronous system can be considered as labelled with the set of labels $L = pt$ containing a single element.

Let $(\mathcal{A}, \lambda, L)$ and $(\mathcal{A}', \lambda', L)$ be labelled asynchronous systems. A morphism $(\sigma, \eta) : \mathcal{A} \to \mathcal{A}'$ *preserves labels* , if for all $e \in E$, it satisfies to equality $\lambda(e) = \lambda'(\eta(e))$. In this case, the pair (σ, η) is called a *morphism of labelled asynchronous systems* $(\mathcal{A}, \lambda, L) \to (\mathcal{A}', \lambda', L)$. The category of labelled asynchronous systems with label preserving morphisms is denoted by $\mathcal{A}_L$.

For the definition of the labelled event structure $(E, \leqslant, Con, l)$, we refer the reader to [1] and [3]. By [1], a *pomset* can be defined as a labelled event structure $(E, \leqslant, Con, l)$ in which $(E, \leqslant)$ is an arbitrary partially ordered set with a map $l : E \to L$ and Con consists of all finite subsets of events E. Let $\mathcal{E}_L$ be the category of labelled event structures [3]. The category $\mathcal{E}_L$ admits an inclusion into $\mathcal{A}_L$. Define the category Pom_L, with respect to a labelling set L, to be the full subcategory of $\mathcal{E}_L$ whose objects consist of finite pomsets. The embedding $\mathcal{E}_L \to \mathcal{A}_L$ described in [3] allows us consider Pom_L as the subcategory of the category $\mathcal{A}_L$, as well as Pom_L-open morphisms and Pom_L-bisimilar objects of $\mathcal{A}_L$.

These definitions are not suitable for using. Fortunately, there is a characterization of Pom_L-open morphisms of labelled asynchronous morphisms. The following statement is a reformulation of the characterization of Pom_L-open morphisms given in [3, Prop.16]:

Proposition 2. *A morphism* $(\sigma, \eta) : (\mathcal{A}, \lambda, L) \to (\mathcal{A}', \lambda', L)$ *of labelled asynchronous systems is* Pom_L*-open if and only if the morphism* $(\sigma, \eta) : \mathcal{A} \to \mathcal{A}'$ *is open and preserves labels.*

This proposition allows us to mean by Pom_L-open morphisms the open morphisms, preserving labels.

Proposition 3. *Let* $(\mathcal{A}, \lambda, L)$ *and* $(\mathcal{A}', \lambda', L)$ *be labelled asynchronous systems. If* $(\mathcal{A}, \lambda, L)$ *and* $(\mathcal{A}', \lambda', L)$ *are* Pom_L*-bisimilar, then for every* $a_1 \in E$ *satisfying* $s_0 \cdot a_1 \in S$*, there exists* $a'_1 \in E'$ *such that the following two properties hold:*

- $s'_0 \cdot a'_1 \in S'$*;*
- *labelled asynchronous systems* $(\mathcal{A}(s_0 \cdot a_1), \lambda, L)$ *and* $(\mathcal{A}(s'_0 \cdot a'_1), \lambda', L)$ *are* Pom_L*-bisimilar.*

Proof. Given labelled asynchronous systems are Pom_L-bisimilar. Hence, there are $(\mathcal{A}'', \lambda'', L)$ and Pom_L-open morphisms

$$(\mathcal{A}, \lambda, L) \xleftarrow{(\sigma,\eta)} (\mathcal{A}'', \lambda'', L) \xrightarrow{(\sigma',\eta')} (\mathcal{A}', \lambda', L).$$

Morphism (σ, η) is open. It follows by property (ii) of Definition 4 that there exists a transition (s''_0, a''_1, s''_1) satisfying conditions $\eta(a''_1) = a_1$ and $\sigma(s''_1) = s_1$ (Fig. 2). In other words, there exists $a''_1 \in E''$ such that $\eta(a''_1) = a_1$ and $\sigma(s''_0 \cdot a''_1) = s_1$. By Proposition 1, the morphism $(\sigma, \eta) : \mathcal{A}''(s''_1) \to \mathcal{A}(s_1)$ is open. The map σ' of the

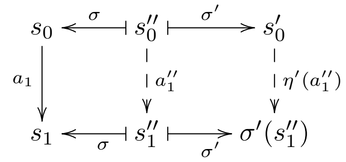


FIGURE 2. To the construction of open morphisms.

morphism $(\sigma', \eta') : \mathcal{A}'' \to \mathcal{A}'$ is total. It follows that $\sigma'(s''_1) \in S'$. By Proposition 1, the morphism $(\sigma', \eta') : \mathcal{A}''(s''_1) \to \mathcal{A}'(\sigma'(s''_1))$ is open. The morphisms (σ, η) and (σ', η') preserve labels. By putting $a'_1 = \eta'(a''_1)$ and $s'_1 = \sigma'(s''_1)$, we obtain the desired. □

Corollary 1. *Let* $(\mathcal{A}, \lambda, L)$ *and* $(\mathcal{A}', \lambda', L)$ *be* Pom_L*-bisimilar labelled asynchronous systems. For every* $w = a_1 \cdots a_k \in E^*$ *with* $k \geqslant 0$ *satisfying the condition* $s_0 \cdot w \in S$*, there exists a word* $w' = a'_1 \cdots a'_k \in E'^*$ *such that the following two properies hold:*

- $s'_0 \cdot w' \in S'$*;*
- *the labelled asynchronous systems* $(\mathcal{A}(s_0 \cdot w), \lambda, L)$ *and* $(\mathcal{A}'(s'_0 \cdot w'), \lambda', L)$ *are* Pom_L*-bisimilar.*

Proof. For $k = 0$, the word w is empty, that is $w = 1$. Taking $w' = 1$, we get the Pom_L-bisimilar labelled asynchronous systems $(\mathcal{A}, \lambda, L)$ and $(\mathcal{A}', \lambda', L)$. For $k = 1$, the assertion follows from Proposition 3. Assuming that the assertion is true for some $k > 0$, we can prove by Proposition 3, that it holds for $k+1$. So, it is true for all $k \geqslant 0$. □

3.3. **Open maps and surjectivity.** Let $\mathcal{A}$ be an asynchronous system. Denote by $Q_0(\mathcal{A}) = S(s_0)$ the set of all reachable states $s \in S$. For every $n > 0$, we consider sets

$$Q_n(\mathcal{A}) = \{(s, e_1, \cdots, e_n) \in S(s_0) \times E^n \mid s \cdot e_1 \cdots e_n \in S \;\&\; (e_i, e_j) \in I \text{ for all } 1 \leqslant i < j \leqslant n\}$$

Let $(\sigma, \eta) : \mathcal{A} \to \mathcal{A}'$ be a morphism of asynchronous system. If $\eta : E \to E'$ is total, then for all $n \geqslant 0$ the maps $Q_n(\sigma, \eta) : Q_n(\mathcal{A}) \to Q_n(\mathcal{A}')$ are defined by the formula

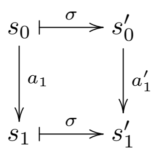
$$Q_n(\sigma, \eta)(s, e_1, \cdots, e_n) = (\sigma(s), \eta(e_1), \cdots, \eta(e_n)).$$

Proposition 4. *If a morphism $(\sigma, \eta) : \mathcal{A} \to \mathcal{A}'$ is open, then for all $n \geqslant 0$, the maps $Q_n(\sigma, \eta) : Q_n(\mathcal{A}) \to Q_n(\mathcal{A}')$ are surjective.*

Proof. Prove for $n = 0$. We have $\sigma(s_0) = s_0'$. If s' is reachable, then there exists a path

$$\sigma(s_0) = s_0' \overset{a_1'}{\to} s_1' \overset{a_2'}{\to} \dots \overset{a_k'}{\to} s_k' = s'.$$

There are $a_1 \in E$ and $s_1 \in S$ for which $\eta(a_1) = a_1'$ and $(\sigma, \eta)(s_0 \overset{a_1}{\to} s_1) = (s_0' \overset{a_1'}{\to} s_1')$:



We have $\sigma(s_1) = s_1'$. There are $a_2 \in E$ and $s_2 \in S$ satisfying $\sigma(s_2) = s_2'$ and $\eta(a_2) = a_2'$ and so on. By induction, we obtain $s_k \in S$ such that $\sigma(s_k) = s_k' = s'$. Therefore, $\sigma : S(s_0) \to S'(s_0')$ is surjective.

For $n = 1$, the map $Q_1(\sigma, \eta) : \{(s, e_1) | s e_1 \in S\} \to \{(\sigma(s), e_1') | \sigma(s) e_1' \in S'\}$ is surjective by property (ii) of open morphisms.

Let $n \geqslant 2$. For each $s \in S(s_0)$, consider the set

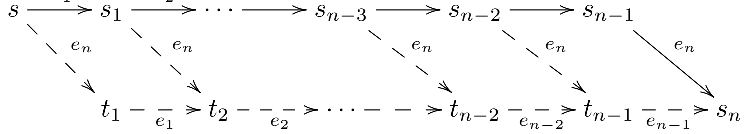
$$Q_n(\mathcal{A}, s) = \{(s, e_1, \cdots, e_n) \in \{s\} \times E^n \mid s \cdot e_1 \cdots e_n \in S \;\&\; (e_i, e_j) \in I \text{ for all } 1 \leqslant i < j \leqslant n\}$$

and

$$Q_n(\mathcal{A}', \sigma(s)) = \{(\sigma(s), e_1', \cdots, e_n') \in \{\sigma(s)\} \times E'^n | \sigma(s) \cdot e_1' \cdots e_n' \in S' \;\&\; (e_i', e_j') \in I \text{ for all } 1 \leqslant i < j \leqslant n.\}$$

For any $(\sigma(s), e_1', \cdots, e_n') \in Q_n(\mathcal{A}', \sigma(s))$, there are $e_1, e_2, \dots, e_n \in E$ for which $s_1 = s \cdot e_1 \in S$, $s_2 = s \cdot e_1 \cdot e_2 \in S$, $\dots$, $s_n = s \cdot e_1 \cdots e_n \in S$, wherein $\eta(e_1) = e_1'$, $\dots$, $\eta(e_n) = e_n'$.

By induction on n, we will prove that $(e_i, e_j) \in I$ for all $1 \leqslant i < j \leqslant n$. For this purpose, we assume that $(e_i, e_j) \in I$ for all $1 \leqslant i < j \leqslant n - 1$. We will show that $(e_i, e_n) \in I$ for all $1 \leqslant i \leqslant n - 1$.



Since $(s_{n-2}, e_{n-1}, s_{n-1}) \in$ Tran, $(s_{n-1}, e_n, s_n) \in$ Tran, and $(\eta(e_{n-1}), \eta(e_n)) \in I'$, we have by the property (iii) of open morphisms that $(e_{n-1}, e_n) \in I$. By Axiom (ii) for a state space, there is $t_{n-1} \in S$ such that $(s_{n-2}, e_n, t_{n-1}) \in$ Tran and $(t_{n-1}, e_{n-1}, s_n) \in$ Tran. It follows from $(\eta(e_{n-2}), \eta(e_n)) \in I'$, that $(e_{n-2}, e_n) \in I$. Again by Axiom (ii), there is $t_{n-2} \in S$ such that $(s_{n-3}, e_n, t_{n-2}) \in$ Tran and $(t_{n-2}, e_{n-2}, t_{n-1}) \in$ Tran. By the property (iii), it follows from $(\eta(e_{n-3}), \eta(e_n)) \in I'$, that $(e_{n-3}, e_n) \in I$, and so on. In the end, we obtain $(e_i, e_n) \in I$ for all $1 \leqslant i \leqslant n-1$. Consequently $(e_i, e_j) \in I$ for all $1 \leqslant i < j \leqslant n$. Thus, $(s, e_1, \ldots, e_n) \in Q_n(\mathcal{A}, s)$. Therefore for every $(s', e'_1, \ldots, e'_n) \in Q_n(\mathcal{A})$, there is $(s, e_1, \ldots, e_n) \in Q_n(\mathcal{A})$ mapped to $(s', e'_1, \ldots, e'_n) \in Q_n(\mathcal{A})$. □

Remark 1. *The converse is not true. There are morphisms (σ, η), for which the map $Q_n(\sigma, \eta)$ is surjective for all $n \geqslant 0$, but the (σ, η) is not Pom_{pt}-open. For example, $S = \{s_0\}$, $E = \{a, b, c\}$, $I = \{(a, b), (b, a)\}$, $S' = \{s'_0\}$, $E' = \{a', b'\}$, $I' = \{(a', b'), (b', a')\}$. Figure 3 shows the independence graphs and the map $\eta : E \to E'$. We have $(\eta(b), \eta(c)) \in I'$, but $(b, c) \notin I$. Hence, the morphism (σ, η) is not open.*

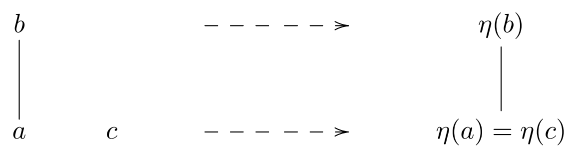


FIGURE 3. The map $\eta : E \to E'$

Corollary 2. *If $(\sigma, \eta) : \mathcal{A} \to \mathcal{A}'$ is open, then for each reachable state $s \in S$, the maps $Q_n(\mathcal{A}(s)) \to Q_n(\mathcal{A}'(\sigma(s)))$ are surjective for all $n \geqslant 0$.*

4. HOMOLOGY GROUPS OF ASYNCHRONOUS SYSTEMS

We introduce the homology groups of labelled asynchronous systems. We will prove that bisimulation equivalence is stronger than homological equivalence.

4.1. Computing homology groups of simplicial schemes.

A *simplicial scheme* $(A, \mathfrak{M})$ consists of a set A of *vertices* and a set $\mathfrak{M}$ of finite nonempty subsets $S \subseteq A$ satisfying the following conditions

- $(\forall a \in A)\ \{a\} \in \mathfrak{M}$,
- $(\forall S, S' \subseteq A)\ S \in \mathfrak{M}\ \&\ S' \subseteq S \Rightarrow S' \in \mathfrak{M}$.

The elements of $\mathfrak{M}$ are called *simplices*. For $n \geqslant 0$, a simplex S is called *n-dimensional* or *n-simplex* if number $|S|$ of its elements equals $n+1$.

Let $(A, \mathfrak{M})$ be a simplicial scheme. For the computing its *homology groups* $H_n(A, \mathfrak{M})$, we define an arbitrary total order relation on A. Consider the complex

$$0 \leftarrow \mathbb{Z}\mathfrak{M}_0 \stackrel{d_1}{\leftarrow} \mathbb{Z}\mathfrak{M}_1 \stackrel{d_2}{\leftarrow} \mathbb{Z}\mathfrak{M}_2 \leftarrow \cdots \leftarrow \mathbb{Z}\mathfrak{M}_{n-1} \stackrel{d_n}{\leftarrow} \mathbb{Z}\mathfrak{M}_n \leftarrow \cdots$$

where $\mathfrak{M}_n = \{(a_0, a_1, \ldots, a_n) | a_0 < a_1 < \cdots < a_n\ \&\ \{a_0, a_1, \ldots, a_n\} \in \mathfrak{M}\}$. Elements of $\mathfrak{M}_n$ are called *ordered n-simplices*. Here $\mathbb{Z}\mathfrak{M}_n$ denotes the free Abelian group generated by ordered n-simplices. The differentials d_n are defined on ordered n-simplices by the formula

$$d_n(a_0, a_1, \ldots, a_n) = \sum_{i=0}^{n} (-1)^i (a_0, \ldots, \widehat{a_i}, \ldots, a_n)$$

where $\widehat{a_i}$ denotes the operation of removing the symbol a_i from the tuple. We will suppose that the sets of n-simplices are finite. In this case, the differentials d_n can be specified using integer matrices.

Each column of the matrix for d_n corresponds to a tuple $(a_0, a_1, \ldots, a_n) \in \mathfrak{M}_n$. Each string corresponds to $(a_0, \ldots, a_{n-1}) \in \mathfrak{M}_{n-1}$. For each column $(a_0, a_1, \ldots, a_n)$ and string $(a_0, \ldots, \widehat{a_i}, \ldots, a_n)$, at their intersection, the entry equals $(-1)^i$. Other entries of the matrix equal 0. For calculating the homology groups, each matrix d_n is reduced to the Smith normal form. The homology groups $H_n = Ker(d_n)/Im(d_{n+1})$ of this complex is equal to

$$\mathbb{Z}^{|\mathfrak{M}_n| - rank(d_n) - rank(d_{n+1})} \oplus \mathbb{Z}/\delta_1 \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/\delta_r \mathbb{Z}$$

where $r = rank(d_{n+1})$ and $\delta_1, \cdots, \delta_r$ is the non-zero diagonal entries of the Smith normal form for the matrix d_{n+1}.

4.2. **Homology groups of labelled asynchronous systems.** Let $(\mathcal{A}, \lambda, L)$ be a labelled asynchronous system.

Introduce homology groups of the labelled asynchronous systems. For this purpose, consider the simplicial scheme $(\lambda^+ E, \mathfrak{M})$ whose vertices are the elements $\lambda(a)$, where $a \in E$ are elements for which there are $s, s' \in S(s_0)$ satisfying $(s, a, s') \in$ Tran. Thus

$$\lambda^+ E = \{\lambda(a) \mid (\exists s \in S(s_0)) s \cdot a \in S\}.$$

Simplices are finite sets $\{\lambda(a_1), \ldots, \lambda(a_k)\}$, $k \geqslant 1$, for which the following two conditions hold:

- $(a_i, a_j) \in I$, for all $1 \leqslant i < j \leqslant k$;
- there are $s \in S(s_0)$ for which $s \cdot a_1 \cdots a_k \in S$.

Remark 2. *We give an explanation to the construction of simplices.*

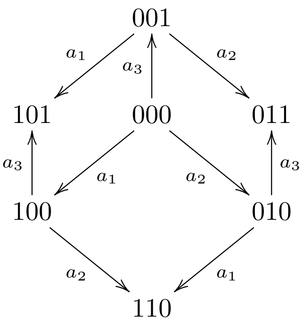
(i) *For every $(s, a_1, \ldots, a_k) \in Q_k(\mathcal{A})$, we include the set $\{\lambda(a_1), \ldots, \lambda(a_k)\}\}$ in $\mathfrak{M}$.*
(ii) *If the elements are duplicated in $\{\lambda(a_1), \ldots, \lambda(a_k)\}$, then we remove them. For example $\{a, b, a, c, a, b\} = \{a, b, c\}$.*

Definition 5. Homology groups $H_n(\mathcal{A}, \lambda, L)$ *of a labelled asynchronous system is the homology groups* $H_n(\lambda^+ E, \mathfrak{M})$ *of the constructed simplicial scheme.*

Example 1. *Consider an asynchronous system $\mathcal{A} = (S, s_0, E, I, \mathrm{Tran})$ where $S = \{000, 001, 010, 011, 100, 101, 110\}$, $s_0 = 000$, $E = \{a_1, a_2, a_3\}$, and*

$$I = \{(a_1, a_2), (a_2, a_1), (a_1, a_3), (a_3, a_1), (a_2, a_3), (a_3, a_2)\}.$$

Transitions correspond to arrows of the diagram:



Let $L = E$ *and let the label function* $\lambda : E \to L$ *is defined as* $\lambda(a) = a$ *for all* $a \in E$. *The simplicial scheme consists of vertices* $E = \{a_1, a_2, a_3\}$ *and simplices* $\{a_1, a_2\}$, $\{a_1, a_3\}$, $\{a_2, a_3\}$. *Define the order on vertices by* $a_1 < a_2 < a_3$. *Homology groups is computed by the complex*

$$0 \leftarrow \mathbb{Z}\{a_1, a_2, a_3\} \stackrel{d_1}{\leftarrow} \mathbb{Z}\{(a_1, a_2), (a_1, a_3), (a_2, a_3)\} \leftarrow 0$$

Matrix for d_1 *equals*

$$\begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{ccc} (a_1,a_2) & (a_1,a_3) & (a_2,a_3) \end{array} \\ \left(\begin{array}{ccc} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{array} \right) \end{array}$$

The Smith normal form for d_1 *equals*

$$\begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{ccc} (a_1,a_2) & (a_1,a_3) & (a_2,a_3) \end{array} \\ \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right) \end{array}$$

It follows that $H_0(\mathcal{A}, \lambda, L) = \mathbb{Z}^{3-0-2} \oplus \mathbb{Z}/1\,\mathbb{Z} \oplus \mathbb{Z}/1\,\mathbb{Z} \cong \mathbb{Z}$, $H_1(\mathcal{A}, \lambda, L) = \mathbb{Z}^{3-2-0} \cong \mathbb{Z}$. *Other homology groups equal* 0.

The complex for computing groups $H_n(\mathcal{A}(s), \lambda, L)$ *for* $s = 001$ *has unique non-zero term* $\mathbb{Z}\{a_1, a_2\}$. *It follows*

$$H_n(\mathcal{A}(s), \lambda, L) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & \text{if } n = 0, \\ 0, & \text{if } n > 0. \end{cases}$$

The complex for computing $H_n(\mathcal{A}(s), \lambda, L)$ *for* $s = 011$ *consists of zeros. Therefore* $H_n(\mathcal{A}(011), \lambda, L) = 0$ *for all* $n \geqslant 0$.

Theorem 1. *If labelled asynchronous systems* $(\mathcal{A}, \lambda, L)$ *and* $(\mathcal{A}', \lambda', L)$ *are* Pom_L*-bisimilar, then their homology groups are isomorphic.*

Proof. Denote by $\mathfrak{M}$ and $\mathfrak{M}'$ the simplicial schemes corresponded to the labelled asynchronous systems. If the labelled asynchronous systems are Pom_L-bisimilar, then there is a labelled asynchronous system together with the morphisms

$$(\mathcal{A}, \lambda, L) \stackrel{(\sigma,\eta)}{\longleftarrow} (\mathcal{A}'', \lambda'', L) \stackrel{(\sigma',\eta')}{\longrightarrow} (\mathcal{A}', \lambda', L)\ .$$

Let $P^f(L)$ be the set of all finite subsets of L. Consider a maps $\lambda_n : Q_n(\mathcal{A}) \to P^f(L)$ acting as $\lambda(s, a_1, \ldots, a_n) = \{\lambda(a_1), \ldots, \lambda(a_n)\}$. The function λ can have equal values. Hence, the set $\{\lambda(a_1), \ldots, \lambda(a_n)\}$ can contain $< n$ elements

For $n = 0$, we let $\lambda_0(s) = \emptyset$. The pairs (σ, η) and (σ', η') are morphisms of asynchronous systems. Hence, the following diagram is commutative

$$\begin{array}{ccccc} Q_n(\mathcal{A}) & \xleftarrow{Q_n(\sigma,\eta)} & Q_n(\mathcal{A}'') & \xrightarrow{Q_n(\sigma',\eta')} & Q_n(\mathcal{A}') \\ & \searrow^{\lambda_n} & \downarrow{\scriptstyle \lambda''_n} & \swarrow_{\lambda'_n} & \\ & & P^f(L) & & \end{array}$$

By Proposition 4 the maps $Q_n(\sigma, \eta)$ and $Q_n(\sigma', \eta')$ are surjective. Thus, we have the equalities $Im(\lambda_n) = Im(\lambda''_n) = Im(\lambda'_n)$. Consequently the simplicial schemes $\mathfrak{M}$ and $\mathfrak{M}'$ are equal. Therefore, the groups $H_n(\lambda^+ E, \mathfrak{M})$ and $H_n(\lambda'^+ E', \mathfrak{M}')$ are isomorphic. □

Corollary 3. *Let $(\mathcal{A}, \lambda, L)$ and $(\mathcal{A}', \lambda', L)$ be Pom_L-bisimilar asynchronous systems. For each $w = a_1 \cdots a_k \in E^*$, $k \geqslant 0$, satisfying $s_0 \cdot w \in S$ there is a word $w' = a'_1 \cdots a'_k \in E'^*$ such that $s'_0 \cdot w' \in S'$ and*

(1) $$(\forall n \geqslant 0)\ H_n(\mathcal{A}(s_0 \cdot w), \lambda, L) \cong H_n(\mathcal{A}'(s'_0 \cdot w'), \lambda', L).$$

Proof. By Proposition 1, in this case for the word w, there exists w' for which $(\mathcal{A}(s_0 \cdot w), \lambda, L)$ and $(\mathcal{A}'(s'_0 \cdot w'), \lambda', L)$ are Pom_L-bisimilar. Application of Theorem 1 to the obtained labelled asynchronous systems leads us to desired isomorphism of the homology groups. □

5. Homology groups of labelled Petri nets

Recall some definitions from theory of Petri nets. Then consider homology groups of labelled Petri nets and prove that for each simplicial scheme, there is a labelled Petri net homological equivalent to this simplicial scheme.

5.1. **Petri nets.** For a finite set P, any function $M : P \to \mathbb{N} = \{0, 1, 2, \ldots\}$ can be considered as a vector with components $M(p)$, $p \in P$. let $\mathbb{N}^P$ denotes a set of all functions $M : P \to \mathbb{N}$. For any $M_1, M_2 \in \mathbb{N}^P$, define a sum $M_1 + M_2$ as a function with values $(M_1 + M_2)(p) = M_1(p) + M_2(p)$ for all $p \in P$. Let $M_1 \geqslant M_2$ if $M_1(p) \geqslant M_2(p)$ for all $p \in P$. If $M_1 \geqslant M_2$, then we can define a *difference* $M_1 - M_2$ as the function with the values $M_1(p) - M_2(p)$. Define a *scalar product* by $M_1 \cdot M_2 = \sum_{p \in P} M_1(p) M_2(p)$.

A *Petri net* $\mathcal{N} = (P, T, pre, post, M_0)$ consists of finite sets P and T with two maps $pre : T \to \mathbb{N}^P$, $post : T \to \mathbb{N}^P$ and a function $M_0 : P \to \mathbb{N}$ called *initial marking*. Elements $p \in P$ are called *places*, and $t \in T$ are *events*. A *marking* is an arbitrary function $M : P \to \mathbb{N}$.

A Petri net can be given as a directed graph whose vertices are places depicted by circles, and events depicted by rectangles. Every arrow goes from an event to a place or from a place to an event. For any $t \in T$, the number of entering into it arrows equals $pre(t)(p)$ and the number of arrows outgoing from t equals $post(t)(p)$. The initial marking is given by drawing the points in each place. These points are called *tokens*. The number of tokens in a place p is equal to $M_0(p)$. If $M_0(p) = 0$, then the place is empty.

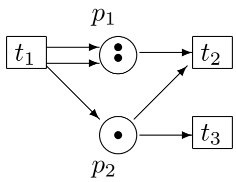


FIGURE 4. Example of Petri net

Fig. 4 shows a Petri net $\mathcal{N} = (P, T, pre, post, M_0)$ where $P = \{p_1, p_2\}$, $T = \{t_1, t_2, t_3\}$. The values $pre(t_i)(p_j)$ and $post(t_i)(p_j)$, $1 \leqslant i \leqslant 3$, $1 \leqslant j \leqslant 2$, are equal to the entries of the matrices

$$(pre(t_i)(p_j)) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad (post(t_i)(p_j)) = \begin{pmatrix} 2 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

5.2. **Labelled asynchronous system for a Petri net and its homology groups.** Let $\mathcal{N} = (P, T, pre, post, M_0)$ be a Petri net. Similar to how it is done in [14] for elementary Petri nets, consider a corresponding asynchronous system $\mathcal{A}(\mathcal{N}) = (S, s_0, E, I, \mathrm{Tran})$, with $S = \mathbb{N}^P$, $s_0 = M_0$, $E = T$. The independence relation I consists of pairs $(e_1, e_2) \in T \times T$ for which the scalar product $(pre(e_1) + post(e_1)) \cdot (pre(e_2) + post(e_2))$ equals 0. This means that e_1 and e_2 do not have common input or output places. The set Tran consists of triples (M, e, M') where M and M' are markings and $e \in T$ satifies two following conditions

- $M \geqslant pre(e)$,
- $M - pre(e) + post(e) = M'$.

If $(M, e, M') \in \mathrm{Tran}$, then we say that the marking M' is obtained from M *by operation of event* $e \in T$. For example, for Petri net in Fig. 4, we have $pre(t_2) \geqslant M_0$. The operation of the event t_2 leads to the new marking $M_1 = M_0 - pre(t_2) + post(t_2)$ (Fig. 5).

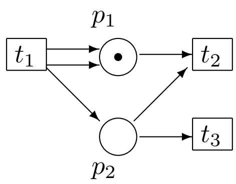


FIGURE 5. The marking obtained by operation of the event t_2

Let L be an arbitrary nonempty set. A Petri net $\mathcal{N}$ with a function $\lambda : T \to L$ is called *labelled*. The asynchronous system $\mathcal{A}(\mathcal{N})$ corresponding to $\mathcal{N}$ has the set of events $E = T$. Hence, for any labelled Petri nets, it is defined the labelled asynchronous system $(\mathcal{A}(\mathcal{N}), \lambda, L)$.

Definition 6. *Let $(\mathcal{N}, \lambda, L)$ be a labelled Petri net. Its homology groups $H_n(\mathcal{N}, \lambda, L)$ are defined as $H_n(\mathcal{A}(\mathcal{N}), \lambda, L)$, $n \geqslant 0$.*

Example 2. *Consider the Petri net $\mathcal{N} = (P, T, pre, post, M_0)$, in Fig. 6. Let $L = E = \{t_1, t_2, t_3, t_4\}$, $\lambda(t_i) = t_i$, for all $1 \leqslant i \leqslant 4$. The relation I contains the*

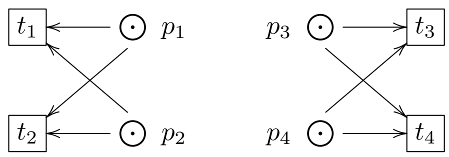


FIGURE 6. Example of computing the homology groups of Petri net

pairs (t_1, t_3), (t_1, t_4), (t_2, t_3), (t_2, t_4), (t_3, t_1), (t_3, t_2), (t_4, t_1), (t_4, t_2). The simplicial scheme $(E, \mathfrak{M})$ has the following sets of simplices

$$\mathfrak{M}_0 = \{t_1, t_2, t_3, t_4\}, \mathfrak{M}_1 = \{(t_1, t_3), (t_1, t_4), (t_2, t_3), (t_2, t_4)\},$$

and $\mathfrak{M}_n = \emptyset$ for $n \geqslant 2$. We get the following complex for the computing the homology groups of the labelled Petri nets:

$$0 \leftarrow \mathbb{Z}^4 \xleftarrow{d_1} \mathbb{Z}^4 \leftarrow 0.$$

The differential d_1 is given by the matrix

$$\begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{array} \begin{array}{c} \begin{array}{cccc} (t_1,t_3) & (t_1,t_4) & (t_2,t_3) & (t_2,t_4) \end{array} \\ \begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ +1 & 0 & +1 & 0 \\ 0 & +1 & 0 & +1 \end{pmatrix} \end{array}$$

Its Smith normal form has the diagonal entries $(1, 1, 1, 0)$. Consequently

$$H_0(\mathcal{N}, \lambda, L) \cong H_1(\mathcal{N}, \lambda, L) = \mathbb{Z} \text{ and } H_n(\mathcal{N}, \lambda, L) = 0 \text{ for all } n \geqslant 2.$$

Let us find out what the homology groups can have a Petri net.

Let $(X, \mathfrak{M})$ be a simplicial scheme. A *barycentric subdivision* $(E, \mathfrak{M}')$ of $(X, \mathfrak{M})$ is a simplicial scheme with the set of vertices $E = \mathfrak{M}$ and where $\mathfrak{M}'$ consists of all finite sets of simplices $\{\sigma_0, \ldots, \sigma_n\}$, $n \geqslant 0$, totally ordered by the relation $\subseteq$. It means that there is a permutation $(\sigma_{i_0}, \ldots, \sigma_{i_n})$ such that $\sigma_{i_0} \subset \sigma_{i_1} \subset \ldots \subset \sigma_{i_n}$. It is well known that homology groups of $(E, \mathfrak{M}')$ are isomorphic to homology groups of $(X, \mathfrak{M})$.

Let (E, I) be a set with an independence relation. Consider a simplical scheme $(E, \mathfrak{M}(E, I))$ where $\mathfrak{M}(E, I)$ is the set of all non-empty finite subsets $\{e_0, \ldots, e_n\} \subseteq E$ consisting of mutually independent elements.

Lemma 1. *For any simplicial scheme $(X, \mathfrak{M})$, there exists a set E with an independence relation I such that $H_n(X, \mathfrak{M}) \cong H_n(E, \mathfrak{M}(E, I))$ for all $n \geqslant 0$.*

Proof. Let $(E, \mathfrak{M}')$ be the barycentric subdivision of $(X, \mathfrak{M})$. We prove that $(E, \mathfrak{M}')$ coincides with the simplicial scheme $(E, \mathfrak{M}(E, I))$ where $I \subseteq E \times E$ is the relation such that

$$(\sigma, \sigma') \in I \Leftrightarrow \sigma \subset \sigma' \vee \sigma' \subset \sigma.$$

Each $\{\sigma_0, \ldots, \sigma_n\} \in \mathfrak{M}(E, I)$ consists of simplices satisfying $\sigma_i \subset \sigma_j \ \vee \ \sigma_j \subset \sigma_i$, for all $0 \leqslant i \neq j \leqslant n$. Hence $\{\sigma_0, \ldots, \sigma_n\}$ is totally ordered by $\subseteq$. It follows that $\{\sigma_0, \ldots, \sigma_n\} \in \mathfrak{M}'$. Consequently $\mathfrak{M}(E, I) \subseteq \mathfrak{M}'$. The inclusion $\mathfrak{M}' \subseteq \mathfrak{M}(E, I)$

is obvious. Thus $\mathfrak{M}' = \mathfrak{M}(E, I)$. Therefore, $H_n(E, \mathfrak{M}(E, I)) \cong H_n(E, \mathfrak{M}') \cong H_n(X, \mathfrak{M})$. □

A sequence of Abelian groups A_k, $k \geqslant 0$, is called to be *finite* if there is $n \geqslant 0$ such that $A_k = 0$ for all $k > n$.

Theorem 2. *For an arbitrary finite sequence of finitely generated Abelian groups A_0, A_1, A_2, … where A_0 is free and is not equal to 0, there exists a labelled Petri net such that its kth homology groups are isomorphic to A_k for all $k \geqslant 0$.*

Proof. In this case by [17, Chapter 4, Exercise C-7], there exists a compact polyhedron with homology groups A_k for all $k \geqslant 0$. Compact polyhedra are precisely the topological spaces admitting triangulations [17, Chapter 3, Corollary 20]. Hence, there exists a simplicial scheme $(X, \mathfrak{M})$ the homology groups of which are isomorphic to A_k.

We will construct a Petri net with homology groups A_k. By Lemma 1, there exists a set E with an independence relation I such that for all $k \geqslant 0$ there are isomorphisms $H_k(E, \mathfrak{M}(E, I)) \cong H_k(X, \mathfrak{M}) \cong A_k$ for all $k \geqslant 0$.

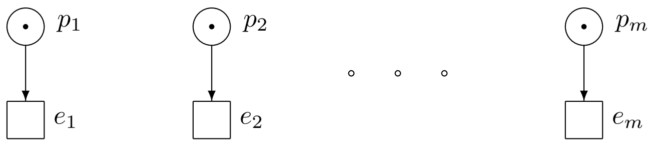


FIGURE 7. The constructing of a Petri net

Denote the elements of E by e_1, e_2, …, e_m where $m = |E|$. Consider the Petri net depicted in Fig. 7. It consists of places p_i, connected with the events e_i by the arrows where $i = 1, 2, \ldots, m$. The initial marking is defined as $s_0(p_i) = 1$ for all $i = 1, 2, \ldots, m$. For every $(e_i, e_j) \notin I$, we make the events e_i and e_j to be dependent by adding the place p_{ij} with two arrows as shown in Fig. 8.

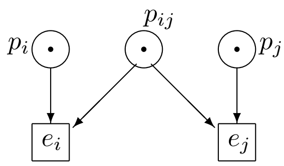


FIGURE 8. Adding arrows to the Petri net

Let $L = E$ and let the label function defined as $\lambda(e_i) = e_i$ for all $i = 1, \ldots, m$. For every $e_i \in E$, we have $s_0 \cdot e_i \in S$. It follows that the set of vertices of a simplicial scheme corresponding to the Petri net is equal to E. For each nonempty subset $\{e_{i_0}, \dots, e_{i_n}\} \subseteq E$ consisting of mutually independent elements, we have $s_0 \cdot e_{i_0} \cdots e_{i_n} \in S$. Consequently the simplicial scheme corresponding to the Petri net is equal to $(E, \mathfrak{M}(E,I))$. Thus, $H_k(\mathcal{N}, \lambda, L) = A_k$ for all $k \geqslant 0$. □

Corollary 4. *For any finite sequence of finitely generated Abelian groups A_0, A_1, A_2, ... where A_0 is free and non-zero, there is a labelled asynchronous system the kth homology groups of which are isomorphic to A_k for all $k \geqslant 0$.*

References

[1] A. Joyal, M. Nielsen and G. Winskel, *Bisimulation from open maps*, Information and Computation, **127** (1996), 164–185. MR1407994

[2] R. Milner, *Communication and concurrency.* International Series in Computer Science, Prentice Hall, New York, 1989. Zbl 0683.68008

[3] M. Nielsen and G. Winskel, *Petri nets and bisimulation*, Theoretical Computer Science, **153**:1-2 (1996), 211–244. MR1365741

[4] M. Herlihy and N. Shavit, *The Topological Structure of Asynchronous Computability*, Journal of ACM, **46**:6 (1999), 858–923. MR1753036

[5] E. Goubault and T.P. Jensen, *Homology of higher dimensional automata*, Lecture Notes in Computer Science, **630**, Springer, Berlin, 1992. 254–268. MR1251524

[6] E. Goubault, *The Geometry of Concurrency,* Ph.D. Thesis, Ecole Normale Supérieure, 1995.

[7] L. Fajstrup, M. Raußen, E.Goubault. *Algebraic topology and concurrency*, Theoretical Computer Science **357**:1-3 (2006), 241–278. MR2242768

[8] E. Goubault, E. Haucourt and S. Krishnan, *Covering space theory for directed topology*, Theory and Applications of Categories, **22**:9 (2009), 252–268. MR2520971

[9] U. Fahrenberg, A. Legay, *History-Preserving Bisimilarity for Higher-Dimensional Automata via Open Maps*, Electronic Notes in Theoretical Computer Science, **538** (2013), 165–178.

[10] A. Husainov, *On the homology of small categories and asynchronous transition systems*, Homology, Homotopy and Applications, **6**:1 (2004), 439–471. MR2118495

[11] A.A. Khusainov, V.E. Lopatkin, I.A. Treshchev, *Studying a mathematical model of parallel computation by algebraic topology methods*, Journal of Applied and Industrial Mathematics, **3**:3 (2009), 353–363. MR2535256

[12] A.A. Husainov, *The cubical homology of trace monoids*, Far Eastern Mathematical Journal, **12**:1 (2012), 108–122
http://mi.mathnet.ru/eng/dvmg/v12/i1/p108 MR2946821

[13] A.A. Husainov, *The Homology of Partial Monoid Actions and Petri Nets*, Applied Categorical Structures, **21**:6 (2013), 587–615. MR3127362

[14] G. Winskel and M. Nielsen, *Models for Concurrency*, in: Abramsky, Gabbay and Maibaum, eds., Handbook of Logic in Computer Science, **4**, Oxford University Press, Oxford, 1995. 1–148. MR1365754

[15] M.A. Bednarczyk, *Categories of Asynchronous Systems,* Ph.D. thesis, University of Sussex, Report No. 1/88, 1988.

[16] V. Diekert, Y. Métivier, *Partial Commutation and Traces*, in: Handbook of formal languages, **3**, Springer, New York, 1997. 457–533. MR1470025

[17] E.H. Spanier, *Algebraic topology*, McGraw-Hill Book Company, New York, 1966. MR0210112

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