S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 11, стр. 981–987 (2014) УДК 512.5
MSC 13A99

ON SPIN(7) MONGE–AMPÈRE TYPE EQUATIONS

DMITRY V. EGOROV

Abstract. In this paper we define the Monge–Ampère type equation on spaces equipped with the Spin(7)-structure. We express the equation in octonionic coordinates and prove its ellipticity.

Keywords: Monge–Ampère equation, Spin(7)-manifolds

1. Introduction

The classical Monge–Ampère type equation is a second-order nonlinear equation of the form $\det \mathrm{Hess}(\varphi) = F$, where Hess is the Hessian matrix of the real valued function φ. One can consider not only the real Hessian $\partial^2\varphi/\partial x^i \partial x^i$, but also its counterpart matrices whose elements lie in different normed division algebras over reals.

The real Monge–Ampère equation was first considered by Monge in 1784 and later by Ampère in 1820 in study of optimal transport problem. Since then, the Monge–Ampère type equations arise in a variety of geometrical problems. For example, in 1978 Yau proved existence of solutions for the complex Monge–Ampère type equation ($\mathrm{Hess} = \partial^2\varphi/\partial z^i \partial \bar{z}^j$) on compact Kähler manifolds and this led to the topological characterization of manifolds that possess the Kähler metric with vanishing Ricci curvature [1].

The quaternionic and two-dimensional octonionic cases are first considered by Alesker [3, 4, 5]. Harvey and Lawson generalized a form of the Monge–Ampère type operator from determinant of the Hessian to any symmetric polynomial of its eigenvalues [6].

Egorov, D.V. On Spin(7) Monge–Ampère type equations.

This work is supported in part by the Russian Federation president grants (MK–1526.2013.1 and NSh–4382.2014.1); RFBR grant 12-01-00873-a; «Dynasty» foundation grant.

Received November, 13, 2014, published December, 19, 2014.

In this article we consider another way to extend the notion of the Monge–Ampère type equations. Namely, we consider the Monge–Ampere equations that arise on manifolds with explicit holonomy group Spin(7). For each holonomy group G in the above there exists corresponding G-structure or equivalently differential form of certain type. The point for us is that each of these differential forms induces a volume form. Since determinant in the Monge–Ampère equation arises from calculation of the volume form this exactly fits our needs.

Next step in definition of new Monge–Ampère equations is to generalize the differential operator. We use the following definition, which generalizes the operator d^c from complex geometry.

Definition 1.1 ([7]). Let ϕ be a differential form. By d^ϕ denote a differential operator acting on k-form α by the following formula

$$d^\phi \alpha = \phi \wedge \delta\alpha - (-1)^k \delta(\phi \wedge \alpha),$$

where δ is the Hodge dual to d.

Definition 1.2. Let V be a real oriented vector space of dimension 8. Let $\{v_i\}_{i=0}^7$ be an oriented basis of V and let $\{v^i\}_{i=0}^7$ be a dual basis of V^*. Put

$$\begin{aligned}\Omega_0 = v^{0135} - v^{0146} - v^{0236} - v^{0245} + v^{0127} + v^{0347} + v^{0567} - v^{2467} + v^{1457} + \\ v^{2357} + v^{1367} + v^{1234} + v^{1256} + v^{3456},\end{aligned} \tag{1}$$

where $v^{ijkl} = v^i \wedge v^j \wedge v^k \wedge v^l$. A 4-form Ω on V is called Spin(7)-form if there exists $A \in GL^+(V):\ A^*\Omega = \Omega_0$.

This definition is generalized to the smooth manifolds case in the straightforward way. The Spin(7)-form induces the Riemannian metric g on M.

Definition 1.3. Let Ω be a Spin(7)-form on smooth manifold M. If Ω is closed, then holonomy group of the induced Riemannian metric is a subgroup of Spin(7).

Let Ω be a closed Spin(7)-form. We consider a nonlinear Monge–Ampère equation:

$$(d^\Omega d\varphi) \wedge (d^\Omega d\varphi) = e^F \Omega \wedge \Omega, \tag{2}$$

where φ and F are real-valued functions.

If Ω is the Kähler form, then $d^\Omega \equiv d^c$ and we get the complex Monge–Ampère equation:

$$(d^\Omega d\varphi)^n = e^F \Omega^n. \tag{3}$$

Remark 1.4. Since Spin(7)-form induces the Riemannian metric, operator d^Ω is well defined. In the Kähler case one starts from the Hermitian metric h in complex space and defines the Kähler form as imaginary part of h. Then one can prove that d^Ω doesn't depend on the Riemannian metric.

2. The Spin(7) Monge–Ampère equation in coordinates.

In this subsection we show that equation (2) takes form of det Hess in octonionic coordinates.

Let $\mathbb{O}$ be an algebra of octonions and let $e_0 = 1, e_2, \dots, e_7$ be its basis elements over $\mathbb{R}$ so that any octonion $o = x^0e_0 + \ldots + x^7e_7$, where $x^i \in \mathbb{R}$. By $\bar{o} = x^0e_0 - x^1e_1 \ldots - x^7e_7$ denote a conjugate element to o. Note that $\mathbb{O}$ is non-commutative and non-associative algebra. However, subalgebra generated by any two elements is associative.

Let $\mathbb{H}$ be a quaternion algebra generated by $1, i, j, k$ over $\mathbb{R}$ and let $\imath : \mathbb{H} \to \mathbb{O}$ be an injective algebra homomorphism. By π and π_* denote projections of $\mathbb{O}$ to $\imath(\mathbb{H})$ and factor $\mathbb{O}/\imath(\mathbb{H})$ respectively. For any octonion o there exists unique decomposition $o = \pi(o) + \pi_*(o)$ provided injection $\imath$.

Assume that $\mathbb{H}$ has quaternionic coordinate $q = y^0 + y^1 i + y^2 j + y^3 k$. By $\partial/\partial q = \partial/\partial y^0 - i\partial/\partial y^1 - j\partial/\partial y^2 - k\partial/\partial y^3$ denote a quaternionic Dolbeault operator on $\mathbb{H}$. Let us denote the push-forward by $\imath$ of $\partial/\partial q$ by $\partial/\partial o'$ and let $\partial/\partial o'' = \partial/\partial o - \partial/\partial o'$, where $\partial/\partial o = e_0\partial/\partial x^0 - \sum_{\alpha=1}^{7} e_\alpha \partial/\partial x^\alpha$ is the octonionic Dolbeault operator.

Let us distinguish 7 injective homomorphisms, which we call *canonical*. These are $\imath : (1, i, j, k) \to (1, e_\alpha, e_\beta, e_\gamma)$, where (α, β, γ) are the following triples:

$$(1,3,5), (1,6,4), (2,6,3), (2,5,4), (1,2,7), (3,4,7), (5,6,7). \tag{4}$$

Proposition 2.1. *Let X be an open subset of $\mathbb{O}$ and let φ be a real-valued function on X. Let Ω be a real Spin(7)-form on $\mathbb{O}$ that induces the standard Euclidean metric:*

$$\Omega = \frac{1}{24}(do \wedge (d\bar{o})) \wedge \overline{(do \wedge (d\bar{o}))}, \tag{5}$$

where $do = dx^0 e_0 + \ldots + dx^7 e_7$. Then

$$L_\Omega(\varphi) = \log \sum_{\imath : \mathbb{H} \to \mathbb{O}} \det \begin{pmatrix} \frac{\partial^2 \varphi}{\partial o' \partial \bar{o}'} & \frac{\partial^2 \varphi}{\partial o' \partial \bar{o}''} \\ \frac{\partial^2 \varphi}{\partial o'' \partial \bar{o}'} & \frac{\partial^2 \varphi}{\partial o'' \partial \bar{o}''} \end{pmatrix}, \tag{6}$$

where summation is taken over 7 canonical homomorphisms.

Proof. Octonionic 2×2 matrix A is called octonionic Hermitian if $A = \bar{A}^T$. This means that diagonal elements are real and off-diagonal are conjugate to each other so that the determinant of the octonionic Hermitian matrices is well defined. Proof follows from direct calculations. □

3. The Spin(7) Monge–Ampère operator is elliptic.

We need the following definition to establish the ellipticity property of equation (2).

Definition 3.1 ([8]). Let X be an open set in $\mathbb{R}^n$. A twice differentiable function $\varphi : X \to \mathbb{R}$ is called *r-plurisubharmonic*, $1 \leq r \leq n$ if for any combination $(i_1, \ldots, i_r)$ of pairwise distinct elements of the set $(1, \ldots, n)$

$$\lambda_{i_1} + \ldots + \lambda_{i_r} \geq 0,$$

where λ are eigenvalues of the Hessian of φ.

For example, if $r = 1$, then r-plurisubharmonic functions are (twice differentiable) convex functions. If $r = n$, then these are subharmonic functions. Note that a r-plurisubharmonic function is $(r+1)$-plurisubharmonic as well.

Definition 3.2. A function φ is called *strictly $*$-plurisubharmonic* if all inequalities in the definition of the $*$-plurisubharmonicity are strict, where $*$ denotes some special kind of plurisubharmonicity used in this paper.

Proposition 3.3. *Equation* (2) *is elliptic, if the solution is restricted to the class of strictly* 7*-plurisubharmonic functions.*

Proof. The matrix of the principal symbol at the point φ reads as: $\sigma(\varphi) = -H + \operatorname{tr} H \cdot I$, where H is the Hessian of φ and I is the identity matrix. Then σ is strictly positive definite if and only if φ is a strictly 7-plurisubharmonic function. □

4. Positive forms

The complex Monge–Ampère equation is elliptic if restricted to the class of the plurisubharmonic functions. The operator dd^c maps any plurisubharmonic function to non-degenerate positive 2-form. Positivity of the form means that it defines the Kähler metric with the background complex structure. Let us define positive 4-forms in real 8-space.

Definition 4.1. Let V be a real oriented vector 8-space. We say that a real 4-form Ω on V is *positive*, if for any non-zero $v \in \Lambda^2 V$

$$(v \lhd \Omega) \wedge (v \lhd \Omega) \wedge \Omega > 0,$$

where[1] we identify $\Lambda^8 V^*$ with $\mathbb{R}$ by using the orientation.

Any Spin(7) -form is positive [9].

Conjecture 4.2. *Let* Ω *be a positive* 4*-form such that* Ω *is self-dual:* $*\Omega = \Omega$*, where* $*$ *is the Hodge star of the inner product on* V *induced by* Ω*. Then* Ω *is the Spin*(7) *form.*

Proposition 4.3. *Let* Ω *be the Spin(7) form and let* $\tilde{\Omega}$ *be a positive* 4*-form. Then* $\Omega \wedge \tilde{\Omega} > 0$.

Proof. We can assume that Ω takes the form of (1) and therefore equips V with the standard Euclidean inner product $\langle \cdot, \cdot \rangle$. Octonionic structure on $V \otimes \mathbb{O}$ gives rise to 7 linear operators I_α on V: $I_\alpha^2 = -1$. Let us denote by the same symbol a 2-form $\langle \cdot, I_\alpha \cdot \rangle$. Then $I_\alpha^\# \lhd \Omega = I_\alpha$, where $I_\alpha^\#$ is the 2-vector dual to the form I_α. Since $\tilde{\Omega}$ is positive,

$$(I_\alpha^\# \lhd \Omega) \wedge (I_\alpha^\# \lhd \Omega) \wedge \tilde{\Omega} = I_\alpha \wedge I_\alpha \wedge \tilde{\Omega} > 0.$$

From the formula $\sum_\alpha I_\alpha \wedge I_\alpha = \Omega$ the proof follows. □

However, if $\varphi : X \to \mathbb{R}$ is 7-plurisubharmonic, then generally the form $\Omega \wedge dd^\Omega \varphi$ is not positive. For example, consider a quadratic polynomial $Q_{ij} x^i x^j$ with matrix $Q = \operatorname{diag}(1, -1, 1, -1, 1, -1, 1, 1)$. Let us further restrict the class of solutions to ensure that $dd^\Omega \varphi \wedge \Omega$ is a positive form.

Definition 4.4 ([6])**.** An affine real 4-plane P in $\mathbb{O}$ directed by 4-vector v is *calibrated* by Ω if $\Omega(v) = 1$. A real function $\varphi : X \to \mathbb{R}$ is called Ω*-plurisubharmonic* if it is subharmonic on any plane calibrated by Ω. If Ω is the Spin(7) -form, then Ω-calibrated plurisubharmonic function is called *Cayley plurisubharmonic.*

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[1] The symbol "$\lhd$" denotes the contraction of vector and form. More recognizable symbols are contained in some advanced TEX libraries that are not allowed to use in this journal.

Let $\imath : \mathbb{H} \to \mathbb{O}$ be some injective algebra homomorphism. Consider an affine quaternionic line in $\mathbb{O}$:

$$a + b\imath(q), \tag{7}$$

where $q \in \mathbb{H}$ and $a, b \in \mathbb{O}$.

Proposition 4.5. *There exists a one-to-one correspondence between the set of affine quaternionic lines and the set of Ω-calibrated planes, where Ω is some Spin(7)-form.*

Proof. For any quaternionic line (7) the real directing 4-vector $v = B \wedge IB \wedge JB \wedge KB$, where $B = b_{\mathbb{R}}$; I, J, and K are real linear transformations induced by multiplication from right by $\imath(i)$, $\imath(j)$, and $\imath(k)$. Suppose $o : o^2 = -1$ is arbitrary octonion that is not an element of subalgebra generated by $\imath(i)$ and $\imath(j)$. Then $\imath(i)$, $\imath(j)$, and o generate algebra $\mathbb{A}$ isomorphic to $\mathbb{O}$. We can construct a Spin(7)-form $\tilde{\Omega}$ by formula (5) for $\mathbb{A}$. Then $\tilde{\Omega}(v) = 1$.

Conversely, suppose $\Omega(v_0 \wedge v_1 \wedge v_2 \wedge v_3) = 1$ for some given Spin(7)-form Ω. Construct three real linear transformations I_α on $V/\langle v_0, v_\alpha\rangle$ respectively by the following formula:

$$\Omega(v_0 \wedge v_\alpha \wedge u \wedge I_\alpha u) = 1, \quad \alpha = 1, 2, 3.$$

Since Ω is the Spin(7)-form, operators I_α are not degenerate. By definition, put $I_\alpha v_0 = v_\alpha$, $\;I_\alpha v_\alpha = -v_0$. Operators I_α span an algebra isomorphic to $\mathbb{H}$. Then v takes the form $v_0 \wedge I_1 v_0 \wedge I_2 v_0 \wedge I_3 v_0$, i.e. directing vector of some quaternionic line. □

Proposition 4.6. *A real function $\varphi : X \to \mathbb{R}$ is Cayley plurisubharmonic if and only if for any affine quaternionic line:*

$$\begin{pmatrix} \frac{\partial^2\varphi}{\partial o'\partial\bar{o}'} & \frac{\partial^2\varphi}{\partial o'\partial\bar{o}''} \\ \frac{\partial^2\varphi}{\partial o''\partial\bar{o}'} & \frac{\partial^2\varphi}{\partial o''\partial\bar{o}''} \end{pmatrix} \geq 0, \tag{8}$$

where $\partial/\partial o'$ is the push-forward by $\imath$ of the quaternionic Dolbeault operator and $\partial/\partial o'' = \partial/\partial o - \partial/\partial o'$

Proof. Restrict φ to the quaternionic line (7) and apply $\Delta = \frac{\partial^2}{\partial q\partial\bar{q}}$. Since on any quaternionic line φ is subharmonic, we have

$$\frac{\partial^2\varphi}{\partial q\partial\bar{q}}\varphi(a + (b' + b'')\imath(q)) \geq 0,$$

where b' and b'' are the projection of b to $\imath(\mathbb{H})$ and $\mathbb{O}/\imath(\mathbb{H})$ respectively. Since φ and $\partial^2\varphi/\partial q\partial\bar{q}$ are real valued,

$$\frac{\partial^2\varphi}{\partial q\partial\bar{q}}\varphi(a + (b' + b'')\imath(q)) =$$

$$\operatorname{Re}\left[\frac{\partial}{\partial\bar{q}}\left\{\frac{\partial}{\partial q}\varphi(a + b'q + b''q)\right\}\right] = \operatorname{Re}\left[\frac{\partial}{\partial\bar{q}}\left\{\frac{\partial\varphi}{\partial o'}b' + \frac{\partial\varphi}{\partial o''}b''\right\}\right] =$$

$$\operatorname{Re}\left[\frac{\partial^2\varphi}{\partial o'\partial\bar{o}'}b'\bar{b}' + \frac{\partial^2\varphi}{\partial o'\partial\bar{o}''}b'\bar{b}'' + \frac{\partial^2\varphi}{\partial o''\partial\bar{o}'}b''\bar{b}' + \frac{\partial^2\varphi}{\partial o''\partial\bar{o}''}b''\bar{b}''\right] \geq 0.$$

Last inequality holds if and only if the octonionic Hermitian matrix (8) is positive definite [5]. Problem with the non-associativity are surpassed by the fact that $\operatorname{Re}(a(bc)) = \operatorname{Re}((ab)c)$. □

Proposition 4.7. *If a function $\varphi : X \to \mathbb{R}$ is Cayley plurisubharmonic, then φ is 7-plurisubharmonic.*

Proof. The sum of octonionic Hermitian quadratic forms with matrices (8) over all canonical homomorphisms is equal to the real quadratic form with matrix $-H + \operatorname{tr} H \cdot I$, where H is the real Hessian of φ. $\square$

The converse is not true. For example, the real quadratic polynomial with the following symmetric matrix is 7-plurisubharmonic but not Cayley plurisubharmonic

$$\begin{pmatrix} 7 & 5 & 2 & 1 & 6 & 9 & 5 & 1 \\ 5 & 6 & 6 & 5 & 3 & 1 & 4 & 4 \\ 2 & 6 & 5 & 7 & 7 & 5 & 5 & 9 \\ 1 & 5 & 7 & 0 & 1 & 3 & 4 & 1 \\ 6 & 3 & 7 & 1 & 2 & 0 & 6 & 1 \\ 9 & 1 & 5 & 3 & 0 & 4 & 1 & 3 \\ 5 & 4 & 5 & 4 & 6 & 1 & 7 & 2 \\ 1 & 4 & 9 & 1 & 1 & 3 & 2 & 3 \end{pmatrix}.$$

Proposition 4.8. *If $\varphi : X \to \mathbb{R}$ is strictly Cayley plurisubharmonic, then $\Omega \wedge dd^{\Omega}\varphi > 0$, where Ω is of the form* (5).

Proof. From the fact that any Cayley plurisubharmonic function is subharmonic and the formula

$$\Omega \wedge dd^{\Omega}\varphi = \Delta\varphi$$

the proof follows. $\square$

The author would like to thank M. Verbitsky for useful conservations and support.

References

[1] S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation, I*, Communications on pure and applied mathematics, **31** (1978), 339–411. MR0480350

[2] E. Calabi, *On Kähler manifolds with vanishing canonical class*, Algebraic geometry and topology. A symposium in honor of S. Lefschetz (1957), 78–89. MR0085583

[3] S. Alesker, *Quaternionic Monge–Ampere equations*, The Journal of Geometric Analysis, **13** (2003), 205–238. MR1967025

[4] S. Alesker and M. Verbitsky, *Quaternionic Monge–Ampère equation and Calabi problem for HKT-manifolds*, Israel Journal of Mathematics, **176** (2010), 109–138. MR2653188

[5] S. Alesker, *Plurisubharmonic functions on the octonionic plane and* Spin(9)*-invariant valuations on convex sets*, Journal of Geometric Analysis, **18** (2012), 651–686.

[6] F.R. Harvey and H. B. Lawson Jr., *An introduction to potential theory in calibrated geometry*, American Journal of Mathematics, **131** (2009), 893–944. MR2543918

[7] M. Verbitsky, *Manifolds with parallel differential forms and Kähler identities for G_2-manifolds*, Journal of Geometry and Physics, **61** (2011), 1001–1016. MR2782476

[8] H. Wu, *Manifolds of partially positive curvature*, Indiana Univ. Math. J., **36** (1987), 525–548. MR0905609

[9] S. Karigiannis, *Some notes on G_2 and* Spin(7) *geometry*, arXiv preprint math/0608618, 2006.

Dmitry Vladimirovich Egorov
Ammosov Northeastern federal university,
Kulakovsky str. 48,
677000, Yakutsk, Russia
E-mail address: egorov.dima@gmail.com