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UNRECOGNIZABILITY BY SPECTRUM OF FINITE SIMPLE ORTHOGONAL GROUPS OF DIMENSION NINE

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Abstract. The spectrum of a finite group is the set of its elements orders. A group G is said to be unrecognizable by spectrum if there are infinitely many pairwise non-isomorphic finite groups having the same spectrum as G. We prove that the simple orthogonal group $O_9(q)$ has the same spectrum as $V \rtimes O_8^-(q)$ where V is the natural 8-dimensional module of the simple orthogonal group $O_8^-(q)$, and in particular $O_9(q)$ is unrecognizable by spectrum. Note that for $q = 2$, the result was proved earlier by Mazurov and Moghaddamfar.

Keywords: spectrum, element order, orthogonal group, finite simple group.

1. Introduction

The *spectrum* $\omega(G)$ of a finite group G is the set of its element orders. We call groups G and H *isospectral* if $\omega(G) = \omega(H)$. Let $h(G)$ be the number of pairwise non-isomorphic finite groups isospectral to G. A group G is said to be *unrecognizable* by spectrum if $h(G)$ is infinite. A finite group with a nontrivial normal soluble subgroup is always unrecognizable [1, 2], so of prime interest is the problem of finiteness of $h(G)$ for a nonabelian simple group G.

Our notation of finite nonabelian simple groups and finite classical groups follows [3]. In particular, $O_{2n+1}(q)$ and $O_{2n}^{\varepsilon}(q)$ denote the groups $\Omega_{2n+1}(q)$ and $P\Omega_{2n}^{\varepsilon}(q)$, which are usually simple, while the full orthogonal groups are denoted by $GO_{2n+1}(q)$ and $GO_{2n}^{\varepsilon}(q)$. Recall that the order of the center of $\Omega_{2n}^{\varepsilon}(q)$ is $(4, q^n - \varepsilon)/(2, q - \varepsilon)$, so $O_{4n}^-(q) = \Omega_{4n}(q)$.

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There is a conjecture attributed to Mazurov that every finite group isospectral to a finite simple group L of Lie type of sufficiently large Lie rank is an almost simple group with socle isomorphic to L, and in particular $h(L)$ is finite for a such L. At present the conjecture is proved with the following restrictions on L: $L \neq {}^3D_4(2)$ if L is exceptional, the Lie rank of L is least 44 if L is linear or unitary, and the Lie rank of L is at least 31 if L is symplectic or orthogonal; furthermore, it was conjectured that the bounds 44 and 31 can be replaced by 4 and 5 respectively, which are exact bounds in some strict sense (see [4] for details). On the other hand, the only known unrecognizable symplectic or orthogonal group of Lie rank 4 is $S_8(2) \simeq O_9(2)$: Mazurov and Moghaddamfar [5] showed that $S_8(2)$ is isospectral to an extension of a nontrivial 2-group by $O_8^-(2)$. In the present paper we generalize this result by the following.

Theorem 1. *Let V be the natural 8-dimensional module of $O_8^-(q)$. Then $\omega(O_9(q)) = = \omega(V \rtimes O_8^-(q))$. In particular, $O_9(q)$ is unrecognizable by spectrum.*

2. Preliminaries

If a and b are positive integers, then (a, b) and $[a, b]$ denote respectively the greatest common divisor and least common multiple of a and b. If p is a prime, then $\omega_{p'}(G)$ is the set of element orders of G coprime to p. By writing $q-\varepsilon$ with $\varepsilon \in \{+,-\}$ we mean the number $q-\varepsilon 1$.

By $\operatorname{diag}(d_1, \dots, d_k)$ we denote the diagonal matrix with $d_1, \dots, d_k$ on the diagonal (each d_i can be a number or a diagonal matrix itself). If g is a unipotent element of $GL(V)$, then we write $\oplus_i J_i^{k_i}$ for the Jordan form of g on V, where J_i denotes a unipotent Jordan block of length i and the sum is over values of i such that $k_i > 0$.

Lemma 1. *We have the following isomorphisms.*

(i) $Sp_2(q) \simeq SL_2(q)$.

(ii) $GO_2^\varepsilon(q) \simeq D_{2(q-\varepsilon)}$, $SO_2^\varepsilon \simeq (q-\varepsilon).(2, q)$, *and* $\Omega_2^\varepsilon(q) \simeq (q-\varepsilon)/(2, q-1)$.

(iii) $\Omega_4^+(q) \simeq SL_2(q) \circ SL_2(q)$ *and the natural module for $\Omega_4^+(q)$ is isomorphic to the tensor product of two copies of the natural modules of $SL_2(q)$, one for each direct factor in the preimage group $SL_2(q) \times SL_2(q)$.*

(iv) $\Omega_4^-(q) \simeq L_2(q^2)$ *and the natural module for $\Omega_4^-(q)$ is isomorphic to the tensor product of the natural module M for $SL_2(q^2)$ and the image of M under the automorphism $\sigma : x \mapsto x^q$ of $GF(q^2)$.*

Proof. Parts (i) and (ii) are well known; see for instance [6, Proposition 2.9.1]. Parts (iii) and (iv) are [7, Lemma 1.12.3]. □

Lemma 2. *Let q be a power of a prime p. Then $\omega(\Omega_9(q))$ is the set of all divisors of the following numbers:*

(i) $(q^4 \pm 1)/(2, q-1)$, $(q^2 \pm q + 1)(q^2 - 1)/(2, q-1)$, $p(q^3 \pm 1)/(2, q-1)$, $p(q^2 + 1)(q \pm 1)/(2, q-1)$, $p(q^2 - 1)$;

(ii) $4(q^2 \pm 1)$, $8(q \pm 1)$ *if* $p = 2$;

(iii) $9(q^2 \pm 1)/2$ *if* $p = 3$;

(iv) $25(q \pm 1)/2$ *if* $p = 5$;

(v) 49 *if* $p = 7$.

Proof. This follows from [8, Corollary 3] if q is even and [8, Corollary 6] if q is odd. □

Lemma 3. *Let q be a power of a prime p. Then $\omega(\Omega_8^-(q))$ is the set of all divisors of the following numbers:*

(i) $(q^4 \pm 1)/(2, q-1)$, $(q^2 \pm q+1)(q^2-1)/(2, q-1)$, $p(q^2+1)(q \pm 1)/(2, q-1)$, $p(q^2-1)$;
(ii) $4(q^2-1)$, 8 *if* $p = 2$;
(iii) $9(q \pm 1)$ *if* $p = 3$;
(iv) 25 *if* $p = 5$.

Proof. In even characteristic, this follows from [8, Corollary 4] (there is an evident misprint in Part (6) of this corollary: l must be odd if $\varepsilon = +$ and even if $\varepsilon = -$). In odd characteristic, we use the description of reductive subgroups of $SO_8^-(q)$ given in [8, Proposition 3.3] and then follow the line of the proof of [8, Corollary 8] (we do not apply [8, Corollary 8] itself since it contains some non-obvious misprints). □

The following is an immediate corollary of Lemmas 2 and 3.

Lemma 4. *Let q be a power of a prime p. Then $\omega(\Omega_9(q)) = \omega(\Omega_8^-(q)) \cup M(q)$, where $M(q)$ is the set of all divisors of the following numbers:*

(i) $p(q^3 \pm 1)/(2, q-1)$;
(ii) $4(q^2+1), 8(q \pm 1)$ *if* $p = 2$;
(iii) $9(q^2 \pm 1)/2$ *if* $p = 3$;
(iv) $25(q \pm 1)/2$ *if* $p = 5$;
(v) 49 *if* $p = 7$.

The following result is rather well-known, but we could not find a convenient reference.

Lemma 5. *Let V be a vector space over a finite field of characteristic $r > 0$, $G \leq GL(V)$ and $H = V \rtimes G$ be the natural semidirect product of V by G. Let $g \in G$, $g = us$ be the Jordan decomposition of g with u unipotent and s semisimple, and $W = C_V(s)$. Then the coset Vg of H contains an element of order $r|g|$ if and only if the restriction of u to W has a Jordan block of size $|u|$.*

Proof. Let $|g| = n$, $|u| = k$ and $|s| = m$. Since $vg_1 \cdot vg_2 = (v + vg_1)g_1g_2$, it follows that $(vg)^n = v(1 + \cdots + g^{n-1})$ for $v \in V$. Since $\langle g^k \rangle = \langle s \rangle$, it follows that

$$1 + \cdots + g^{n-1} = \frac{g^n - 1}{g^k - 1} \cdot \frac{g^k - 1}{g - 1} = \frac{s^m - 1}{s - 1} \cdot \frac{g^k - 1}{g - 1}.$$

The element s is semisimple, so $V = W \times V(s-1)$, and we can choose $w \in W$ and $v_1 \in V$ such that $v = w + v_1(s-1)$. Then

$$v(1 + \cdots + g^{n-1}) = (w + v_1(s-1))\frac{s^m - 1}{s - 1} \cdot \frac{g^k - 1}{g - 1} = mw\frac{u^k - 1}{u - 1} = mw(u-1)^{k-1}.$$

Thus Vg contains an element of order rn if and only if there is an element $w \in W$ such that $w(u-1)^{k-1} \neq 0$, or equivalently the restriction of u to W has a Jordan block of size k. □

We conclude this section with two results concerning Jordan forms of elements of $\Omega_{2n}^{\varepsilon}(q)$ on its natural module. We will need the representation of $\Omega_{2n}^{\varepsilon}(q)$ as a subgroup in the centralizer of a suitable Frobenius morphism of a suitable simple linear algebraic group (cf. [9, Chapter 7]). Throughout the paper K denotes an algebraically closed field of characteristic p, where p is the defining characteristic of the orthogonal group under consideration, $\overline{V}$ denotes a $2n$-dimensional vector space over K, $SO_{2n}(K) = SO_{2n}(\overline{V})$ denotes the connected component of the full orthogonal group $GO_{2n}(\overline{V})$, and σ denotes a Frobenius morphism of $G = SO_{2n}(K)$ such that $G_\sigma = C_G(\sigma)$ is equal to $SO^{\varepsilon}_{2n}(q)$ if p is odd and $\Omega^{\varepsilon}_{2n}(q)$ if $p = 2$. Also we write K^* to denote the multiplicative group of K.

Lemma 6. *Let $n \geq 2$ and $\varepsilon \in \{+, -\}$. The group $\Omega^{\varepsilon}_{2n}(q)$ contains a unipotent element with the following Jordan form:*

(i) $J_{2n-1} \oplus J_1$ *if q is odd;*
(ii) $J_{2n-2} \oplus J_2$ *if q is even.*

Proof. If $n = 2$, this follows from Lemma 1. Let us assume that $n \geq 3$.

Let q be odd. It is well known that $\Omega^{\varepsilon}_{2n}(q)$ contains a unipotent element with Jordan form $\oplus_i J_i^{k_i}$ if and only if k_i is even for each even i and, in addition, $k_i \neq 0$ for some odd i when $\varepsilon = -$ (see [10, pp. 36–39] or [9, Corollary 3.6(ii) and Theorem 7.1(i)]). This yields (i).

Let q be even. We use the results of [9, 7.2]. Recall that $\Omega^{\varepsilon}_{2n}(q) = G_\sigma$ where $G = SO_{2n}(K)$ and σ is a suitable Frobenius morphism of G. Regular unipotent elements of G has Jordan form $J_{2n-2} \oplus J_2$ (see [9, p. 61]), and by [9, Theorem 7.3(i)], the group G_σ contains some regular unipotent element.

□

To describe Jordan forms of semisimple elements of orthogonal groups, we use the description of their maximal tori from [11]. By [11, Proposition 4.3] for q odd and [11, Proposition 3.1] together with [11, Section 5] for q even, it follows that every maximal torus of $\Omega^{-}_{2n}(q)$ is conjugate in $SO_{2n}(K)$ to a group of the form

$$\{\mathrm{diag}(t_1^{k_1}, \ldots, t_s^{k_s}, t_1^{-k_1}, \ldots, t_s^{-k_s}) \mid k_1 + k_2 + \cdots + k_s \equiv 0 \,(\mathrm{mod}(2, q-1))\},$$

where for $1 \leq i \leq s$

$$t_i = \mathrm{diag}(\lambda_i, \lambda_i^q, \ldots, \lambda_i^{q^{n_i-1}}), \lambda_i \in K^*, |\lambda_i| = q^{n_i} - \varepsilon_i \text{ for } n_i > 0 \text{ and } \varepsilon_i \in \{+, -\},$$

with $n_1 + n_2 + \cdots + n_s = n$ and the number of i with $\varepsilon_i = -$ being odd. And conversely, for every signed partition of n with odd number of 'minus' parts, there is a maximal torus of $\Omega^{-}_{2n}(q)$ conjugate to a subgroup of the described form. Let us, for brevity, write $(n_1, \ldots, n_k, \overline{n}_{k+1}, \ldots, \overline{n}_s)$ for a partition $(n_1, \ldots, n_s)$ with $\varepsilon_i = +$ for $1 \leq i \leq k$ and $\varepsilon_i = -$ for $k + 1 \leq i \leq n$ and write $D(\lambda_1, \ldots, \lambda_n)$ to denote $\mathrm{diag}(\lambda_1, \ldots, \lambda_n, \lambda_1^{-1}, \ldots, \lambda_n^{-1})$. Applying the above result and notation to $\Omega^{-}_8(q)$, we obtained the following.

Lemma 7. *The structure of the maximal tori of $\Omega^{-}_8(q)$ is given in Table* 1.

3. Proof of Theorem 1

Let q be a power of a prime p, $S = \Omega^{-}_8(q)$, V be the natural 8-dimensional module of S, and $L = \Omega_9(q)$. Denote the product $V \rtimes S$ by H. We begin with two auxiliary lemmas, before proving $\omega(H) = \omega(L)$.

Lemma 8. *Let $g \in S$, $(|g|, p) = 1$ and $d = \dim C_V(g)$.*

(i) *If $d \geq 5$ then $|g|$ divides one of the numbers $(q \pm 1)/(2, q-1)$.*
(ii) *If $d \geq 3$ then $|g|$ divides one of the numbers $(q^2 \pm 1)/(2, q-1)$.*
(iii) *If $d \geq 1$ then $|g|$ divides one of the numbers $q^2 - 1$, $(q^3 \pm 1)/(2, q-1)$, and $(q^2 + 1)(q \pm 1)/(2, q-1)$.*

TABLE 1. Structure of maximal tori of $\Omega_8^-(q)$

Partition	Elements of the torus	Conditions
$(\bar{4})$	$D(\lambda_1^{k_1}, \lambda_1^{qk_1}, \lambda_1^{q^2k_1}, \lambda_1^{q^3k_1})$	$\|\lambda_1\| = q^4+1$; if q is odd then k_1 is even
$(1, \bar{3})$	$D(\lambda_1^{k_1}, \lambda_2^{k_2}, \lambda_2^{qk_2}, \lambda_2^{q^2k_2})$	$\|\lambda_1\| = q-1$, $\|\lambda_2\| = q^3+1$; if q is odd then k_1+k_2 is even
$(2, \bar{2})$	$D(\lambda_1^{k_1}, \lambda_1^{qk_2}, \lambda_2^{k_2}, \lambda_2^{qk_2})$	$\|\lambda_1\| = q^2-1$, $\|\lambda_2\| = q^2+1$; if q is odd then k_1+k_2 is even
$(3, \bar{1})$	$D(\lambda_1^{k_1}, \lambda_1^{qk_1}, \lambda_1^{q^2k_1}, \lambda_2^{k_2})$	$\|\lambda_1\| = q^3-1$, $\|\lambda_2\| = q+1$; if q is odd then k_1+k_2 is even
$(1, 1, \bar{2})$	$D(\lambda_1^{k_1}, \lambda_2^{k_2}, \lambda_3^{k_3}, \lambda_3^{qk_3})$	$\|\lambda_1\| = \|\lambda_2\| = q-1$, $\|\lambda_3\| = q^2+1$; if q is odd then $k_1+k_2+k_3$ is even
$(2, 1, \bar{1})$	$D(\lambda_1^{k_1}, \lambda_1^{qk_1}, \lambda_2^{k_2}, \lambda_3^{k_3})$	$\|\lambda_1\| = q^2-1, \lambda_2 = q-1$, $\|\lambda_3\| = q+1$; if q is odd then $k_1+k_2+k_3$ is even
$(\bar{1}, \bar{1}, \bar{2})$	$D(\lambda_1^{k_1}, \lambda_2^{k_2}, \lambda_3^{k_3}, \lambda_3^{qk_3})$	$\|\lambda_1\| = \|\lambda_2\| = q+1$, $\|\lambda_3\| = q^2+1$; if q is odd then $k_1+k_2+k_3$ is even
$(1, \bar{1}, \bar{1}, \bar{1})$	$D(\lambda_1^{k_1}, \lambda_2^{k_2}, \lambda_3^{k_3}, \lambda_4^{k_4})$	$\|\lambda_1\| = \|\lambda_2\| = \|\lambda_3\| = q-1$ $\|\lambda_4\| = q+1$; if q is odd then $k_1+k_2+k_3+k_4$ is even
$(1, 1, 1, \bar{1})$	$D(\lambda_1^{k_1}, \lambda_2^{k_2}, \lambda_3^{k_3}, \lambda_4^{k_4})$	$\|\lambda_2\| = \|\lambda_3\| = \|\lambda_4\| = q+1$ $\|\lambda_1\| = q-1$; if q is odd then $k_1+k_2+k_3+k_4$ is even

Proof. We may assume that $g \neq 1$. By Lemma 7, the element g is conjugate to a diagonal matrix $D = D(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ with λ_i specified in the last column of Table 1.

Let $d \geq 5$. Then g has at least five eigenvalues equal to 1, whence at least three of λ_i are equal to 1. By Table 1, we may assume that $D = D(\lambda^k, 1, 1, 1)$, where $|\lambda| = q \pm 1$ and k is even if q is odd. Hence $|g|$ divides $(q \pm 1)/(q-1, 2)$.

Let $d \geq 3$. By the above, it is sufficient to handle the case $d = 4$, and so $D = D(\lambda_1^{k_1}, \lambda_2^{k_2}, 1, 1)$, where $|\lambda_1| = q^{n_1} - \varepsilon_1$, $|\lambda_2| = q^{n_2} - \varepsilon_2$ for some integers $n_1 \geq n_2$ and $\varepsilon_1, \varepsilon_2 \in \{+, -\}$. By Table 1, we have that $n_1 = 2$ or $n_1 = 1$. In the first case $\lambda_2 = \lambda_1^q$, $n_1 = n_2 = 2$, $\varepsilon_1 = \varepsilon_2$ and $k_1 = k_2$. Moreover, if q is odd then k_1 is even. Thus $|g|$ is a divisor of $(q^2 - \varepsilon_1)/(q-1, 2)$ as required. If $n_1 = n_2 = 1$, then $|g|$ divides $[q - \varepsilon_1, q - \varepsilon_2]$ and so divides $(q^2 - 1)/(q-1, 2)$.

TABLE 2. Unipotent classes for $\Omega_8^-(q)$, q even

Decomposition	R
$W(1)^4$	$GO_8^-(q)$
$W(3)+W(1)$	$GO_2^+(q)\times GO_2^-(q)$
$W(2)+W(1)^2$	$Sp_2(q)\times GO_4^-(q)$
$W(2)+V(2)^2$	$Sp_2(q)$
$W(1)^2+V(2)^2$	$Sp_4(q)$
$W(1)+V(4)+V(2)$	$Sp_2(q)$
$V(4)$	1
$V(6)+V(2)$	1

Let now $d\geq 1$. We may assume that $d=2$ and so $D=D(\lambda_1^{k_1},\lambda_2^{k_2},\lambda_3^{k_3},1)$, where $|\lambda_i|=q^{n_i}-\varepsilon_i$ for some integers n_i, $\varepsilon_i\in\{+,-\}$ for $1\leq i\leq 3$. For convenience, we suppose that $n_3\geq n_2\geq n_1$. Let $n_1=3$. Then $k_1=k_2=k_3$, $n_1=n_2=n_3$, $\lambda_2=\lambda_1^q$, $\lambda_3=\lambda_1^{q^2}$ and k_1 is even if q is odd. So $|g|$ divides $(q^3-\varepsilon_1)/(q-1,2)$. Let $n_1=2$. Then $n_2=n_1$, $\varepsilon_1=\varepsilon_2$, $n_3=1$, $k_1=k_2$ and $\lambda_2=\lambda_1^q$, therefore, $|g|$ divides $(q^2+1)(q-\varepsilon_3)/(q-1,2)$ if $\varepsilon_1=+$ and q^2-1 otherwise. If $n_1=n_2=n_3=1$ then $|g|$ divides $(q^2-1)/(q-1,2)$. □

Lemma 9. *Let q be even and $u\in S$ be unipotent.*

- (i) *If $|u|=8$ then u has Jordan form $J_6\oplus J_2$ on V.*
- (ii) *If $|u|=4$ and u has a Jordan block of size 4 on V, then $\omega_{2'}(C_S(u))$ consists of divisors of $q\pm 1$.*
- (iii) *If $|u|=2$ and $C_S(u)$ contains an element s of order m_1m_2 where $m_1\neq 1$ is a divisor of $q-1$ or $q+1$ and $m_2\neq 1$ is a divisor of q^2+1, then $C_V(s)=0$.*

Proof. Let $G=SO_{2n}(K)$, σ be a Frobenius morphism of G, so that $G_\sigma=\Omega_{2n}^-(q)$, and u be a unipotent element of G. According to [9, Lemma 6.2], under the action of u, the vector space $\overline{V}$ decomposes into an orthogonal sum of indecomposable modules the form $W(m)$ and $V(m)$, which can be defined as follows: $W(m)$ corresponds to a unipotent element which is a regular element in a Levi subgroup $SL_m(K)$ of G, $V(m)$ to a regular element in $Sp_m(K)$ which lies in an orthogonal group $GO_m(K)$. The element u induces $J_m\oplus J_m$ on $W(m)$ and J_m on $V(m)$ (see [9, Table 4.1]).

The unipotent classes of G are listed in [9, Table 8.5a] together with the corresponding number of G_σ-classes. We extract from this list those classes u^G for which $u^G\cap G_\sigma$ is not empty, and give them in the first column of Table 2. By [9, Theorem 7.3(iii)], the centralizer $C_{GO_8^-(q)}(u')$, where $u'\in u^G\cap G_\sigma$, is an extension of a 2-group by a group R, which is a direct product of symplectic and orthogonal groups. The second column of Table 2 gives R (it turns out that all these groups are isomorphic even when the number of G_σ-classes is larger than 1).

Now Parts (i) and (ii) follow from Table 2 and the fact that a semisimple element of $Sp_2(q)\simeq SL_2(q)$ has order dividing $q-1$ or $q+1$.

Suppose that u, s and m_1, m_2 are as in (iii). Since $m_1m_2 \notin \omega(Sp_4(q))$ (see, for example, [12, Lemma 7]), we see that u belongs to the class with decomposition $W(2) + W(1)^2$. Furthermore, using results of [13, Section 8], we conclude that u is an involution of type a_2 in notation of [13, p. 16]. By [13, (8.6)], we can choose a basis of V so that every element g in $C_{GO_8^-(q)}(u)$ be of the form

$$\begin{pmatrix} X(g) & & \\ A(g) & Y(g) & \\ B(g) & C(g) & X(g) \end{pmatrix}$$

where $X(g) \in Sp_2(q)$, $Y(g) \in GO_4^-(q)$, and $A(g)$, $B(g)$, $C(g)$ are some matrices of sizes 4×2, 2×2, 2×4 respectively. Since $m_2 \notin \omega(Sp_2(q))$ and $m_1m_2 \notin \omega(GO_4^-(q))$, it follows that $|X(s)| = m_1$ and $|Y(s)| = m_2$. But then the eigenvalues of $|X(s)|$ are $\lambda_1, \lambda_1^{-1}$ where $|\lambda_1| = m_1$ and the eigenvalues of $|Y(s)|$ are $\lambda_2, \lambda_2^q, \lambda_2^{-1}, \lambda_2^{-q}$ where $|\lambda_2| = m_2$ (see Lemma 1). Thus none of the eigenvalues of s is equal to 1, and so $C_V(s) = 0$. □

Now we are ready to prove Theorem 1. First we show that $\omega(H) \subseteq \omega(L)$. Assume that $h \in H$ and $|h| \notin \omega(L)$. By Lemma 5, it follows that $h = vus$ where $v \in V$, u and s are commuting unipotent and semisimple elements of S respectively, and the Jordan form of the restriction of u to $C_V(s)$ has a Jordan block of size $|u|$, in particular $\dim C_V(s) \geq |u|$.

Suppose that $|u| = 1$. Since $\dim C_V(s) \geq 1$, Lemma 8 implies that $|s|$ divides one of the numbers $(q^3 \pm 1)/(2, q-1)$, $(q^2+1)(q\pm 1)/(2, q-1)$, and q^2-1. Hence $|h|$ divides one of the numbers $p(q^3\pm 1)/(2, q-1)$, $p(q^2+1)(q\pm 1)/(2, q-1)$, and $p(q^2-1)$, and so $|h| \in \omega(L)$, contrary to our assumption.

Suppose that $|u| > 1$ and p is odd. Then $p \leq |u| \leq \dim C_V(s) \leq 8$ and so $|u| = p$ and $p \leq 7$. If $p = 7$, then Table 1 implies that $\dim C_V(s) = 8$, so $|s| = 1$ and $|h| = 49$. If $p = 5$, then by Lemma 8, we have that $|h|$ divides $25(q\pm 1)/2$. Similarly, if $p = 3$ then $|h|$ divides $9(q^2\pm 1)/2$. In any case $|h| \in \omega(L)$, a contradiction.

Let finally $|u| > 1$ and $p = 2$. If $|u| = 8$, then $|s| = 1$ by Lemma 3. Moreover, u has Jordan form $J_6 \oplus J_2$ by Lemma 9. Hence $|h| = 8$. If $|u| = 4$, then the restriction of u to $C_V(s)$ has a Jordan block of size 4, and by Lemma 9, it follows that $|s|$ divides $q\pm 1$. Therefore $|h|$ divides $8(q\pm 1) \in \omega(L)$.

Let $|u| = 2$. By Lemma 3, the order of s divides q^2-1 or $(q^2+1)(q\pm 1)$. If $|s|$ divides q^2-1 or q^2+1 then $|h| \in \omega(L)$. Hence $|s| = m_1m_2$, where $m_1 \neq 1$ divides $q\pm 1$ and $m_2 \neq 1$ divides q^2+1. Now Lemma 9 yields $C_V(s) = 0$, a contradiction.

To prove that $\omega(L) \subseteq \omega(H)$, it is sufficient to show that $M(q) \subseteq \omega(H)$, where $M(q)$ is defined in Lemma 4.

By Table 1, there exists an element s in S with Jordan form $D(1, \lambda^k, \lambda^{qk}, \lambda^{q^2k})$, where $|\lambda| = q^3+\varepsilon$ and $k = (q-1, 2)$. Note that $|s| = (q^3+\varepsilon)/(q-1, 2)$ and take $w \in C_V(s)\setminus\{0\}$. Then $|ws| = p(q^3+\varepsilon)/(q-1, 2)$. Thus $p(q^3\pm 1)/(q-1, 2) \in \omega(H)$, and the proof is complete for $p > 7$.

Let $p = 7$. By Lemma 6, there is a unipotent element with Jordan form $J_1 \oplus J_7$ in S. By Lemma 5, it follows that $49 \in \omega(H)$.

Let $p = 5$. Let W be a nondegenerate 6-dimensional subspace of V of sign ε and N be the stabilizer of W in S. Then N includes a subgroup $A \times B$ where $A \simeq \Omega_6^\varepsilon(q)$ acts in a natural way on W and trivially on $W^\perp$, while $B \simeq \Omega_2^{-\varepsilon}(q)$ acts in a natural way on $W^\perp$ and trivially on W. By Lemma 6, there is a unipotent element u with Jordan form $J_5 \oplus J_1$ in A. Taking an element s of order $(q+\varepsilon)/2$ in B and applying Lemma 5, we see that $25(q+\varepsilon)/2 \in \omega(H)$.

Let $p = 2$. We construct a subgroup of the form $\Omega_6^{\varepsilon}(q) \times \Omega_2^{-\varepsilon}(q)$ as above and take a unipotent element with Jordan form $J_4 \oplus J_2$ in $\Omega_6^{\varepsilon}(q)$. Then we again choose s as an element of order $(q+\varepsilon)/2$ in $\Omega_2^{-\varepsilon}(q)$ and conclude that $8(q+\varepsilon) \in \omega(H)$.

Similarly, if $p = 3$, then taking a subgroup of the form $\Omega_4^{+}(q) \times \Omega_4^{-}(q)$, a unipotent element with Jordan form $J_3 \oplus J_1$ in $\Omega_4^{+}(q)$ and elements of orders $(q^2 \pm 1)/2$ in $\Omega_4^{-}(q)$ results in $9(q^2 \pm 1)/2 \in \omega(H)$.

Thus $\omega(L) = \omega(H)$ and the proof is complete.

References

[1] W.J. Shi, *The characterization of the sporadic simple groups by their element orders*, Algebra Colloq., **1**:2 (1994), 159–166. MR1272290

[2] V.D. Mazurov, *Recognition of finite groups by a set of orders of their elements*, Algebra Logic, **37**:6 (1998), 371–379. MR1680412

[3] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, and R.A. Wilson, *Atlas of finite groups*, Clarendon Press, Oxford, 1985. MR0827219

[4] A.V. Vasil′ev, M.A. Grechkoseeva, *On the structure of finite groups isospectral to finite simple groups*, Preprint (2014), arXiv:1409.8086 [math.GR].

[5] V.D. Mazurov, A.R. Moghaddamfar, *The recognition of the simple group $S_8(2)$ by its spectrum*, Algebra Colloq., **13**:4 (2006), 643–646. MR2260110

[6] P. Kleidman, M. Liebeck, *The subgroup structure of the finite classical groups*, London Mathematical Society Lecture Note Series, vol. **129**, Cambridge University Press, Cambridge, 1990. MR1057341

[7] J. Bray, D. Holt, C. Roney-Dougal, *The maximal subgroups of the low-dimensional finite classical groups*, London Mathematical Society Lecture Note Series, vol. **407**, Cambridge University Press, Cambridge, 2013. MR3098485

[8] A.A. Buturlakin, *Spectra of finite symplectic and orthogonal groups*, Siberian Adv. Math., **21**:3 (2011), 176–210. MR2789956

[9] M.W. Liebeck, G.M. Seitz, *Unipotent and nilpotent classes in simple algebraic groups and Lie algebras*, Mathematical Surveys and Monographs, vol. **180**, American Mathematical Society, Providence, RI, 2012. MR2883501

[10] G.E. Wall, *On the conjugacy classes in the unitary, symplectic and orthogonal groups*, J. Austr. Math. Soc., **3** (1963), 1–62. MR0150210

[11] A.A. Buturlakin, M.A. Grechkoseeva, *The cyclic structure of maximal tori in finite classical groups*, Algebra Logic, **46**:2 (2007), 73–89. MR2356522

[12] V.D. Mazurov, M.C. Xu, H.P. Cao, *Recognition of the finite simple groups $L_3(2^m)$ and $U_3(2^m)$ by their element orders*, Algebra Logic, **39**:5 (2000), 324–334. MR1805756

[13] M. Aschbacher, G.M. Seitz, *Involutions in Chevalley groups over fields of even order*, Nagoya Math. J., **63** (1976), 1–91. MR0422401

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