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ON SWITCHING NONSEPARABLE GRAPHS WITH SWITCHING SEPARABLE SUBGRAPHS

E.A. BESPALOV

Abstract. A graph of order $n \geq 4$ is called switching separable if its modulo-2 sum with some complete bipartite graph on the same set of vertices is divided into two mutually independent subgraphs, each having at least two vertices. We describe all switching nonseparable graphs of order n whose induced subgraphs of order $(n-1)$ are all switching separable. In particular, such graphs exist only if n is odd. This leads to the following essential refinement of the known test on switching separability, in terms of subgraphs: if all order-$(n-1)$ subgraphs of a graph of order n are separable, then either the graph itself is separable, or n is odd and the graph belongs to the two described switching classes.

Keywords: Two-graph, switching of graph, switching separability, Seidel switching, n-ary quasigroup.

1. Introduction

A graph of order $n \geq 4$ is called switching separable if its sum modulo 2 with some complete bipartite graph on the same set of vertices is divided into two mutually independent subgraphs in two or more vertices. As was shown in [4], if removing one or two vertices of a graph G always leads to a separable graph, then G itself is separable. For a weaker hypothesis, when removing only one vertex always leads to a separable graph, the same conclusion cannot be derived, which was illustrated by an infinite series of examples. The goal of this work is more detailed studying of the last case and the characterization of all examples of nonseparable graphs such that

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removing any vertex results in a separable subgraph. Moreover, it will be shown that such graphs exist only for odd number of vertices, which leads to an essential refinement of the separability test from [4].

Tests for switching separability of graphs are a more simple model of the corresponding assertions about the reducibility of n-ary quasigroups (for the definition and basic properties of n-ary quasigroups see, e.g., [1]; in combinatorics, equivalent objects are known as Latin hypercubes, see, e.g., [7]). Also, examples of nonseparable graphs can be utilized to construct examples of irreducible (undecomposable to a repetition-free composition) n-ary quasigroups. The relationship between the switching separability of graphs and the reducibility of n-ary quasigroups and Boolean functions built from these graphs is discussed in [4]. In [3], [6], and [2], statements and examples for n-ary quasigroups which are similar to statements and examples for graphs from [4] are given. This theory has played a significant role in the characterization of n-ary quasigroups (Latin hypercubes) of order 4 [5]. The results of the current work also have a potential to be generalized to similar statements for n-ary quasigroups and to be applied for the characterization of different classes of these objects.

In Section 2, we give definitions, auxiliary statements and formulate the main theorem. The theorem is proven in Section 3. In Section 4, from the main theorem, we derive another separability test, in terms of separability of $(n-2)$-subgraphs.

2. Auxiliaries and the main theorem

In this paper, we consider only simple (without loops and multiple edges) undirected graphs and only induced subgraphs. Let U be a set of vertices of a graph $G = (V,E)$. *Switching*, or *U-switching*, of the graph G is the graph $G_U = (V, E\Delta E_{U,V\backslash U})$, where $K_{U,V\backslash U} = (V, E_{U,V\backslash U})$ is the complete bipartite graph with parts $U, V\backslash U \subseteq V$ (for generality, we allow one of the parts to be empty). The graph switching, also known as Seidel switching, was introduced in [9]; there is a one-to-one correspondence between the switching classes and the so-called two-graphs [8].

A set W of vertices of a graph $G = (V,E)$ is said to be *isolable* if some switching of G contains no edges joining W with $V\backslash W$. A graph of order n is called *switching separable* (we will often omit the word "switching") if there exists an isolable set of vertices W of cardinality at least 2 and at most $|V| - 2$.

Before stating the main theorem, we define a series of graphs G_n, $n \geq 5$ odd. One of the vertex in G_n is isolated. The remaining vertices form a bipartite graph with parts $\{x_1, \dots, x_{(n-1)/2}\}$ and $\{y_1, \dots, y_{(n-1)/2}\}$. Vertices x_i and y_j are adjacent if and only if i is odd and $i \leq j$, or j is odd and $j \leq i$.

Theorem 1. *If removing any vertex of a graph G of order n always leads to a switching separable subgraph of G, then G is either switching separable, or n is odd and G isomorphic to a switching of G_n or a switching of the complement of G_n.*

Corollary 1. *Every switching nonseparable graph of even order n has a switching nonseparable subgraph of order $n-1$.*

We need the following auxiliary statements.

Lemma 1 ([4]). *If all subgraphs of orders $n-1$ and $n-2$ of a graph G of order n are separable, then G is a separable graph.*

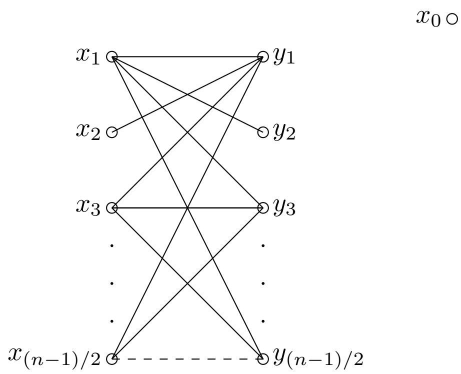

FIG. 1

Lemma 2 ([4]). *Let W be a subset of the vertex set V of a graph $G = (V, E)$. The following assertions are equivalent:*

(1) *The set W is isolable in G.*
(2) *For any distinct x and y from W, z and t from $V \backslash W$, the number of edges from $\{\{x, z\}, \{x, t\}, \{y, z\}, \{y, t\}\} \cap E$ is even.*
(3) *There exist W_1, W_2, V_1, V_2 such that $W = W_1 \cup W_2$, $V \backslash W = V_1 \cup V_2$ and every vertex of W_i, $i = 1, 2$, is adjacent with every vertex of V_{3-i}, and is not adjacent to any vertex of V_i.*

Lemma 3. *Let $D = (V, E)$ be a separable graph of order $n \geq 5$ with an isolable vertex set W, where $2 \leq |W| \leq n/2$. Let there exist a vertex a such that the graph $D \backslash \{a\}$ is not separable. Then $W = \{a, b\}$ for some vertex b.*

Proof. Assume the statement does not hold. That is, $|W \backslash \{a\}| \geq 2$. Then each of $W \backslash \{a\}$ and $(V \backslash W) \backslash \{a\}$ has at least two vertices, and by Lemma 2 the graph $D \backslash \{a\}$ is separable, which contradicts the hypothesis. □

Lemma 4. *If a subset W of vertices is isolable in G, then W is isolable in the complement of G. This also means that if a graph is separable, then its complement is also separable.*

Proof. The statement is straightforward from items 1 and 2 of Lemma 2. □

3. Proof of the theorem

We will try to characterize all nonseparable graphs of order n such that all subgraphs of order $n - 1$ are separable.

Let a graph $G = (V, E)$ of order $n \geq 5$, odd or even, meet the following properties:

(1) G is a nonseparable graph.
(2) For every vertex a of G, the graph $G \backslash \{a\}$ is separable.

Then, by Lemma 1, G has at least one nonseparable subgraph of order $n - 2$. That is, for some vertices a and b, the graph $G \backslash \{a, b\}$ is nonseparable. Without loss of generality we assume that a is an isolated vertex of G (we can always isolate it by switching the set of vertices adjacent with a). In a separable graph $G \backslash \{a\}$, there is a nonseparable subgraph $G \backslash \{a, b\}$; so, by Lemma 3, there is a vertex $c \neq b, a$ such the set $\{b, c\}$ is isolable in the graph $G \backslash \{a\}$. Then, in accordance with items

1 and 3 of Lemma 2, there are several admissible options for the sets W_1, W_2, $W_1 \cup W_2 = \{b, c\}$, and V_1, V_2, $V_1 \cup V_2 = V\backslash\{a, b, c\}$ (see Figure 2):
0) $W_1 = \{b, c\}$, $V_1 = V\backslash\{a, b, c\}$, $W_2 = V_2 = \emptyset$;
1) $W_1 = \{b\}$, $W_2 = \{c\}$, $V_1 = V\backslash\{a, b, c\}$, $V_2 = \emptyset$;
2) $W_1 = \{b\}$, $W_2 = \{c\}$, $V_2 = V\backslash\{a, b, c\}$, $V_1 = \emptyset$;
3) $W_1 = \{b, c\}$, $V_2 = V\backslash\{a, b, c\}$, $W_2 = V_1 = \emptyset$;
4) $W_1 = \{b, c\}$, $W_2 = \emptyset$, $V_1, V_2 \neq \emptyset$;
5) $W_1 = \{b\}$, $W_2 = \{c\}$, $V_1, V_2 \neq \emptyset$.

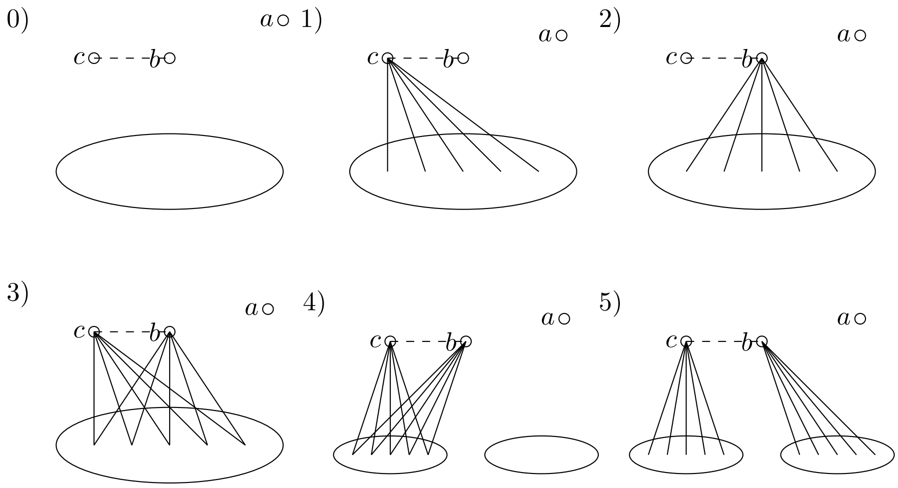

FIG. 2. here and below, the figure does not indicate possible edges between vertices illustrated at the same height

Only the last variant, where all the sets W_1, W_2, V_1, V_2 are nonempty, does not contradict to the nonseparability of the graph G. Indeed, in cases 0, 1, and 2, $\{a, b, c\}$ is isolable in G; in cases 3 and 4, $\{b, c\}$ is isolable in G. Thus, the only admissible case is case 5. Denote $V_b = V_2$ and $V_c = V_1$, see Figure 3.

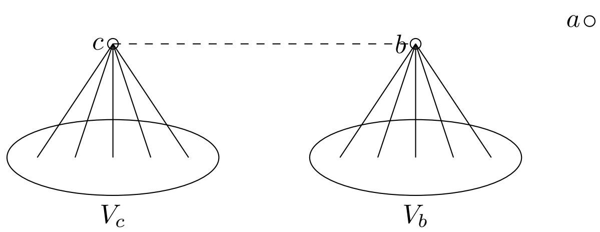

FIG. 3

The graph $G\backslash\{b\}$ has a nonseparable subgraph $G\backslash\{a, b\}$; by Lemma 3, the graph $G\backslash\{b\}$ has an isolable set $\{a, d\}$ for some vertex d. If $d = c$, then we have a contradiction with item 2 of Lemma 2 ($(x, y, z, t) = (a, c, f, g)$, where $f \in V_b$, and $g \in V_c$). Thus, d either belongs to V_c and is adjacent to all vertices of $V\backslash\{a, b, d\}$, or belongs to V_b and is not adjacent to any vertex of $V\backslash\{a, b, d\}$. The $\{b\}$-switching of the graph $G\backslash\{a, c\}$ is isomorphic to $G\backslash\{a, b\}$; consequently, $G\backslash\{a, c\}$ is not separable, too. Similarly, there exists a vertex e that either belongs to V_c and is not adjacent to any vertex of $V\backslash\{a,c,e\}$, or belongs to V_b and is adjacent with all vertices of $V\backslash\{a,c,e\}$. Moreover, d and e cannot belong to V_c or V_b simultaneously (otherwise there is a contradiction with the edge $\{d,e\}$). Therefore, the graph G assumes one of the forms illustrated in Figure 4. We will consider only case 1, as case 2 can be reduced to it by switching the set $\{a\}$ and taking the complement.

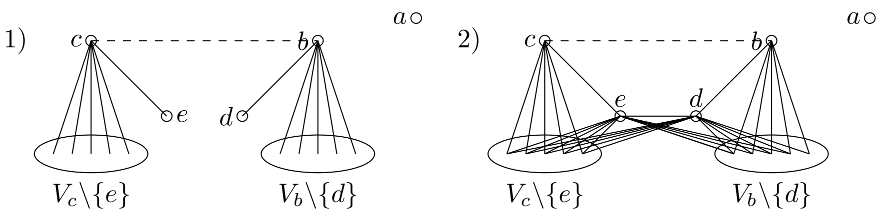

FIG. 4

Consider the graph $G\backslash\{d\}$. It is separable. Consequently, by items 1 and 2 of Lemma 2, its vertices can be colored in two colors, say black and white, in such a way that the following two properties are satisfied. At first, there are at least two vertices of each color. At second, between each pair of black vertices and each pair of white vertices, there are even number of edges. Suppose, to be definite, that a is black. Consider the variants to color b and c.

(1) If b and c are white, then neither $V_b\backslash\{d\}$ nor V_c can contain a black vertex f (by item 2 of Lemma 2, $(x,y,z,t)=(b,c,a,f)$), which contradicts the separability of $G\backslash\{d\}$.
(2) If b and c are black, then all white vertices are in $V_b\backslash\{d\}\cup V_c$. If both $V_b\backslash\{d\}$ and V_c contain white vertices, say f and g, respectively, then we have a contradiction with item 2 of Lemma 2, $(x,y,z,t)=(a,b,f,g)$. Otherwise, the set of white vertices is isolable in G (by item 2 of Lemma 2), which contradicts the nonseparability of G.
(3) If b is black and c is white, then the rest W of white vertices (similarly to the previous case) belongs either to $V_b\backslash\{d\}$, or to V_c. The set W is isolable in G, as it satisfies the condition of item 3 of Lemma 2; hence, W consists of only one vertex, say z. If $z\neq e$, then $(x,y,z,t)=(c,z,a,e)$ does not satisfy the condition of item 2 of Lemma 2. So, the set of white vertices is $\{c,e\}$, and it can be seen by items 1 and 3 of Lemma 2 that this set is isolable in G, contradicting the nonseparability of G.
(4) If b is white and c is black, then, similarly to the previous case, there is only one other white vertex, say f. There is an edge $\{b,c\}$ if and only if f belongs to V_c (by item 2 of Lemma 2, taking $x=a$, $y=c$, $z=b$, $t=f$). By Lemma 2 with $x=b$, $y=f$, $z=a$, we see that f is adjacent with all vertices of $V_b\backslash\{d,f\}$, and with none of $V_c\backslash\{f\}$.

We have seen that only the case 4 is not contradictory.

Analogously, by removing a vertex e, we have a situation similar to one described in item 4. That is, there is a vertex g, adjacent with all vertices of $V_c\backslash\{e,g\}$ and with none of $V_b\backslash\{g\}$.

Moreover, there is an edge $\{b,c\}$ if and only if g belongs to V_b. Consequently, f and g cannot belong to V_b or V_c simultaneously.

If $g = d$, then any vertex h from $V_c\backslash\{e\}$ is not in agree with item 2 of Lemma 2 ($x = a$, $y = h$, $z = c$, $t = g = d$); we conclude that $V_c\backslash\{e\}$ is empty, and $f = e$. Then, similarly, $V_b\backslash\{d\}$ is empty too. Then G has 5 vertices and isomorphic to G_5. Analogously, $f = e$ leads to G_5.

As a result, if G has more than 5 vertices, it should be of one of types shown in Figure 5.

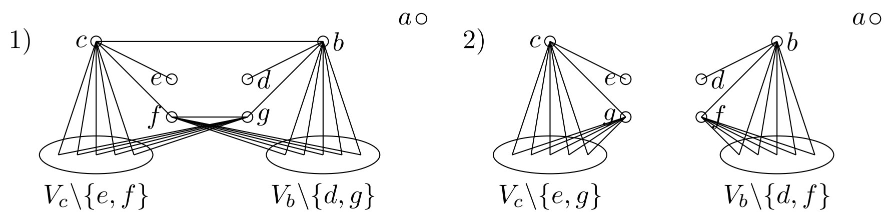

FIG. 5

Consider the case 1.

Proposition 1. *For any $k \le \max(|V_c|+1, |V_b|+1)$, there exist k vertices $x_1, \dots, x_k$ from $V_c \cup \{b\}$ and k vertices $y_1, \dots, y_k$ from $V_b \cup \{c\}$ such that:*

- *the vertices x_i and y_j are adjacent if and only if i is odd, and $i \le j$, or j is odd and $j \le i$;*
- *for every i, the vertex x_i is not adjacent with any vertex from $V_c \cup \{b\}$ and the vertex y_i is not adjacent with any vertex from $V_b \cup \{c\}$;*
- *for every i and every z from $V_b \cup \{c\}\backslash\{y_1, \dots, y_k\}$ (from $V_c \cup \{b\}\backslash\{x_1, \dots, x_k\}$), the vertex x_i (y_i, respectively) is adjacent with z if and only if i is odd.*

Proof. We prove by induction. For $k = 3$, the statement holds ($x_0 = a$, $x_1 = b$, $x_2 = e$, $x_3 = f$, $y_1 = c$, $y_2 = d$, $y_3 = g$, see Figure 5).

Assume the statement holds for $k = i - 1 < \max(|V_c| + 1, |V_b| + 1)$; let us prove it for $k = i$. Without loss of generality we assume that $V_c\backslash\{x_2, \dots, x_{i-1}\}$ has at least one vertex. Consider the graph $G\backslash\{x_{i-1}\}$. It is separable, and we can color its vertices into two colors, black and white, in accordance with some isolable set and its complement, each color corresponding to at least two vertices. Suppose, to be definite, that x_0 is black. Consider several simple claims.

1. Every black vertex is either adjacent with all white vertices, or not adjacent to any white vertex. Indeed, since this is true for the isolated black vertex x_0, by Lemma 2 this is true for each other black vertex.

2. The vertices x_1 and y_1 cannot both be white. This is evident from the previous claim.

3. The vertices x_1 and y_1 cannot be white and black, respectively. Indeed, if it is so and y_2 is black, then by claim 1 we have that x_1 is the only white vertex. If y_2 is white, then we again have a contradiction with claim 1, as the black vertex y_1 is adjacent with the white x_1, but not with the white y_2.

4. Similarly, x_1 and y_1 cannot be black and white, respectively.

5. Thus, x_1 and y_1 are black.

6. The white vertices all belong to one of V_c and V_b. This is straightforward from claim 1, if we consider the black vertex x_1.

7. The white vertices belong to V_b. Indeed, if this is not true, then they all belong to V_c. Then, sinse x_{i-1} is not adjacent with any vertex from V_c, the set of white vertices is separable in G. A contradiction.

8. There are no two white vertices among $y_2, \ldots, y_{i-1}$. Indeed, for every l and m satisfying $2 \leq l < m \leq i-1$, the vertex x_{l+1} is adjacent with only one of y_l and y_m. If $l+1 \neq i-1$ then x_{l+1} is black, and y_l and y_m cannot both be white by claim 1. If $l+1 = i-1$, then $l+1 = m$, and considering any black vertex from $V_c \backslash \{x_2, \ldots, x_{i-1}\}$ (here we use that this set is nonempty) leads to the same conclusion.

9. There are no two white vertices in $V_b \backslash \{y_2, \ldots, y_{i-1}\}$. Indeed, it is easy to see that set of white vertices in $V_b \backslash \{y_2, \ldots, y_{i-1}\}$ is separable in G, as any other vertex is either adjacent with every vertex from this set or not adjacent with every vertex from this set.

10. Therefore, there is exactly one white vertex among $y_2, \ldots, y_{i-1}$; there is exactly one white vertex in $V_b \backslash \{y_2, \ldots, y_{i-1}\}$.

11. The white vertex among $y_2, \ldots, y_{i-1}$ is y_{i-2}. Indeed, if $l = (i-1)$, then x_{i-1} is either adjacent or not adjacent with all white vertices, If $l \neq (i-1), (i-2)$, then we find a contradiction with claim 1 by considering the black vertex x_{l+1}.

The white vertex different from y_{i-2} can be chosen as y_i. Since it is adjacent with the same vertices (except x_{i-1}) as y_{i-2}, it satisfies the necessary properties.

We also see that there is at least one vertex in $V_b \backslash \{y_2, \ldots, y_{i-1}\}$. Then, we can find x_i in a similar way, considering $G \backslash \{y_{i-1}\}$. $\square$

In particular, taking the maximal possible value of k, we get $n = 2k+1$.

Now consider case 2 (Figure 5).

Proposition 2. *For any $k \leq |V_b|+1$, there exist k vertices $x_1, \ldots, x_k$ from $V_b \cup \{b\}$ such that for every i from 1 to k:*

- *for all $j > i$, the vertex x_i is adjacent with x_j if and only if i is odd;*
- *for all x from $V_b \cup \{b\} \backslash \{x_1, \ldots, x_k\}$, the vertex x_i is adjacent with x if and only if i is odd.*
- *x_i is not adjacent with any vertex from $V_c \cup \{c\}$.*

Proof. The proof is rather similar to that of Proposition 1; the most essential differences are in the proofs of claims 3 and 9 below. For $k = 3$, the statement holds ($x_0 = a$, $x_1 = b$, $x_2 = d$, $x_3 = f$, see Figure 5).

Assume the statement holds for $k = i-1 < |V_b|+1$; let us prove it for $k = i$. Consider the graph $G \backslash \{x_{i-1}\}$. It is separable, and we can color its vertices into two colors, black and white, in accordance with some isolable set and its complement, each color corresponding to at least two vertices. Suppose, to be definite, that x_0 is black. Consider several simple claims.

1. Every black vertex is either adjacent with all white vertices, or not adjacent to any white vertex. Indeed, since this is true for the isolated black vertex x_0, by Lemma 2 this is true for each other black vertex.

2. The vertices x_1 and c cannot both be white. This is obvious from claim 1.

3. The vertices x_1 and c cannot be white and black, respectively. Indeed, if it is so and x_2 is black, then by claim 1 we have that x_1 is the only white vertex. Thus, x_2 is white. Then all vertices of V_c are black (if some vertex v from V_c is white, then the black vertex c adjacent with v and not adjacent with x_2, contradicting claim 1). Moreover, all vertices of $V_b \backslash \{x_{i-1}\}$ are white (if there some vertex w is

black then w is adjacent with x_1 and not adjacent with x_2, contradicting claim 1). Then, since x_{i-1} is not adjacent with any vertex of $V_c \cup c$, we see that $V_b \cup \{b\}$ is isolable in G, a contradiction.

4. Similarly, x_1 and c cannot be black and white, respectively.

5. Thus, x_1 and c are black.

6. The white vertices all belong to one of V_c and V_b. This is straightforward from claim 1, if we consider the black vertex x_1.

7. The white vertices belong to V_b. Indeed, if this is not true, then they all belong to V_c. Then, sinse x_{i-1} is not adjacent with any vertex from V_c, the set of white vertices is isolable in G. A contradiction.

8. There are no two white vertices among $x_2, \ldots, x_{i-2}$. Indeed, assume the contrary. Let l and m be the two minimal values such that x_l and x_m are white. If $m > l+1$, then x_{l+1} is black and adjacent with only one of x_l and x_m; so, we have a contradiction with claim 1. Thus, $m = l+1$. As follows from claim 1, there is no black vertex in $V_b \backslash \{x_2, \ldots, x_{m+1}, x_{i-1}\}$. Then, the set of white vertices together with x_{i-1} is isolable in G. We get a contradiction.

9. There are no two white vertices in $V_b \backslash \{x_2, \ldots, x_{i-1}\}$. Indeed, it is easy to see that the set of white vertices from $V_b \backslash \{x_2, \ldots, x_{i-1}\}$ is isolable in G.

10. Therefore, there is exactly one white vertex among $x_2, \ldots, x_{i-2}$; there is exactly one white vertex in $V_b \backslash \{x_2, \ldots, x_{i-1}\}$.

11. The white vertex x_l among $x_2, \ldots, x_{i-2}$ is x_{i-2}. Indeed, if $l < i-2$, then we find a contradiction with claim 1 by considering the black vertex x_{l+1}.

The white vertex different from x_{i-2} can be chosen as x_i. Since it is adjacent with the same vertices (except x_{i-1}) as x_{i-2}, it satisfies the necessary properties. □

Then, taking $k = |V_b| + 1$, we see that $V_b \cup \{b\}$ is isolable in G. The contradiction with the nonseparability of G shows that case 2 (Figure 5) is not possible.

We see from Proposition 1 that n is odd and G is isomorphic to G_n. The proof of Theorem 1 is over.

Remark 1. *It should be noted that G_n is nonseparable and all its subgraphs of order $n-1$ are separable. This is straightforward from Theorem 1 and the fact that such graphs exist* [2]*; however, we will show this independently.*

Assume G_n is separable and consider the corresponding black-and-white coloring of its vertices where the isolated vertex x_0 is black. Then every black vertex is connected with every pair of white vertices by the even number (0 *or* 2) *of edges. The vertices x_1 and y_1 cannot both be white (otherwise, x_0 is the only black vertex). Thus, x_1 or y_1 is black, and we see that all white vertices belong to the same part, say $\{x_1, \ldots, x_{(n-1)/2}\}$. But for two white vertices x_i and x_j, $i < j$, the black vertex y_{i+1} is adjacent with only one of them. We have a contradiction, proving that G_n is not separable.*

It remains to consider the subgraphs of order $n-1$. In $G \backslash \{x_0\}$, the set $\{x_1, y_1\}$ is isolable. In $G \backslash \{x_1\}$, the set $\{x_0, y_2\}$ is isolable (similarly, $\{x_0, x_2\}$ is isolable in $G \backslash \{y_1\}$). The set $\{y_{(n-3)/2}, y_{(n-1)/2}\}$ is isolable in $G \backslash \{x_{(n-1)/2}\}$ (similarly, the set $\{x_{(n-3)/2}, x_{(n-1)/2}\}$, in $G \backslash \{y_{(n-1)/2}\}$). Finally, for every $i \in \{2, \ldots, (n-3)/2\}$, the set $\{y_{i-1}, y_{i+1}\}$ is isolable in $G \backslash \{x_i\}$ (similarly, $\{x_{i-1}, x_{i+1}\}$, in $G \backslash \{y_i\}$).

4. The case of separable $(n-2)$ subgraphs

Theorem 2. *If in a graph G of order $n \geq 7$ all subgraphs of order $n-2$ are separable, then G is separable.*

We will need the following auxiliary statement about the graph G_n, defined in Section 2.

Lemma 5. *The graph $G_{2k+1}\backslash\{x_0, x_1\}$ is isomorphic to G_{2k-1}.*

Proof. The vertices x_0, x_1, x_2, x_3, x_4, x_5, … have the degrees 0, k, 1, $k-1$, 2, $k-2$, …, respectively. That is, the set of degrees are all integers from 0 to k. Similarly, the set of degrees of y_1, …, y_k are all integers from 1 to k. It is easy to see that all bipartite graph with parts of size $k+1$ and k and pairwise different degrees in each part are isomorphic.

For the vertices x_2, x_3, …, x_k of the graph $G_{2k+1}\backslash\{x_0, x_1\}$, the set of degrees are all integers from 1 to $k-1$. The vertices y_1, y_2, y_3, y_4 in the same graph have the degrees $k-1$, 0, $k-2$, 1, …, respectively; this set of degrees are all integers from 0 to $k-1$. Hence, $G_{2k+1}\backslash\{x_0, x_1\}$ is isomorphic to G_{2k-1}. □

of Theorem 2. Assume the contrary, that G is not separable. Then, by Lemma 1, it has a nonseparable subgraph $G\backslash\{z\}$ of order $n-1$. Without loss of generality we assume that some vertex different from z is isolated in G. By Theorem 1, $(n-1)$ is odd, and we can assume that $G\backslash\{z\} = G_{n-1}$, see Figure 1, where we do not know how the vertex z is connected with the other vertices.

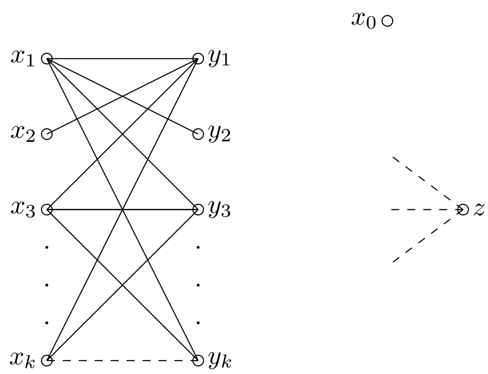

Fig. 6

Consider the graph $G\backslash\{x_k, y_k\}$, where $k = (n-2)/2$. By the hypothesis of the theorem, the graph it is separable, while the subrgaph $G\backslash\{x_k, y_k, z\} = G_{n-3}$ is nonseparable. By Lemma 3, the vertex z has a "mate" vertex x such that $\{z, x\}$ is isolable in $G\backslash\{x_k, y_k\}$.

Now consider the graph $G\backslash\{x_0, x_1\}$. By Lemma 5, the subgraph $G\backslash\{x_0, x_1, z\}$ is isomorphic to G_{n-3}; consequently, it is nonseparable. Hence, the vertex z has a "mate", say y, such that $\{z, y\}$ is isolable in $G\backslash\{x_0, x_1\}$.

Consider the case when there is edge $\{z, y_2\}$. Graph $G\backslash\{x_k, y_{k-1}, z\}$ is isomorphic to nonseparable graph $G\backslash\{x_k, y_k, z\}$ as in $G\backslash\{z\}$ the vertices y_k and y_{k-1} are different only in vertex x_k. Analogous, graph $G\backslash\{x_{k-1}, y_k, z\}$ is isomorphic to nonseparable graph $G\backslash\{x_k, y_k, z\}$. Then, in this graphs z has "mates" $\tilde{x}, \tilde{y}$ such that

$\{\tilde{x}, z\}$ is isolable in $G\backslash\{x_k, y_{k-1}\}$ and $\{\tilde{y}, z\}$ is isolable in $G\backslash\{x_{k-1}, y_k\}$. If $\tilde{x} \neq y_2$ and $\tilde{x} \neq x_0$, then by item 2 of lemma 2 vertex y_2 must be adjacent with x, because it adjacent with z, and then $x = x_1$. Analogous, if $\tilde{y} \neq x_0$ and $\tilde{y} \neq y_2$ then $\tilde{y} = x_1$. If $\tilde{x} = x_0(y_2)$ and $\tilde{y} = x_1$ then contradictory with the edge $\{z, x_2(y_1)\}$. If $\tilde{x} = x_0$ and $\tilde{y} = y_2$ then contradictory with vertex x_2 (As $\{z, y_2\}$ isolable in $G\backslash\{x_{k-1}, y_k, z\}$ there is edge $\{x_1, z\}$ and no edge $\{z, x_2\}$. As $\{z, x_0\}$ is isolable $G\backslash\{x_k, y_{k-1}, z\}$ there is edge $\{z, x_2\}$. Contradiction.) Then $x = y$ and $\{z, x\}$ is isolable in G. Further we can assume that there no edge $\{z, y_2\}$ in graph G.

Note that x and y are different (otherwise $\{z, x\}$ is isolable in G). Suppose $n \geq 10$ and consider subcases for x and y.

First assume $x = x_0$. Then, as $\{z, x_0\}$ is isolable in $G\backslash\{x_k, y_k\}$, the vertex z is either adjacent with all vertices from $V\backslash\{x_k, y_k, x_0, z\}$ or not adjacent with all vertices from $V\backslash\{x_k, y_k, x_0, z\}$. Since $\{z, y\}$ is isolable in $G\backslash\{x_0, x_1\}$, The vertex y is also adjacent with all vertices from $V\backslash\{x_k, y_k, x_0, x_1, y, z\}$ or not adjacent with all of them. The only vertex with this property is y_2. Thus, $y = y_2$. Utilizing Lemma 2, we see that z is adjacent or is not adjacent with x_2, x_k, and y_k simultaneously. Finally, we find that $\{z, x_0\}$ is isolable in G, a contradiction.

The case $y = y_2$ is similar (we conclude that $x = x_0$ and have the same contradiction).

If $x \neq x_0$ and $y \neq y_2$ then x and z are connected with the same vertices in graph $G\backslash\{x_k, y_k\}$ and y and z are connected with the same vertices in graph $G\backslash\{x_0, x_1\}$ and as a consequence x and y are connected with the same vertices in graph $G\backslash\{x_0, x_1, x_k, y_k, z\}$.

If x and y belong to $V\backslash\{x_0, x_1, x_k, y_k\}$, then x and y are connected with the same vertices and $\{x, y\}$ is isolable in $G\backslash\{x_0, x_1, x_k, y_k, z\}$; but this graph has at least 5 vertices ant it is nonseparable (Lemma 5), a contradiction. Let x and y belong to $\{x_1, x_k, y_k\}$. If $x = x_1$ and $y = y_k(x_k)$ then y_2 connected with x_1 and not connected with $y_k(x_k)$. Contradiction. Let $x = x_1$ and $y \neq y_k(x_k)$. Then x_1 and y are connected with the same vertices in graph $G\backslash\{x_k, y_k, z\}$ and $\{x, y\}$ is isolable in this graph but this graph is nonseparable. Let $y = y_k(x_k)$. Then in graph $G\backslash\{x_0, x_1, x_k, y_k, z\} \cup \{y\}$ x and y have a same degree and lying in same part. Consequently, $x = y_{k-1}(x_{k-1})$. But then $\{z, y_k(x_k)\}$ is isolable in G.

Now consider the case $n = 8$. By Theorem 1, it is sufficient to consider the case when for some vertex z the graph $G\backslash\{z\}$ is a switching of G_7 or of its complement. Some switching of the complement of G_7 is a cycle. Thus, we can assume that G is the cycle and some vertex z, where we do not know how z is connected with the vertices of the cycle. Moreover, we can assume that the degree of z is 0, 1, 2, or 3 (we can achieve this by switching the vertex z). There are only 9 such graphs, up to isomorphism. It is straightforward to check that 2 of them are separable, while each of the remaining 7 graphs has a nonseparable subgraph of order 6. $\square$

Remark 2. *For $n = 6$ the statement of Theorem 1 is not true; indeed, all graphs of order 4 are separable, while nonseparable graphs of order 6 exist.*

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Evgeny Andreevich Bespalov
Novosibirsk State University,
Pirogova 2,
630090, Novosibirsk, Russia
Sobolev Institute of Mathematics,
pr. Koptyuga, 4,
630090, Novosibirsk, Russia
E-mail address: bespalovpes@mail.ru