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SMALL CYCLES IN THE STAR GRAPH

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Abstract. The Star graph is the Cayley graph on the symmetric group Sym_n generated by the set of transpositions $\{(12),(13),\dots,(1n)\}$. These graphs are bipartite, they do not contain odd cycles but contain all even cycles with a sole exception 4-cycles. We characterize all distinct 6– and 8–cycles by their canonical forms as products of generating elements. The number of these cycles in the Star graph is also given.

Keywords: Cayley graphs; Star graph; cycle embedding; product of generating elements

1. Introduction

The Star graph $S_n = Cay(Sym_n, t)$, $n \geqslant 2$, is the Cayley graph on the symmetric group Sym_n of permutations $\pi = [\pi_1\pi_2\dots\pi_n]$, where $\pi_i = \pi(i)$, $1 \leqslant i \leqslant n$, with the generating set $t = \{t_i \in Sym_n : 2 \leqslant i \leqslant n\}$ of all transpositions $t_i = (1\,i)$ transposing the 1st and ith elements of a permutation π when multiplied on the right, i.e. $[\pi_1\pi_2\dots\pi_{i-1}\pi_i\pi_{i+1}\dots\pi_n]t_i = [\pi_i\pi_2\dots\pi_{i-1}\pi_1\pi_{i+1}\dots\pi_n]$. It is a connected bipartite $(n-1)$–regular graph of order $n!$ and diameter $diam(S_n) = \lfloor\frac{3(n-1)}{2}\rfloor$ [1]. Since this graph is bipartite it does not contain odd cycles but it contains all even l–cycles where $l = 6, 8, \dots, n!$ [3]. The hamiltonicity of this graph was also shown in papers on generating all permutations by transpositions [6, 8].

The graph $S_n, n \geqslant 3$, has the hierarchical structure such that it contains n copies of $S_{n-1}(i), 1 \leqslant i \leqslant n$, where each $S_{n-1}(i)$ has the vertex set $V_i = \{[\pi_1\dots\pi_{n-1}i]$, where $\pi_k \in \{1,\dots,n\}\backslash\{i\} : 1 \leqslant k \leqslant n-1\}$, $|V_i| = (n-1)!$, and the edge set $E_i = \{\{[\pi_1\dots\pi_{n-1}i],[\pi_1\dots\pi_{n-1}i]\,t_j\} : 2 \leqslant j \leqslant n-1\}$, $|E_i| = \frac{(n-1)!(n-2)}{2}$. Any two copies $S_{n-1}(i), S_{n-1}(j), i \neq j$, are connected by $(n-2)!$ edges presented as

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$\{[i\pi_2 \dots \pi_{n-1} j],\ [j\pi_2 \dots \pi_{n-1} i]\}$, where $[j\pi_2 \dots \pi_{n-1} i] = [i\pi_2 \dots \pi_{n-1} j]t_n$. Transpositions $t_j, 2 \leqslant j \leqslant n-1$, define *internal edges* in all n copies $S_{n-1}(i), 1 \leqslant i \leqslant n$, and the transposition t_n defines *external edges* between copies. Copies $S_{n-1}(i)$ are also called $(n-1)$*–copies.* A vertex in S_n is identified with the permutation corresponding to this vertex.

In this paper the characterization of small cycles in the Star graph is given and the number of distinct 6− and 8−cycles is obtained. An explicit description of cycles is given by their representation via a product of generating elements. First this representation was used in [5] to characterize small cycles in the Pancake graph.

A sequence of transpositions $C_\ell = t_{i_0} \dots t_{i_{\ell-1}}$, where $2 \leqslant i_j \leqslant n$, and $i_j \neq i_{j+1}$ for any $0 \leqslant j \leqslant \ell - 1$, such that $\pi\, t_{i_0} \dots t_{i_{\ell-1}} = \pi$, where $\pi \in Sym_n$, is said to be *a form of a cycle C_ℓ of length ℓ* in the Star graph. A cycle C_ℓ of length ℓ is called an ℓ–cycle. It is evident that any ℓ–cycle can be presented by $2\,\ell$ forms (not necessarily different) with respect to a vertex and a direction. *The canonical form C_ℓ of an ℓ–cycle* is called a form with a lexicographically maximal sequence of indices $i_0 \dots i_{l-1}$. For cycles of a form $C_\ell = t_a t_b \cdots t_a t_b$, where $\ell = 2\,k$, and $t_a t_b$ appears k times, we write $C_\ell = (t_a t_b)^k$. In particular, $S_3 \cong C_6$ with the following canonical representation:

$$C_6 = t_3 t_2 t_3 t_2 t_3 t_2 = (t_3 t_2)^3. \tag{1}$$

The main results of this paper are the following two theorems giving the complete description of 6− and 8−cycles in the Star graph.

Theorem 1. *Each of vertices of $S_n, n \geqslant 3$, belongs to $\binom{n-1}{2}$ distinct 6–cycles of the following canonical form:*

$$C_6 = (t_k\, t_i)^3, \quad 2 \leqslant i < k \leqslant n. \tag{2}$$

In total, there are $\frac{(n-2)(n-1)n!}{12}$ distinct 6–cycles in S_n.

Theorem 2. *Each of vertices of $S_n, n \geqslant 4$, belongs to $3(n-3)(n-2)(n-1)$ distinct 8–cycles of the following canonical forms:*

$$C_8^1 = t_k\, t_i\, t_j\, t_i\, t_k\, t_i\, t_j\, t_i, \quad 2 \leqslant i \neq j \leqslant k-1; \tag{3}$$

$$C_8^2 = t_k\, t_j\, t_i\, t_j\, t_k\, t_i\, t_j\, t_i, \quad 2 \leqslant i \neq j \leqslant k-1; \tag{4}$$

$$C_8^3 = t_k\, t_j\, t_i\, t_k\, t_j\, t_k\, t_i\, t_j, \quad 2 \leqslant i \neq j \leqslant k-1; \tag{5}$$

$$C_8^4 = t_k\, t_j\, t_k\, t_i\, t_k\, t_j\, t_k\, t_i, \quad 2 \leqslant i < j \leqslant k-1, \tag{6}$$

where $4 \leqslant k \leqslant n$. There are $\frac{3\,(n-3)(n-2)(n-1)\,n!}{8}$ distinct 8–cycles in S_n.

Proofs of theorems above are based on the hierarchical structure of the Star graph $S_n, n \geqslant 3$. If $n = 3$ then $S_3 \cong C_6$, where C_6 is described by (1). The graph S_4 has four copies of $S_3 \cong C_6$ described by (1), and it may contain 6–cycles with representations differ from (1). Such 6–cycles of S_4 should consist of paths within copies of S_3 together with external edges between these copies. In general, any 6–cycle (8–cycle) of $S_k, k \geqslant 4$, must consist of paths within subgraphs that are isomorphic to S_{k-1} for some $4 \leqslant k \leqslant n$, joined by external edges between these subgraphs. Hence, all 6–cycles (8–cycles) of $S_n, n \geqslant 4$, could be found recursively by considering 6–cycles (8–cycles) within each $S_k, 4 \leqslant k \leqslant n$, consisting of vertices from some copies of S_{k-1}. This approach is used in proofs of Theorem 1 and Theorem 2. To get the main results, we need technical lemmas presented in the next section.

2. Technical lemmas

Any permutation $\pi = [\pi_1 \ldots \pi_i \ldots \pi_j \ldots \pi_n]$ can be written as a sequence of singleton and multiple segments $[\pi_i \ldots \pi_j]$ consisting of all elements that lie between π_i and π_j inclusive. We use characters from $\{i,\, j,\, k,\, p\}$ to denote singletons and characters from $\{\alpha,\, \beta,\, \gamma\}$ to denote multiple segments. For instance, a permutation $\pi = [k\, \pi_2\, \pi_3\, \pi_4\, i\, \pi_6\, \pi_7\, \pi_8\, j]$ can be written as $\pi = [k\, \alpha\, i\, \beta\, j]$, where $\alpha = [\pi_2\, \pi_3\, \pi_4]$, $\beta = [\pi_6\, \pi_7\, \pi_8]$. We also write $\pi = [k\, i\, j]^*$ when it is not important to specify multiple segments in a permutation $\pi = [k\, \alpha\, i\, \beta\, j]$. Without loss of generality, we further let $\pi(i) = i$ in a permutation $\pi = [k\, i\, j]^*$, then $\pi\, t_i = [k\, i\, j]^*\, t_i = [i\, k\, j]^* = \tau$, and $\tau\, t_i = [i\, k\, j]^*\, t_i = [k\, i\, j]^* = \pi$, correspondingly. We use $|\pi|$ to denote the number of elements in a permutation π.

The *distance* $d = d(\pi, \tau)$ between two vertices π, τ in S_n is defined as the least number of transpositions transforming π into τ, i.e. $\pi\, t_{i_1}\, t_{i_2}\, \ldots\, t_{i_d} = \tau$. Let $\overline{\pi} = \pi\, t_n$ and $\overline{\tau} = \tau\, t_n$.

Lemma 1. *Let permutations $\pi \neq \tau$ belong to the same $(n-1)$–copy of S_n, $n \geqslant 3$, and $d(\pi, \tau) \leqslant 2$, then $\overline{\pi}$, $\overline{\tau}$ belong to distinct $(n-1)$–copies of the graph.*

Proof. Let $\pi,\, \tau \in S_{n-1}(j)$, $1 \leqslant j \leqslant n$. If $d(\pi, \tau) = 1$ then for $\pi = [k\, i\, j]^*$ we have $\tau = [i\, k\, j]^*$, where $i \neq j \neq k$. Since the first elements of π and τ are different then $\overline{\pi}$ and $\overline{\tau}$ belong to the distinct copies $S_{n-1}(k)$, $S_{n-1}(i)$. If $d(\pi, \tau) = 2$ then there is a permutation ω in $S_{n-1}(j)$ adjacent to π and τ. Permutations π and τ are obtained from ω by multiplication on different (not equal to t_n) transpositions on the right. Thus, the first elements of π and τ should be different hence $\overline{\pi}$ and $\overline{\tau}$ should be different, i.e. they belong to the distinct $(n-1)$–copies of S_n. □

An independent set D of vertices in a graph is called an *efficient dominating set* if each vertex not in D is adjacent to exactly one vertex in D. Existence of efficient dominating sets in Cayley graphs on the symmetric groups was considered in [2, 4, 7]. The sets $D_k = \{[k\, \pi_2\, \ldots\, \pi_n] \,:\, \pi_j \in \{1, \ldots, n\} \backslash \{k\}, 2 \leqslant j \leqslant n\}$, $|D_k| = (n-1)!$, $1 \leqslant k \leqslant n$, are efficient dominating sets in the Star graph S_n, $n \geqslant 3$. Distances between vertices from the same efficient dominating set in this graph were investigated in [4]. In particular, all paths of length three between two vertices of the same efficient dominating set in $S_n, n \geqslant 3$, are presented in the following lemma.

Lemma 2. *Two vertices $\pi,\, \tau \in D_k$, $k \in \{1, \ldots, n\}$, are at distance three from each other in the graph $S_n, n \geqslant 3$, if and only if $\tau = \pi\, t_i\, t_j\, t_i$, when $2 \leqslant i \neq j \leqslant n$.*

Proof. Let $\pi = [k\, i\, j]^* \in D_k$. There are only two ways to reach a vertex $\tau \in D_k$:

$$[k\, i\, j]^* \xrightarrow{t_i} [i\, k\, j]^* \xrightarrow{t_j} [j\, k\, i]^* \xrightarrow{t_i} [k\, j\, i]^*, \tag{7}$$

$$[k\, i\, j]^* \xrightarrow{t_j} [j\, i\, k]^* \xrightarrow{t_i} [i\, j\, k]^* \xrightarrow{t_j} [k\, j\, i]^*. \tag{8}$$

Then by (7) and (8) we get $\tau = \pi\, t_i\, t_j\, t_i$, where $2 \leqslant i \neq j \leqslant n$. Converse statement of lemma is obvious. □

3. Proof of Theorem 1: 6–cycles

If $n = 3$ then $S_3 \cong C_6$ and the 6–cycle is described as:

$$[1\,2\,3] \xrightarrow{t_3} [3\,2\,1] \xrightarrow{t_2} [2\,3\,1] \xrightarrow{t_3} [1\,3\,2] \xrightarrow{t_2} [3\,1\,2] \xrightarrow{t_3} [2\,1\,3]^* \xrightarrow{t_2} [1\,2\,3],$$

that corresponds to the canonical form $C_6 = (t_3\,t_2)^3$.

If $n > 3$ then for a permutation $\pi = [j\,\alpha\,i\,\beta\,k] = [j\,i\,k]^*$ from a $(k-1)$–copy, where $4 \leqslant k \leqslant n$, let us prove that 6–cycles are formed as follows:

$$[j\,i\,k]^* \xrightarrow{t_i} [i\,j\,k]^* \xrightarrow{t_k} [k\,j\,i]^* \xrightarrow{t_i} [j\,k\,i]^* \xrightarrow{t_k} [i\,k\,j]^* \xrightarrow{t_i} [k\,i\,j]^* \xrightarrow{t_k} [j\,i\,k]^*,$$

that corresponds to the canonical form $C_6 = (t_k\,t_i)^3, 2 \leqslant i < k \leqslant n$, and there is no way to obtain a 6–cycle in the graph.

First, let us prove that a 6–cycle doesn't appear on vertices of two distinct $(k-1)$–copies, where $4 \leqslant k \leqslant n$. Indeed, let $\pi,\, \tau \in S_{k-1}(k)$, $\pi \neq \tau$, then if $\overline{\pi},\, \overline{\tau} \in S_{k-1}(j)$, $j \neq k$, where $\overline{\pi} = \pi\, t_k$, $\overline{\tau} = \tau\, t_k$, hence by Lemma 1 we have $d(\pi,\, \tau) \neq 1$ and $d(\pi,\, \tau) \neq 2$, and so $d(\pi,\, \tau) \geqslant 3$. Suppose that there is a 6–cycle containing vertices $\pi, \tau, \overline{\pi}, \overline{\tau}$. If $d(\pi, \tau) = 3$ then vertices $\overline{\pi},\, \overline{\tau}$ are adjacent in $S_{k-1}(j)$ and by Lemma 1 vertices $\pi = \overline{\pi}\, t_k, \tau = \overline{\tau}\, t_k$ belong to the distinct $(k-1)$–copies but this is not true since $\pi,\, \tau \in S_{k-1}(k)$. If $d(\pi, \tau) = 4$ then $\overline{\pi} = \overline{\tau}$ but this is not possible since $\pi \neq \tau$. Thus, a 6–cycle doesn't appear on vertices of two distinct $(k-1)$–copies, where $4 \leqslant k \leqslant n$.

Now, suppose a 6–cycle appears on vertices of three distinct $(k-1)$–copies. Let $\pi,\, \tau \in S_{k-1}(k)$, $\pi \neq \tau$, and $d(\pi, \tau) \leqslant 2$, then by Lemma 1 vertices $\overline{\pi}, \overline{\tau}$ belong to distinct $(k-1)$–copies. Let us consider two cases. If $d(\pi, \tau) = 2$, then vertices $\pi, \tau, \overline{\pi}, \overline{\tau}$ belong to a 6–cycle if and only if $d(\overline{\pi}, \overline{\tau}) = 2$. However, this is not possible, since by Lemma 1 vertices $\pi = \overline{\pi}\, t_k$, $\tau = \overline{\tau}\, t_k$ should belong to distinct $(k-1)$–copies. If $d(\pi, \tau) = 1$ then for $\pi = [j\,i\,k]^*$ and $\tau = [i\,j\,k]^*$, $i \neq j \neq k$, $\pi, \tau \in S_{k-1}(k)$, we have $\overline{\pi} = [k\,i\,j]^* \in S_{k-1}(j)$, $\overline{\tau} = [k\,j\,i]^* \in S_{k-1}(i)$. The shortest path starting at $\overline{\pi}$ and ending in $S_{k-1}(i)$ should contain vertices $\omega = [i\,k\,j]^*$, $\overline{\omega} = [j\,k\,i]^*$, i.e. $d(\overline{\pi}, \overline{\omega}) = 2$. So, $d(\overline{\tau}, \overline{\omega}) = 1$ and a 6–cycle is presented by $\pi \to \tau \to \overline{\tau} \to \overline{\omega} \to \omega \to \overline{\pi} \to \pi$, that is:

$$[j\,i\,k]^* \xrightarrow{t_i} [i\,j\,k]^* \xrightarrow{t_k} [k\,j\,i]^* \xrightarrow{t_i} [j\,k\,i]^* \xrightarrow{t_k} [i\,k\,j]^* \xrightarrow{t_i} [k\,i\,j]^* \xrightarrow{t_k} [i\,j\,k]^*,$$

which corresponds to the canonical form (2) in Theorem 1.

It is obvious that a 6–cycle doesn't appear on vertices of four and more distinct $(k-1)$–copies, $4 \leqslant k \leqslant n$, since there should be at least four external edges and at least one edge in each of $(k-1)$–copies. Thus, there is the only canonical form, namely, form (2) describing 6–cycles in S_n, $n \geqslant 3$. This form gives the number $\binom{n-1}{2}$ of distinct 6–cycles containing a given vertex. The total number of distinct 6–cycles in $S_n, n \geqslant 4$, is given by $\frac{(n-2)(n-1)n!}{12}$ since there are $n!$ vertices in the graph each of which belongs to $\binom{n-1}{2}$ distinct 6–cycles S_n, $n \geqslant 4$. □

4. Proof of Theorem 2: 8–cycles

To find all 8–cycles passing through a vertex in the graph $S_n, n \geqslant 4$, we use its hierarchical structure by considering recursively 8–cycles within each copy $S_k, 4 \leqslant k \leqslant n$, consisting of vertices from copies of S_{k-1}. It is assumed that any copy of S_{k-1} has at least two vertices of a cycle, since each vertex has a unique external edge. We obtain canonical forms of 8–cycles and count their numbers.

Case 1: an 8–cycle within S_k has vertices from two copies of S_{k-1}

Suppose that an 8–cycle is formed on vertices from copies $S_{k-1}(k)$ and $S_{k-1}(p)$, $1 \leqslant k \neq p \leqslant n$. By Lemma 1 if two vertices π and τ, belonging to the same $(k-1)$–copy, are at the distance at most two, then their external neighbours $\overline{\pi}$ and $\overline{\tau}$ should belong to distinct $(k-1)$–copies. Hence, an 8–cycle cannot occur in situations when its two (three) vertices belong to one copy and six (five) vertices belong to another one. Therefore, such an 8–cycle must have four vertices in each of two copies.
Let $\pi, \tau \in S_{k-1}(k)$, $|\pi| = |\tau| = k$, $\overline{\pi}, \overline{\tau} \in S_{k-1}(p)$, where $\overline{\pi} = \pi\, t_k$, $\overline{\tau} = \tau\, t_k$. An 8–cycle containing vertices $\pi, \tau, \overline{\pi}, \overline{\tau}$ might appear if and only if $d(\pi, \tau) = d(\overline{\pi}, \overline{\tau}) = 3$. Moreover, by Lemma 1 vertices π, τ and $\overline{\pi}, \overline{\tau}$ must belong to two different efficient dominating sets. So, if we put $\pi = [p\, i\, j\, k]^*$, $p \neq i \neq j \neq k$, then by Lemma 2 there are following paths of length three in copies $S_{k-1}(k)$ and $S_{k-1}(p)$ (see Fig. 1):

$$\pi = \tau\, t_i\, t_j\, t_i, \;\; \pi = \tau\, t_j\, t_i\, t_j, \;\; i \neq j,$$

$$\overline{\pi} = \overline{\tau}\, t_i\, t_j\, t_i, \;\; \overline{\pi} = \overline{\tau}\, t_j\, t_i\, t_j, \;\; i \neq j.$$

Combining these paths we immediately obtain 8–cycles with canonical forms:

$$C_8^1 = t_k\, t_i\, t_j\, t_i\, t_k\, t_i\, t_j\, t_i, \;\; 2 \leqslant i \neq j \leqslant k-1,$$

$$C_8^2 = t_k\, t_j\, t_i\, t_j\, t_k\, t_i\, t_j\, t_i, \;\; 2 \leqslant i \neq j \leqslant k-1,$$

corresponding to forms (3) and (4) in Theorem 2 for $4 \leqslant k \leqslant n$. Thus, all 8–cycles occurring in the case of two copies are found.

Case 2: an 8–cycle within S_k has vertices from three copies of S_{k-1}

Suppose an 8–cycle is formed on vertices from copies $S_{k-1}(i)$, $S_{k-1}(j)$, $S_{k-1}(k)$, where $1 \leqslant i \neq j \neq k \leqslant n$. There are following possible situations in this case.

<u>$(2+2+4)$–**situation**</u>. Suppose that vertices π^{k_1}, π^{k_2} belong to a copy $S_{k-1}(k)$, vertices π^{j_1}, π^{j_2} belong to a copy $S_{k-1}(j)$, and vertices π^{i_s}, $s = 1, \ldots, 4$, belong to

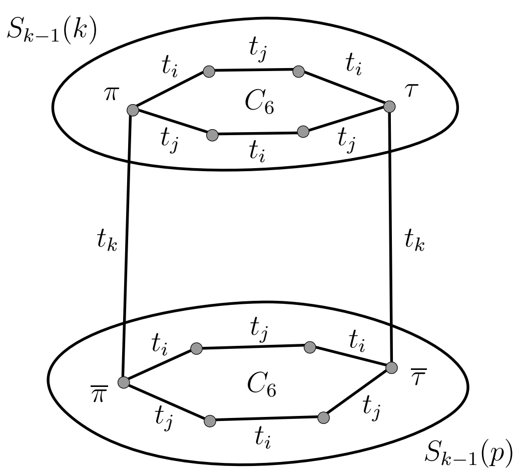

FIG. 1. 8-cycle on two copies of S_{k-1}

a copy $S_{k-1}(i)$, where $1 \leqslant i \neq j \neq k \leqslant n$. Let $\pi^{k_1} = [i\,\alpha\,j\,\beta\,k] = [i\,j\,k]^*$. Since a vertex $\overline{\pi^{k_2}}$ belongs to a copy $S_{k-1}(j)$, then a vertex π^{k_2} is uniquely identified as

$$\pi^{k_2} = \pi^{k_1}\,t_j = [j\,i\,k]^*,$$

and a vertex π^{j_1} coincides with a vertex $\overline{\pi^{k_2}}$, so we have:

$$\pi^{j_1} = [j\,i\,k]^*\,t_k = [k\,i\,j]^*.$$

A vertex $\overline{\pi^{j_2}}$ must belong to a copy $S_{k-1}(i)$, hence a vertex π^{j_2} is also uniquely identified as:

$$\pi^{j_2} = \pi^{j_1}\,t_j = [i\,k\,j]^*.$$

Since

$$\pi^{i_1} = \overline{\pi^{k_1}} = [k\,j\,i]^*, \quad \pi^{i_4} = \overline{\pi^{j_2}} = [j\,k\,i]^*,$$

then $\pi^{i_1} = \pi^{i_4}\,t_j$. Moreover, since there are no 4–cycles in the graph S_n, $n \geqslant 3$, then there is no path of length three between vertices π^{i_1} и π^{i_4}. Hence, an 8–cycle cannot occur in this situation.

$(2+3+3)$–**situation**. Suppose that vertices π^{k_1}, π^{k_2}, π^{k_3} belong to a copy $S_{k-1}(k)$, where π^{k_1} is adjacent to a vertex belonging to a copy $S_{k-1}(i)$, and π^{k_3} is adjacent to a vertex belonging to a copy $S_{k-1}(j)$, where $1 \leqslant i \neq j \leqslant k-1$. Let us put $\pi^{k_2} = [p\,\alpha\,i\,\beta\,j\,\gamma\,k] = [p\,i\,j\,k]^*$, $i < j$. Since a vertex π^{k_1} is adjacent to a vertex from a copy $S_{k-1}(i)$, and π^{k_3} is adjacent to a vertex from a copy $S_{k-1}(j)$, then

$$\pi^{k_1} = \pi^{k_2}\,t_i = [i\,p\,j\,k]^*, \quad \pi^{k_3} = \pi^{k_2}\,t_j = [j\,i\,p\,k]^*,$$

and vertices π^{i_1} and π^{j_1} are presented as:

$$\pi^{i_1} = \pi^{k_1}\,t_k = [k\,p\,j\,i]^*, \quad \pi^{j_1} = \pi^{k_3}\,t_k = [k\,i\,p\,j]^*.$$

There are two cases.

1) Suppose two vertices π^{i_1}, π^{i_2} belong to a copy $S_{k-1}(i)$, and three vertices π^{j_1}, π^{j_2}, π^{j_3} belong to a copy $S_{k-1}(j)$. Since a vertex π^{i_2} is adjacent to a vertex from a copy $S_{k-1}(j)$, then a vertex π^{i_2} is uniquely identified as $\pi^{i_2} = \pi^{i_1}\,t_j = [j\,p\,k\,i]^*$, and hence we have $\pi^{j_3} = \pi^{i_2}\,t_k = [i\,p\,k\,j]^*$. Since there are no 4–cycles in the graph, then there exists a unique path of length two between vertices π^{j_1} and π^{j_3} within a

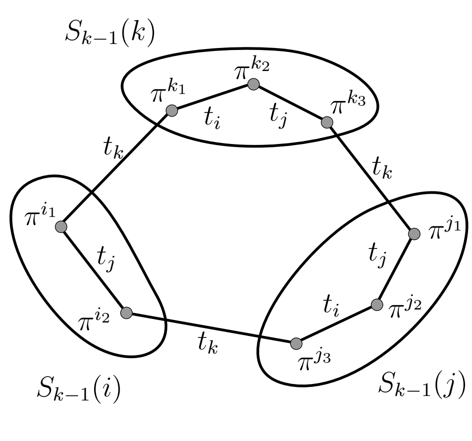

FIG. 2. 8-cycle on three copies of S_{k-1}

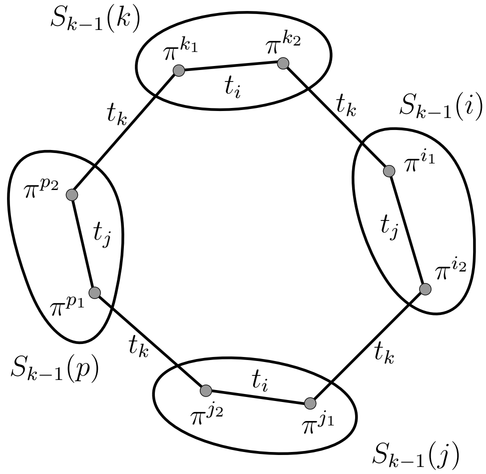

FIG. 3. 8-cycle on three copies of S_{k-1}

copy $S_{k-1}(j)$ presented by $\pi^{j_3} = \pi^{j_1}\, t_j\, t_i$ (see Fig. 2) Hence, in this case an 8–cycle is given by the canonical form

$$C_8^* = t_k\, t_j\, t_k\, t_i\, t_j\, t_k\, t_j\, t_i,\ 2 \leqslant i < j \leqslant k-1,$$

corresponding to the form (5) in Theorem 2, where $4 \leqslant k \leqslant n$.

2) Now suppose, that three vertices π^{i_1}, π^{i_2}, π^{i_3} belong to a copy $S_{k-1}(i)$, and two vertices π^{j_1}, π^{j_2} belong to a copy $S_{k-1}(j)$. Since a vertex π^{j_2} is adjacent to a vertex belonging to a copy $S_{k-1}(i)$, then a vertex π^{j_2} is uniquely identified and presented as $\pi^{j_2} = \pi^{j_1}\, t_i = [i\, k\, p\, j]^*$. Hence, we have $\pi^{i_3} = \pi^{j_2}\, t_k = [j\, k\, p\, i]^*$.

Since there are no 4–cycles in the graph, then there exists a unique path of length two between vertices π^{i_1} и π^{i_3} within a copy $S_{k-1}(i)$ presented as $\pi^{i_3} = \pi^{i_1}\, t_i\, t_j$, and in this case an 8–cycle is given by the canonical form

$$C_8^{**} = t_k\, t_i\, t_k\, t_j\, t_i\, t_k\, t_i\, t_j,\ 2 \leqslant i < j \leqslant k-1,$$

where $4 \leqslant k \leqslant n$, which also corresponds to the canonical form (5) in Theorem 2 after mutual replacement of i and j. Thus, all 8–cycles occurring in the case of three copies are found.

Case 3: an 8–cycle within S_k has vertices from four copies of S_{k-1}

Suppose an 8–cycle is formed on vertices from copies $S_{k-1}(i)$, $S_{k-1}(j)$, $S_{k-1}(p)$, $S_{k-1}(k)$, where $1 \leqslant i \neq j \neq p \leqslant k-1$. There is the only situation in this case when each copy has the only edge belonging to an 8–cycle. Moreover, each of these edges is uniquely defined.

Let $\pi^{k_1} = [p\,\alpha\, i\,\beta\, j\,\gamma\, k] = [p\, i\, j\, k]^*$, where $2 \leqslant i < j \leqslant k-1$, then in sequence we define other vertices of an 8–cycle as follows (see Fig. 3):

$$\begin{gathered}\pi^{k_2} = \pi^{k_1}\, t_i = [i\, p\, j\, k]^*, \quad \pi^{i_1} = \pi^{k_2}\, t_k = [k\, p\, j\, i]^*,\\ \pi^{i_2} = \pi^{i_1}\, t_j = [j\, p\, k\, i]^*, \quad \pi^{j_1} = \pi^{i_2}\, t_k = [i\, p\, k\, j]^*,\\ \pi^{j_2} = \pi^{j_1}\, t_i = [p\, i\, k\, j]^*, \quad \pi^{p_1} = \pi^{j_2}\, t_k = [j\, i\, k\, p]^*,\\ \pi^{p_2} = \pi^{p_1}\, t_j = [k\, i\, j\, p]^*,\end{gathered}$$

and finally we get

$$\pi^{p_2}\, t_k = [p\, i\, j\, k]^* = \pi^{k_1}.$$

So, in this case an 8–cycle is presented by the canonical form:

$$C_8^4 = t_k\, t_j\, t_k\, t_i\, t_k\, t_j\, t_k\, t_i,\ 2 \leqslant i < j \leqslant k-1,$$

corresponding to (6) in Theorem 2, where $4 \leqslant k \leqslant n$.

It is evident that an 8–cycle doesn't appear on vertices of five and more distinct $(k-1)$–copies, $4 \leqslant k \leqslant n$, in the graph S_n, $n \geqslant 4$. Therefore, all canonical forms of 8–cycles in $S_n, n \geqslant 4$, are obtained.

Now we count the total number $N = \sum_{m=1}^{4} N_{C_8^m}$ of distinct 8–cycles passing through a given vertex, where $N_{C_8^m}$ corresponds to the number of distinct 8–cycles described by the canonical form $C_8^m, 1 \leqslant m \leqslant 4$. Let us note that any canonical form of an l–cycle describes l cycles (not necessarily distinct) for a given vertex. For the forms C_8^1, C_8^2 and C_8^3, the following inequalities hold: $1 \leqslant (i-1) \neq (j-1) \leqslant k-2,\ 4 \leqslant k \leqslant n$. This means, for counting forms we have to choose two unordered distinct numbers between 1 and $k-2$, that is $(k-2)(k-3)$, and hence in total we have:

$$A = \sum_{k=4}^{n} (k-2)(k-3) = \sum_{k=1}^{n-3} k(k+1) = \frac{(n-3)(n-2)(n-1)}{3}.$$

By taking into account identical forms, each of the canonical forms C_8^1, C_8^2 gives two distinct cycles, and the form C_8^3 gives four distinct cycles, i.e.:

$$N_{C_8^1} = 2A, \quad N_{C_8^2} = 2A, \quad N_{C_8^3} = 4A.$$

For the form C_8^4 we have $1 \leqslant i-1 \leqslant j-2,\ i+1 \leqslant j \leqslant k-1,\ 4 \leqslant k \leqslant n$, and the first inequality means that for any fixed j there are $j-2$ forms, where $3 \leqslant j \leqslant k-1$. So we have:

$$B = \sum_{k=4}^{n} \sum_{j=3}^{k-1} (j-2) = \sum_{k=4}^{n} \sum_{j=1}^{k-3} j = \sum_{k=4}^{n} \frac{(k-3)(k-2)}{2} = \frac{1}{2} \sum_{k=1}^{n-3} k(k+1) = \frac{1}{2} A.$$

By taking into account identical forms, the canonical form C_8^4 gives two distinct cycles, and we have

$$N_{C_8^4} = 2B = A.$$

Thus, the total number of distinct 8–cycles passing through a given vertex is:

$$N = \sum_{i=1}^{4} N_{C_8^i} = 9A = 3(n-3)(n-2)(n-1),$$

and the total number of distinct 8–cycles in $S_n, n \geqslant 4$, is given by $\frac{n!\, N}{8}$, since there are $n!$ vertices in the graph each of which belongs to N distinct 8–cycles. This proves Theorem 2. $\square$

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