S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 12, стр. 108–121 (2015) УДК 512.5
MSC 13A99

ON STRONG EQUIVALENCE THEOREM FOR ANSWER SET SEMANTICS WITH STRONG NEGATION

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Abstract. We discuss the problem of finding a minimal deductive base for paraconsistent and ordinary answer set semantics satisfying the strong equivalence theorem.

Keywords: answer set, paraconsistency, strong negation, strong equivalence of logic programs, Nelson logic

1. Introduction

In the seminal paper [13], David Pearce proved that the nonclassical logic of here-and-there with strong negation, often denoted by $\mathbf{N}_5$, is closely connected with answer set semantics. Namely, answer sets can be viewed as a certain kind of minimal $\mathbf{N}_5$-models. An important notion of strong equivalence of logic programs relative was introduced in [8]: two programs Π_1 and Π_2 are called strongly equivalent if for any program Π, $\Pi_1 \cup \Pi$ and $\Pi_2 \cup \Pi$ have the same answer sets. It turns out [8] that programs are strongly equivalent wrt answer set semantics if and only if they are equivalent viewed as propositional theories in $\mathbf{N}_5$. This shows that $\mathbf{N}_5$ can be used to reason about answer set programs, and $\mathbf{N}_5$-deduction may be relevant for program transformation and optimisation. In [11], similar results were proved for paraconsistent answer set (admitting inconsistent answer sets) and the logic $\mathbf{N}_9$, the paraconsistent version of the logic of here and there with strong negation.

The logic programs considered in [13, 11] involve two kinds of negation: traditional for logic programming default negation and strong negation based on the idea of explicit or constructible falsity. The concept of constructible falsity assumes that the falsity of atomic statements is given explicitly, and the falsity of complex statements

Odintsov, S.P., Makridin, Z.V., On strong equivalence theorem for answer set semantics with strong negation.

The work is supported by RFBR (project No. 12-01-00168-a).

Received November, 25, 2014, published February, 22, 2015.

is reduced to the truth or falsity of its constituents via a constructive procedure. This concept was introduced into logic by David Nelson [9] to overcome the non-constructivity of intuitionistic negation. Subsequently, his system of constructive logic with strong negation, traditionally denoted by **N3**, was axiomatized by Vorob'ev [18, 19] and studied algebraically by Helena Rasiowa [15, 16]. The concept of constructible falsity agrees well with that of paraconsistency. If the falsity of an atom p represented as $\sim p$, the strong negation of p, is given explicitly, we may admit that both p and $\sim p$ are true. The paraconsistent Nelson's logic **N4** is obtained by deleting the "explosive"axiom $\sim p \to (p \to q)$ from the axiomatics of **N3**.

In logic programming, the strong negation was introduced by Pearce and Wagner [14], and in the setting of answer set programming by Gelfond and Lifschitz [7]. Originally Gelfond and Lifschitz called the second negation *classical*, but it was also based on the idea of explicit falsity. Later Pearce [12] established that answer set inference involving this kind of negation could be viewed as an extension of Nelson's logic **N3**. The idea of explicit falsity makes it natural to consider answer sets inconsistent wrt strong negation producing in this way a paraconsistent answer set semantics (PAS). The study of PAS was initiated in [17] and continued in [1], [11], and in [10].

The relations between the intermediate logic HT of here-and-there and answer sets of programs with default negation, between the logic $\mathbf{N_5}$ and answer sets of programs with two negations (default and strong), and between the logic $\mathbf{N_9}$ and paraconsistent answer sets are very similar. This similarity is catched by the notion of a deductive base of a non-monotonic consequence relation suggested by Dietrich [5].

Let $\mathrel{|\!\sim}$ be an non-monotonic consequence relation, $\approx$ be the equivalence relation between sets of formulas associated with $\mathrel{|\!\sim}$. The equivalence $\approx$ can be defined syntactically: $\Gamma_1 \approx \Gamma_2$ iff $\Gamma_1 \mathrel{|\!\sim} \varphi$ for all $\varphi \in \Gamma_2$ and $\Gamma_2 \mathrel{|\!\sim} \varphi$ for all $\varphi \in \Gamma_1$, or in some other natural way. Further, let L be a monotonic logic defined in the same language with associated consequence relation $\vdash_L$ and equivalence relation $\equiv_L$. The logic L is a *deductive base for* $\mathrel{|\!\sim}$ if the following hold: 1) $\vdash_L \subseteq \mathrel{|\!\sim}$; 2) if $\Gamma \mathrel{|\!\sim} \varphi$ and $\varphi \vdash_L \psi$, then $\Gamma \mathrel{|\!\sim} \psi$; 3) if $\Gamma_1 \equiv_L \Gamma_2$, then $\Gamma_1 \approx \Gamma_2$. We say that L is a *strong* deductive base if it satisfies the additional condition:

$$\Gamma_1 \not\equiv_L \Gamma_2 \quad \Rightarrow \quad \text{there exists } \Gamma \text{ such that } \Gamma_1 \cup \Gamma \not\approx \Gamma_2 \cup \Gamma.$$

In other words, the deductive base L is strong if one can prove the strong equivalence theorem for L.

Each of the logics HT, $\mathbf{N_5}$, and $\mathbf{N_9}$ is a strong deductive base for the non-monotonic consequence relation determined by answer sets, answer sets with two negations, and respectively by paraconsistent answer sets. It is clear that properties 1)–3) are inherited by a smaller logic. Due to this reason it is interesting to find out a maximal deductive base of the respective non-monotonic consequence. It is know that HT is the maximal deductive base for answer set semantics in the class of intermediate logics, as well as that $\mathbf{N_5}$ is the maximal deductive base for the answer sets with two negations in the class of **N3**-extensions. In [10], it was proved that the logic $\mathbf{N_9}$ is a maximal deductive base for paraconsistent answer sets in the class of $\mathbf{N4}^\perp$-extensions, where $\mathbf{N4}^\perp$ is a version of paraconsistent Nelson's logic with additional falsity constant $\perp$, this constant satisfies the axiom $\perp \to p$ and allows to define the intuitionistic negation $\neg\varphi := \varphi \to \perp$.

The property to satisfy a strong equivalence theorem is inherited on the other hand by a greater logic, which leads to a problem of finding out the smallest strong deductive base. For the class of intermediate logics and logic programs without nested implication this problem was solved by De Jong and Hendriks in [4], where it was proved that the intermediate logic **KC** axiomatized modulo intuitionistic logic by the weak law of excluded middle $\neg p \vee \neg\neg p$ is the least intermediate logic satisfying the strong equivalence theorem for the class of logic programs with default negation and without nested implications. Our goal is to transfer this result to logic programs with default and strong negations and to two kinds of answer set semantics, the ordinary and the paraconsistent one. We obtain a natural and predictable result: the logic **N3KC**, the minimal conservative extension of **KC** in the class of **N3**-extensions, is the least strong deductive base for answer set semantics and programs without nested implications, whereas the logic **NKC**, the minimal conservative extension of **KC** in the class of $\mathbf{N4}^{\perp}$-extensions, is the least strong deductive base for PAS and programs without nested implications. Unfortunately, there is no metatheorem which allow to transfer such results directly from one class of programs to the other, and we have to do some technical work, in particular, to obtain a suitable Kripke style semantics for **N3KC** and **NKC**, and to prove finite model property for this logics.

2. Paraconsistent Nelson's logic and its extensions

We consider propositional language $\mathcal{L}^{\perp} := \{\wedge, \vee, \to, \sim, \perp\}$. The formulas of this language are constructed in a usual way from propositional variables or atoms belonging to a fixed set $Prop$ with the help of connectives in $\mathcal{L}^{\perp}$. The set of all formulas is denoted $For(\mathcal{L}^{\perp})$, the set of all atomic formulas, i.e. $Prop \cup \{\perp\}$ is denoted by At. By a *logic* in this language we mean a subset of $For(\mathcal{L}^{\perp})$ closed under the rules of substitution and *modus ponens*. If L is a logic, then $\mathcal{E}L$ denotes the lattice of all extensions of this logic.

With every logic L we associate a consequence relation $\vdash_L$ between sets of formulas and formulas. $\Gamma \vdash_L \varphi$ means that φ can be obtained from elements of $\Gamma \cup L$ via a finite number of applications of *modus ponens*.

Paraconsistent Nelson's logic $\mathbf{N4}^{\perp}$ is the least logic in the language $\mathcal{L}^{\perp}$ containing axioms:

(1) *Axioms of positive logic*:
- P1. $p \to (q \to p)$
- P2. $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$
- P3. $(p \wedge q) \to p$
- P4. $(p \wedge q) \to q$
- P5. $(p \to q) \to ((p \to r) \to (p \to (q \wedge r)))$
- P6. $p \to (p \vee q)$
- P7. $q \to (p \vee q)$
- P8. $(p \to r) \to ((q \to r) \to ((p \vee q) \to r))$

(2) *Strong negation axioms*:
- A1. $\sim\sim p \leftrightarrow p$
- A2. $\sim (p \vee q) \leftrightarrow (\sim p \wedge \sim q)$
- A3. $\sim (p \wedge q) \leftrightarrow (\sim p \vee \sim q)$
- A4. $\sim (p \to q) \leftrightarrow (p \wedge \sim q)$

(3) *Axioms for the constant "absurd"*:

A5. $\bot \to p$
A6. $\sim \bot$.

Notice that we can treat the definable connective $\neg\varphi := \varphi \to \bot$ as the intuitionistic negation, in a sense that $\mathbf{N4}^{\bot}$ is a conservative extension of the intuitionistic logic Int considered in the language $\{\vee, \wedge, \to \bot\}$.

The explosive Nelson's logic **N3** can be defined as $\mathbf{N3} = \mathbf{N4}^{\bot} + \{\sim p \to (p \to q)\}$.

Define the logics **N3KC** and **NKC** as extension of **N3** and respectively of $\mathbf{N4}^{\bot}$ via the axiom:

[K1.] $\neg p \vee \neg\neg p$.

We say that a formula φ is a *negative normal form* (*nnf*), if the symbol $\sim$ occurs in φ only in front of atomic formulas (in front of propositional variables and of constant $\bot$). It is known that every formula can be reduced over $\mathbf{N4}^{\bot}$ to a negative normal form, more exactly, for every $\varphi \in For(\mathcal{L}^{\bot})$ there is a *nnf* ψ such that $\varphi \leftrightarrow \psi \in \mathbf{N4}^{\bot}$.

An atomic formula or its strong negation is called a *literal.* We denote by Lit the set of all literals, i.e., $Lit = At \cup \{\sim p \mid p \in At\}$. It will be convenient to represent a set $\mathbf{S}$ of literals as a pair of sets of atoms (S^+, S^-), where S^+ consists of atoms that occur in $\mathbf{S}$ positively, and S^- contains atoms occurring in $\mathbf{S}$ negatively. More exactly, $S^+ = \mathbf{S} \cap At$ and $S^- = \{p \mid \sim p \in \mathbf{S}\}$. The symbol $\leq$ will denote the set theoretical inclusion of sets of literals. It is obvious, that $\mathbf{S_1} \leq \mathbf{S_2}$ iff $S_1^+ \subseteq S_2^+$ and $S_1^- \subseteq S_2^-$. The set $\mathbf{S}$ of literals is called *consistent* if $S^+ \cap S^- = \varnothing$.

Let $\leq$ be a preorder on W and $K \subseteq W$. We say that K is a *cone wrt* $\leq$ if $x \leq y$ and $x \in K$ imply $y \in K$ for all $x, y \in W$. The set of all cones wrt $\leq$ is denoted as $\langle W, \leq\rangle^+$.

Definition 2.1. A *frame* is a pair $\mathcal{W} = \langle W, \leq\rangle$ consisting of a non-empty set of possible worlds W and a preorder $\leq$ on it.

An $\mathbf{N4}^{\bot}$-*model* (over a frame $\mathcal{W} = \langle W, \leq\rangle$) is a triple $\mu = \langle \mathcal{W}, v^+, v^-\rangle$ consisting of a frame $\mathcal{W}$ and two valuations $v^+, v^- : Prop \to \langle W, \leq\rangle^+$.

If $\mu = \langle \mathcal{W}, v^+, v^-\rangle$ is an $\mathbf{N4}^{\bot}$-model such that $v^+(p) \cap v^-(p) = \varnothing$ for every $p \in Prop$, we say that μ is an **N3**-*model.*

We define the verification and falsification relations between worlds of $\mu = \langle \mathcal{W}, v^+, v^-\rangle$ and formulas of $\mathcal{L}^{\bot}$:

(1) $\mu, w \models^+ p \Leftrightarrow w \in v^+(p), \quad \mu, w \models^- p \Leftrightarrow w \in v^-(p)$;

(2) $\mu, w \models^+ \varphi \vee \psi \Leftrightarrow \mu, w \models^+ \varphi$ or $\mu, w \models^+ \psi$,
$\mu, w \models^- \varphi \vee \psi \Leftrightarrow \mu, w \models^- \varphi$ and $\mu, w \models^- \psi$;

(3) $\mu, w \models^+ \varphi \wedge \psi \Leftrightarrow \mu, w \models^+ \varphi$ and $\mu, w \models^+ \psi$,
$\mu, w \models^- \varphi \wedge \psi \Leftrightarrow \mu, w \models^- \varphi$ or $\mu, w \models^- \psi$;

(4) $\mu, w \models^+ \varphi \to \psi \Leftrightarrow \forall w' \geq w(\mu, w' \models^+ \varphi \Rightarrow \mu, w' \models^+ \psi)$,
$\mu, w \models^- \varphi \to \psi \Leftrightarrow \mu, w \models^+ \varphi$ and $\mu, w \models^- \psi$;

(5) $\mu, w \models^+ \sim \varphi \Leftrightarrow \mu, w \models^- \varphi, \quad \mu, w \models^- \sim \varphi \Leftrightarrow \mu, w \models^+ \varphi$;

(6) $\mu, w \not\models^+ \bot, \quad \mu, w \models^- \bot$.

For a formula φ, we put

$$\mu^+(\varphi) = \{w \in W \mid \mu, w \models^+ \varphi\} \text{ and } \mu^-(\varphi) = \{w \in W \mid \mu, w \models^- \varphi\}.$$

Lemma 2.2 (Monotonicity lemma)**.** *Let* $\mu = \langle W, \leq, v^+, v^- \rangle$ *be an* $\mathbf{N4}^{\perp}$*-model. Then for every* $\varphi \in For(\mathcal{L}^{\perp})$*, we have* $\mu^+(\varphi), \mu^-(\varphi) \in \langle W, \leq \rangle^+$.

A formula φ is *true in* the model μ, $\mu \models \varphi$, if $\mu, w \models^+ \varphi$ for all $w \in W$, i.e., $\mu^+(\varphi) = W$. Let $\mathcal{W} = \langle W, \leq \rangle$ be a frame and $\Gamma \cup \{\varphi\} \subseteq For(\mathcal{L}^{\perp})$. The relation $\Gamma \models_{\mathcal{W}} \varphi$ holds iff for every μ over $\mathcal{W}$ and every $w \in W$, the following implication holds

$$\forall \psi \in \Gamma \ (\mu, w \models \psi) \quad \Rightarrow \quad \mu, w \models \varphi.$$

If $\mathcal{K}$ is a class of frames, then $\Gamma \models_{\mathcal{K}} \varphi$ means that $\Gamma \models_{\mathcal{W}} \varphi$ for all $\mathcal{W} \in \mathcal{K}$.

For a class $\mathcal{K}$ of frames we also define the following sets of formulas:

$$\begin{aligned} L\mathcal{K} &= \{\varphi \in For(\mathcal{L}^{\perp}) \mid \mu \vDash \varphi \text{ for every } \mathbf{N4}^{\perp}\text{-model over every } \mathcal{W} \in \mathcal{K}\} \\ L^3\mathcal{K} &= \{\varphi \in For(\mathcal{L}^{\perp}) \mid \mu \vDash \varphi \text{ for every } \mathbf{N3}\text{-model over every } \mathcal{W} \in \mathcal{K}\} \end{aligned}$$

We write $L\mathcal{W}$ ($L^3\mathcal{W}$) instead of $L\{\mathcal{W}\}$ ($L^3\{\mathcal{W}\}$).

Let $\mu = \langle W, \leq, v^+, v^- \rangle$ be an $\mathbf{N4}^{\perp}$-model over $\mathcal{W} = \langle W, \leq \rangle$ and $K \in \mathcal{W}^+$. We define

$$\mathcal{W}^K := \langle K, \leq \cap K^2 \rangle \text{ and } \mu^K := \langle \mathcal{W}^K, v_K^+, v_K^- \rangle,$$

where $v_K^+(p) = v^+(p) \cap K$ and $v_K^-(p) = v^-(p) \cap K$ for all $p \in Prop$.

Lemma 2.3 (Generated submodel lemma)**.** *Let* $\mu = \langle W, \leq, v^+, v^- \rangle$ *be an* $\mathbf{N4}^{\perp}$*-model and* $K \in \mathcal{W}^+$*. For every* $\varphi \in For(\mathcal{L}^{\perp})$ *and* $x \in K$*, we have*

$$\mu, x \models^+ \varphi \quad \Leftrightarrow \quad \mu^K, x \models^+ \varphi,$$
$$\mu, x \models^- \varphi \quad \Leftrightarrow \quad \mu^K, x \models^- \varphi.$$

To obtain the semantical characterization of the logics **N3KC** and **NKC** we recall main features of the canonical model method. For a logic L extending $\mathbf{N4}^{\perp}$, the notions of L-theory and prime L-theory are defined in a standard way. An *L-theory* is set of formulas containing L and closed under modus ponens. A *prime L-theory* Γ is a non-trivial L-theory satisfying the disjunction property: if $\varphi \vee \psi \in \Gamma$, then $\varphi \in \Gamma$ or $\psi \in \Gamma$.

Lemma 2.4 (Extension lemma)**.** *For every* $L \in \mathcal{E}\mathbf{N4}^{\perp}$ *and a set of formulas* $\Sigma \cup \{\varphi\}$*, if* $\Sigma \nvdash_L \varphi$*, then there is a prime L-theory* $\Gamma \supseteq \Sigma$ *such that* $\Gamma \nvdash_L \varphi$.

Definition 2.5 (Canonical model)**.** *Let L be a logic, extending* $\mathbf{N4}^{\perp}$*. A* canonical L-frame *is a pair* $\mathcal{W}^L = \langle W^L, \leq^L \rangle$*, where*

(1) W^L *is a set of all prime L-theories,*
(2) $\Gamma \leq^L \Delta \Leftrightarrow \Gamma \subseteq \Delta$.

A *canonical L-model* μ^L is the canonical L-frame $\mathcal{W}^L$ with the valuations v^{+L} and v^{-L} defined as follows:

$$\Gamma \in v^{+L}(p) \quad \Leftrightarrow \quad p \in \Gamma;$$
$$\Gamma \in v^{-L}(p) \quad \Leftrightarrow \quad \sim p \in \Gamma$$

It is obvious that a canonical L-model is an $\mathbf{N4}^{\perp}$-model.

Lemma 2.6 (Canonical model lemma)**.** *Let L be a logic extending* $\mathbf{N4}^{\perp}$*. In canonical L-model* μ^L *for every* $\Gamma \in W^L$ *and* $\mathcal{L}^{\perp}$*-formula* φ*, the following equivalences hold:*

$$\mu^L, \Gamma \models^+ \varphi \quad \Leftrightarrow \quad \varphi \in \Gamma;$$
$$\mu^L, \Gamma \models^- \varphi \quad \Leftrightarrow \quad \sim \varphi \in \Gamma.$$

Recall that a logic L is *complete* wrt to a class $\mathcal{K}$ of frames (models) if

$$L = \{\varphi \mid \mathcal{W} \models \varphi(\mu \models \varphi) \text{ for all } \mathcal{W} \in \mathcal{K}(\mu \in \mathcal{K})\}.$$

A logic L is *strongly complete* wrt $\mathcal{K}$ if the relations $\vdash_L$ and $\models_{\mathcal{K}}$ coincide.

Denote the class of all frames as $\mathcal{K}_{all}$, the class of all partially ordered frames as $\mathcal{K}_{po}$, and, finally, the class of all directed partially ordered frames as $\mathcal{K}_{dpo}$. It is well known that $\mathbf{N4}^{\perp}$ is strongly complete wrt $\mathcal{K}_{all}$ and wrt $\mathcal{K}_{po}$, the logic $\mathbf{N3}$ is strongly complete wrt the class of all $\mathbf{N3}$-models and wrt to the class of all $\mathbf{N3}$-models over partially ordered frames. In particular, $\mathbf{N4}^{\perp} = L\mathcal{K}_{all} = L\mathcal{K}_{po}$ and $\mathbf{N3} = L^3\mathcal{K}_{all} = L^3\mathcal{K}_{po}$.

Theorem 2.7. *The logic* $\mathbf{NKC}$ *is strongly complete wrt* $\mathcal{K}_{dpo}$. *The logic* $\mathbf{N3KC}$ *is strongly complete wrt the class of all* $\mathbf{N3}$*-models over frames from* $\mathcal{K}_{dpo}$. *In particular, we have* $\mathbf{NKC} = L\mathcal{K}_{dpo}$ *and* $\mathbf{N3KC} = L^3\mathcal{K}_{dpo}$.

Proof. The fact that $\Gamma \vdash_{\mathbf{NKC}} \varphi$ implies $\Gamma \models_{\mathcal{W}} \varphi$ for $\mathcal{W} \in \mathcal{K}_{dpo}$ can be proved in a standard way by induction on the length of inference from Γ. In fact, it would be enough to prove that $\mathcal{W} \models \neg\varphi \vee \neg\neg\varphi$. Assume that a model μ over $\mathcal{W}$ and a world x of $\mathcal{W}$ are such that $\mu, x \not\models^+ \neg\varphi \vee \neg\neg\varphi$. This means that $\mu, y \models^+ \varphi$ and $\mu, z \models^+ \neg\varphi$ for some $y, z \geq x$. Since $\mathcal{W}$ is directed, there is a world u such that $y, z \leq u$. By Monotonicity lemma $\mu, u \models^+ \varphi \wedge \neg\varphi$, a contradiction.

Assume that $\Gamma \not\vdash_{\mathbf{NKC}} \varphi$. Let $\Gamma' \in W^{\mathbf{NKC}}$ be a prime extension of Γ such that $\varphi \notin \Gamma'$. By Canonical model lemma we have $\mu^{\mathbf{NKC}}, \Gamma' \models^+ \Gamma$ and $\mu^{\mathbf{NKC}}, \Gamma' \not\models^+ \varphi$. It remains to check that the submodel of $\mu^{\mathbf{NKC}}$ generated by the cone $\{\Gamma \in W^{\mathbf{NKC}} \mid \Gamma' \subseteq \Gamma\}$ is directed.

Notice that $\perp \notin \Gamma$ for all $\Gamma \in W^{\mathbf{NKC}}$. Indeed, if $\perp \in \Gamma$, then Γ is trivial by axiom A5. Let $\Gamma_1, \Gamma_2 \in W^{\mathbf{NKC}}$ be such that $\Gamma' \subseteq \Gamma_1, \Gamma' \subseteq \Gamma_2$. Assume that there is no prime theory $\Gamma_3 \in W^{\mathbf{NKC}}$ such that $\Gamma_1, \Gamma_2 \subseteq \Gamma_3$. By Extension lemma this means that $\Gamma_1 \cup \Gamma_2 \vdash_{\mathbf{NKC}} \perp$. Consequently, there is $\varphi \in \Gamma_2$ such that $\Gamma_1 \cup \{\varphi\} \vdash_{\mathbf{NKC}} \perp$. By Deduction theorem $\Gamma_1 \vdash_{\mathbf{NKC}} \neg\varphi$. Since Γ' is an $\mathbf{NKC}$-theory, we have $\neg\varphi \vee \neg\neg\varphi \in \Gamma'$. Since Γ' is prime, either $\neg\varphi \in \Gamma'$, or $\neg\neg\varphi \in \Gamma'$. However, the condition $\neg\varphi \in \Gamma'$ contradicts $\varphi \in \Gamma_2$, whereas $\neg\neg\varphi \in \Gamma'$ contradicts $\neg\varphi \in \Gamma_1$.

We have thus proved the completeness theorem for the logic $\mathbf{NKC}$. To prove the completeness result for $\mathbf{N3KC}$ it is enough to notice that for every logic L extending $\mathbf{N3}$, every prime L-theory is consistent wrt $\sim$. As a consequence the canonical model $\mu^{\mathbf{N3KC}}$ is an $\mathbf{N3}$-model.

□

Now we adopt filtration method for $\mathbf{N4}^{\perp}$-models. Let Φ be a set of $\mathcal{L}^{\perp}$-formulas closed under subformulas.

For a model $\mu = \langle W, \leq, v^+, v^- \rangle$, define on W an equivalence $\equiv_\Phi$ as follows:

$$x \equiv_\Phi y \Leftrightarrow \forall\varphi \in \Phi \left(\mu, x \models^+ \varphi \;\Leftrightarrow\; \mu, y \models^+ \varphi \text{ and } \mu, x \models^- \varphi \;\Leftrightarrow\; \mu, y \models^- \varphi\right).$$

The coset of $x \in W$ wrt $\equiv_\Phi$ is denoted as $[x]_\Phi$, $[x]_\Phi = \{y \in W \mid x \equiv_\Phi y\}$.

A model $\mu' = \langle W', \leq', v'^+, v'^- \rangle$ is called a *filtration* of $\mu = \langle W, \leq, v^+, v^- \rangle$ *wrt* Φ if the following holds:

(1) $W' = \{[x]_\Phi \mid x \in W\}$;
(2) $v'^+(p) = \{[x]_\Phi \mid x \in v^+(p) \text{ and } p \in \Phi\}, p \in Prop$;
(3) $v'^-(p) = \{[x]_\Phi \mid x \in v^-(p) \text{ and } p \in \Phi\}, p \in Prop$;
(4) $x \leq y$ implies $[x]_\Phi \leq' [y]_\Phi$ for all $x, y \in W$;

(5) $[x]_\Phi \leq' [y]_\Phi$ implies
$\forall\varphi \in \Phi \left(\mu, x \models^+ \varphi \;\Rightarrow\; \mu, y \models^+ \varphi \text{ and } \mu, x \models^- \varphi \;\Rightarrow\; \mu, y \models^- \varphi\right),$
for all $x, y \in W$.

Notice that if we define the relation $\leq^g$ as: $[x]_\Phi \leq^g [y]_\Phi$ iff

$$\forall\varphi \in \Phi \left(\mu, x \models^+ \varphi \;\Rightarrow\; \mu, y \models^+ \varphi \text{ and } \mu, x \models^- \varphi \;\Rightarrow\; \mu, y \models^- \varphi\right),$$

we obtain that $\langle W', \leq^g, v'^+, v'^-\rangle$ is a filtration of μ wrt Φ. Thus, at least one filtration exists for every μ and Φ.

Lemma 2.8 (Filtration lemma). *Let $\mu = \langle W, \leq, v^+, v^-\rangle$ be an $\mathbf{N4}^\perp$-model, Φ a set of $\mathcal{L}^\perp$-formulas closed under subformulas. Let $\mu' = \langle W', \leq', v'^+, v'^-\rangle$ be a filtration of μ wrt Φ. Then for every $\varphi \in \Phi$ and $x \in W$ the following equivalences hold*

$$\mu, x \models^+ \varphi \;\Leftrightarrow\; \mu', [x]_\Phi \models^+ \varphi,$$
$$\mu, x \models^- \varphi \;\Leftrightarrow\; \mu', [x]_\Phi \models^- \varphi,$$

This statement can be proved in essentially the same way as for intuitionistic logic.

Theorem 2.9. *We have $\mathbf{NKC} = L\mathcal{K}_{fing}$ and $\mathbf{N3KC} = L^3\mathcal{K}_{fing}$, where $\mathcal{K}_{fing}$ denotes the class of all finite partialy ordered frames with the greatest element.*

Proof. If $\varphi \in \mathbf{NKC}$, then φ is true in all directed frames, in particular in all finite frames with the greatest element.

Assume that $\varphi \notin \mathbf{NKC}$. Then there is a model $\mu = \langle W, \leq, v^+, v^-\rangle$ over a directed frame and a world $x \in W$ such that $\mu, x \not\models^+ \varphi$. Let Φ be the set of all subformulas of φ. Consider some filtration $\mu' = \langle W', \leq', v'^+, v'^-\rangle$ of μ wrt Φ. Then by Filtration lemma we have $\mu', [x]_\Phi \not\models^+ \varphi$. It follows from Item 4 of the filtration definition that μ' is directed. Since Φ is finite, the set W' is finite too. It remains to notice that a finite directed frame contains the greatest element.

The second statement follows from the obvious observation that a filtration of an $\mathbf{N3}$-model is again an $\mathbf{N3}$-model.

□

We have thus proved the finite model property for the logics $\mathbf{NKC}$ and $\mathbf{N3KC}$, which implies, in particular, that these logics are decidable.

Let us consider the frame $\mathcal{W}^{HT} = \langle\{h, t\}, \leqslant\rangle$ of the logic of here-and-there, where $h \leqslant t$, and the extensions of $\mathbf{N4}^\perp$ and of $\mathbf{N3}$ determined by this frame:

$$\mathbf{N_9} = L\mathcal{W}^{HT}, \quad \mathbf{N_5} = L^3\mathcal{W}^{HT}.$$

The choice of the notation is explained by the fact, that the logic $\mathbf{N}_x$ can be determined by an x-element algebra, $x \in \{5, 9\}$, see, e.g. [10]. A model $\mu = \langle\mathcal{W}^{HT}, v^+, v^-\rangle$ over $\mathcal{W}^{HT}$ is called an $\mathbf{N_9}$-model. Define the following four sets of atoms:

$$H^+ = \{p \mid \mu, h \vDash^+ p\},\; H^- = \{p \mid \mu, h \vDash^- p\},$$
$$T^+ = \{p \mid \mu, t \vDash^+ p\},\; T^- = \{p \mid \mu, t \vDash^- p\}.$$

In view of persistency condition on v^+ and v^- we have $H^+ \subseteq T^+$ and $H^- \subseteq T^-$. It is clear that every $\mathbf{N_9}$-model is uniquely determined by this quadruple of sets of atoms $\langle H^+, H^-, T^+, T^-\rangle$ and that every quadruple $\langle H^+, H^-, T^+, T^-\rangle$ with $H^+ \subseteq T^+$ and $H^- \subseteq T^-$ determines an $\mathbf{N_9}$-model. In this case the set H^+ is interpreted as a set of atoms which are true in the world h, the set H^- is interpreted as a set of atoms which are false in the world h, and similarly for the world t. In what follows it will be convenient to divide this quadruple into two pairs $\mathbf{H} = (H^+, H^-)$ and $\mathbf{T} = (T^+, T^-)$, so $\langle \mathbf{H}, \mathbf{T} \rangle = \langle (H^+, H^-), (T^+, T^-) \rangle$. In this way, we identified $\mathbf{N_9}$-models with pairs of sets of literals $\langle \mathbf{H}, \mathbf{T} \rangle$ such that $\mathbf{H} \leq \mathbf{T}$.

An $\mathbf{N_5}$-model is an $\mathbf{N_9}$-model, which is simultaneously an **N3**-model. In this case, we have $v^+(p) \cap v^-(p) = \varnothing$ for all p, which is equivalent to $H^+ \cap H^- = \varnothing$ and $T^+ \cap T^- = \varnothing$. Thus, $\mathbf{N_5}$-models are pairs of consistent sets of literals.

A one-element frame $\mathcal{W}^H = \langle \{h\}, \leqslant \rangle$, where $h \leqslant h$ will determine the logics $\mathbf{B4} = L\mathcal{W}^H$ and $\mathbf{B3} = L^3\mathcal{W}^H$. Obviously, models of logics **B4** and **B3** can be identified with sets of literals, which are arbitrary for **B4** and consistent in case of **B3**. Recall that the logics **B4** and **B3** are extensions of well known four- and three-valued Belnap-Dunn logics [3, 2, 6] via connectives $\to$ and $\bot$.

In what follows it will be convenient for us to identify a set $\mathbf{T}$ of literals with the pair $\langle \mathbf{T}, \mathbf{T} \rangle$ and consider in this way **B4**-models as a proper subset of the set of $\mathbf{N_9}$-models.

With every world $w \in W$ of an arbitrary $\mathbf{N4}^\perp$-model $\mu = \langle W, \leq, v^+, v^- \rangle$ we associate a set of literals verified in this word:

$$lit_\mu(w) = \{p \mid \mu, w \models^+ p\} \cup \{\sim p \mid \mu, w \models^- p\}.$$

Naturally, $lit_\mu(w)$ can be considered as a **B4**-model. In what follows we will also use the denotation $\langle w \rangle_\mu$ for this **B4**-model. The lower index μ will be omitted if it does not lead to a confution.

3. Paraconsistent anwer set semantics

We will consider two versions of answer set semantics for logic programs with two kinds of negation [7], the ordinary one, which admits only consistent sets of literals as answer sets, and the paraconsistent version, which takes into consideration also inconsistent answer sets.

Recall that the strong negation is denoted by '$\sim$' and the second, default negation, usually written as '*not*' will be denoted by '$\neg$'. The rules of disjunctive logic programs will be considered as formulas having the following form

$$L_1 \wedge \ldots \wedge L_m \wedge \neg L_{m+1} \wedge \ldots \wedge \neg L_n \to K_1 \vee \ldots \vee K_k, \tag{1}$$

where $L_i, K_j \in Lit$ and we may have $m = n$ and m or n may be zero. A *disjunctive logic program* Π is a set of such formulas, a program Π is *normal* if $k = 1$ for all rules in Π. The rules of disjunctive programs can be considered as formulas of the language $\mathcal{L}^\perp$, in which case $\neg L_i$ is treated as an abbreviation for $L_i \to \bot$. It will be convenient in this section to consider a definitional extension of the logic $\mathbf{N4}^\perp$ via intuitionistic negation $\neg$ and the constant $\top$ ($\top :=\sim \bot$). So, in this section $\mathcal{L}^\perp := \{\wedge, \vee, \to, \sim, \neg, \bot, \top\}$.

A set $\mathbf{M}$ of literals is a model of a disjunctive program Π if $\mathbf{M}$ is a **B4**-model of Π.

The *Gelfond and Lifschitz reduct (GL-reduct)* $\Pi^{\mathbf{M}}$ of a disjunctive program Π with respect to a set of literals $\mathbf{M}$ is a disjunctive program obtained from Π in two steps. First, we exclude from Π all rules containing $\neg L_i$ with $L_i \in \mathbf{M}$. Second, from the rest of rules we delete all conjunctive terms of the form $\neg L_i$. A set of literals $\mathbf{M}$ is an *answer set* of a program Π, if $\mathbf{M}$ is $\leq$-minimal among all models of the GL-reduct $\Pi^{\mathbf{M}}$. In this way we have defined a paraconsistent answer set semantics

(PAS). The only difference of PAS from the ordinary answer set semantics is that the latter additionally assumes that an answer set must be consistent.

The next statement follows readily from the definition of GL-reduct.

Lemma 3.1. *For every disjunctive program* Π *and set of literals* $\mathbf{M}$*, the following equivalence holds:*

$$\mathbf{M} \models \Pi \quad \textit{iff} \quad \mathbf{M} \models \Pi^{\mathbf{M}}.$$

An $\mathbf{N_9}$-model of the form $\langle \mathbf{T}, \mathbf{T} \rangle$ is called *total.* The relation $\trianglelefteq$ on $\mathbf{N_9}$-models is defined as follows: $\langle \mathbf{H_1}, \mathbf{T_1} \rangle \trianglelefteq \langle \mathbf{H_2}, \mathbf{T_2} \rangle$ iff $\mathbf{H_1} \leq \mathbf{H_2}$ and $\mathbf{T_1} = \mathbf{T_2}$.

Let Γ be an arbitrary subset of $For(\mathcal{L}^{\perp})$. A total model $\langle \mathbf{T}, \mathbf{T} \rangle$ is called an *equilibrium model* of the set of formulas Γ, if $\langle \mathbf{T}, \mathbf{T} \rangle$ is an $\mathbf{N_9}$-model of Γ and it is $\trianglelefteq$-minimal in the class of all $\mathbf{N_9}$-models of Γ. In other worlds, $\langle \mathbf{T}, \mathbf{T} \rangle$ is an equilibrium model of Γ iff $\langle \mathbf{T}, \mathbf{T} \rangle \models_{\mathbf{N_9}} \Gamma$ and there is no $\mathbf{H}$ such that $\mathbf{H} \neq \mathbf{T}$ and $\langle \mathbf{H}, \mathbf{T} \rangle \models_{\mathbf{N_9}} \Gamma$.

A set of equilibrium models of the set of formulas Γ is denoted by $\mathcal{E}l(\Gamma)$, a set of consistent equilibrium models of Γ is denoted as $\mathcal{E}l^3(\Gamma)$.

Since the equilibrium model of an arbitrary set of formulas is total it can be identified with a set of literals. The following fact is important to establish the connection between (consistent) equilibrium models and (consistent) answer sets in case of disjunctive programs.

Lemma 3.2. *Let* Π *be a disjunctive program and* $\langle \mathbf{H}, \mathbf{T} \rangle$ *an* $\mathbf{N_9}$*-model. The following equivalence holds:*

$$\langle \mathbf{H}, \mathbf{T} \rangle \models \Pi \quad \textit{iff} \quad \mathbf{T} \models \Pi^{\mathbf{T}} \textit{ and } \mathbf{H} \models \Pi^{\mathbf{T}}.$$

The next statement was proved in [13] for consistent answer sets and in [11] for arbitrary answer sets.

Theorem 3.3. [13, 11] *For a disjunctive program* Π*, a set* $\mathbf{T}$ *of literals is a (consistent) answer set of* Π *iff the* $\mathbf{N_9}$*-model (*$\mathbf{N_5}$*-model)* $\langle \mathbf{T}, \mathbf{T} \rangle$ *is an equilibrium model of* Π*.*

This statement was generalized in [4] for very general class of programs (without strong negation), namely programs without nested implication. We extend this result to programs with strong negation.

Let $\mathbf{M}$ be a set of literals and φ a *nnf.* A formula $\varphi^{\mathbf{M}}$ is defined by induction as follows:

(1) $\varphi^{\mathbf{M}} = \varphi$, *if* φ *is a literal*
(2) $(\varphi \circ \psi)^{\mathbf{M}} = \varphi^{\mathbf{M}} \circ \psi^{\mathbf{M}}$, *where* $\circ \in \{\vee, \wedge, \rightarrow\}$
(3) $(\neg\varphi)^{\mathbf{M}} = \begin{cases} \perp, \text{ if } \mathbf{M} \models \varphi \\ \top, \text{ if } \mathbf{M} \not\models \varphi. \end{cases}$

For a formula φ, which is not a *nnf*, we put $\varphi^{\mathbf{M}} = (\bar{\varphi})^{\mathbf{M}}$, where $\overline{\varphi}$ is an *nnf* of φ. We assume that some algorithm of reducing a formula to a negative normal form is fixed. If Π is a set of formulas, then $\Pi^{\mathbf{M}} = \{\varphi^{\mathbf{M}} \mid \varphi \in \Pi\}$.

A set $\mathbf{M}$ of literals is an *answer set* of Π if $\mathbf{M}$ is $\leq$-minimal among all $\mathbf{B4}$-models of the GL-reduct $\Pi^{\mathbf{M}}$.

Let $For^{\star}_{\mathcal{L}}$ denote a set of formulas, which can be constructed from literals with the help of connectives of the language $\mathcal{L}$.

Lemma 3.4. *Let $\mathcal{L} = \{\wedge, \vee, \bot, \top\}$. The for a world w of an $\mathbf{N4}^{\bot}$-model $\mu = \langle W, \leq, v^+, v^- \rangle$ and $\varphi, \psi \in For^{\star}_{\mathcal{L}}$*

$$\mu, w \models^{\sigma} \varphi \textit{ iff } \langle w \rangle_{\mu} \models^{\sigma} \varphi; \tag{2}$$

$$\textit{if } \mu, w \models^{\sigma} \varphi \to \psi \text{ , } \textit{then } \langle w \rangle_{\mu} \models^{\sigma} \varphi \to \psi, \tag{3}$$

where $\sigma \in \{+, -\}$.

Proof. The equivalence (2) readily follows from the fact that only one world is involved in the definition of verification or falsification of a formula from $For^{\star}_{\mathcal{L}}$.

To prove (3) we consider first the case $\sigma = +$. If $\langle w \rangle \not\models^{+} \varphi \to \psi$, then $\langle w \rangle \models^{+} \varphi$ and $\langle w \rangle \not\models^{+} \psi$. It follows by (2) that $\mu, w \models^{+} \varphi$ and $\mu, w \not\models^{+} \psi$, whence $\mu, w \not\models^{+} \varphi \to \psi$.

Consider the case $\sigma = -$. If $\langle w \rangle \not\models^{-} \varphi \to \psi$, then either $\langle w \rangle \not\models^{+} \varphi$ or $\langle w \rangle \not\models^{-} \psi$. According to (1) we have $\mu, w \not\models^{+} \varphi$ or $\mu, w \not\models^{-} \psi$, whence $\mu, w \not\models^{-} \varphi \to \psi$. □

Lemma 3.5. *Let $\mathcal{L} = \{\wedge, \vee, \bot, \top\}$, $\varphi, \psi \in For^{\star}_{\mathcal{L}}$. For an arbitrary $\mathbf{N_9}$-model $\langle \mathbf{H}, \mathbf{T} \rangle$, the relation $\langle \mathbf{H}, \mathbf{T} \rangle \models \varphi \to \psi$ holds, if and only if $\mathbf{T} \models \varphi \to \psi$ and $\mathbf{H} \models \varphi \to \psi$.*

Proof. The statement $\langle \mathbf{H}, \mathbf{T} \rangle \models \varphi \to \psi$ is equivalent by definition to the conjunction of two implications: $\langle \mathbf{H}, \mathbf{T} \rangle, h \models^{+} \varphi$ implies $\langle \mathbf{H}, \mathbf{T} \rangle, h \models^{+} \psi$, and $\langle \mathbf{H}, \mathbf{T} \rangle, t \models^{+} \varphi$ implies $\langle \mathbf{H}, \mathbf{T} \rangle, t \models^{+} \psi$. This is equivalent by Lemma 3.4 to implications: $\mathbf{H} \models^{+} \varphi$ implies $\mathbf{H} \models^{+} \psi$, and $\mathbf{T} \models^{+} \varphi$ implies $\mathbf{T} \models^{+} \psi$, which is equivalent in turn to the desired conjunction: $\mathbf{T} \models \varphi \to \psi$ and $\mathbf{H} \models \varphi \to \psi$. □

Lemma 3.6. *Let $\varphi, \psi \in For_{\mathcal{L}^-}$, where $\mathcal{L}^- := \{\wedge, \vee, \sim, \neg, \bot, \top\}$ is a language without implication. For an arbitrary $\mathbf{N_9}$-model $\langle \mathbf{H}, \mathbf{T} \rangle$, we have $\langle \mathbf{H}, \mathbf{T} \rangle \models \varphi \to \psi$ iff $\langle \mathbf{H}, \mathbf{T} \rangle \models (\varphi \to \psi)^{\mathbf{T}}$.*

Proof. By the properties of *nnf*'s we have $\langle \mathbf{H}, \mathbf{T} \rangle \models \varphi \to \psi \;\Leftrightarrow\; \langle \mathbf{H}, \mathbf{T} \rangle \models \overline{\varphi} \to \overline{\psi}$. So we may assume that both φ and ψ are *nnf*'s. Again the statement $\langle \mathbf{H}, \mathbf{T} \rangle \models \varphi \to \psi$ is equivalent by definition to implications: $\langle \mathbf{H}, \mathbf{T} \rangle, h \models^{+} \varphi$ implies $\langle \mathbf{H}, \mathbf{T} \rangle, h \models^{+} \psi$, and $\langle \mathbf{H}, \mathbf{T} \rangle, t \models^{+} \varphi$ implies $\langle \mathbf{H}, \mathbf{T} \rangle, t \models^{+} \psi$. The same holds for $\langle \mathbf{H}, \mathbf{T} \rangle \models \varphi^{\mathbf{T}} \to \psi^{\mathbf{T}}$. Since *nnf*'s of formulas from $For_{\mathcal{L}^-}$ belong to $For^{*}_{\mathcal{L}^-}$, the conclusion of the lemma will follow from the fact that the equivalences:

$$\langle \mathbf{H}, \mathbf{T} \rangle, h \models^{+} \chi \;\text{ iff }\; \langle \mathbf{H}, \mathbf{T} \rangle, h \models^{+} \chi^{\mathbf{T}},$$
$$\langle \mathbf{H}, \mathbf{T} \rangle, t \models^{+} \chi \;\text{ iff }\; \langle \mathbf{H}, \mathbf{T} \rangle, t \models^{+} \chi^{\mathbf{T}}$$

are true for any formula $\chi \in For^{*}(\mathcal{L}^-)$.

These equivalences can be proved by an easy induction on the structure of $\chi \in For^{*}(\mathcal{L}^-)$. The basis of induction is obvious since the GL-reduct of a literal is the same literal. The only non-trivial case we have to consider is the case of intuitionistic negation. Since t is the maximal world of the model $\langle \mathbf{H}, \mathbf{T} \rangle$, for $x \in \{h, t\}$ we have $\langle \mathbf{H}, \mathbf{T} \rangle, x \models^{+} \neg\varphi$ iff $\langle \mathbf{H}, \mathbf{T} \rangle, t \not\models^{+} \varphi$. By the induction hypothesis the latter is equivalent to $\langle \mathbf{H}, \mathbf{T} \rangle, t \not\models^{+} \varphi^{T}$. The GL-reduct $(\neg\varphi)^{\mathbf{T}}$ equals $\top$ iff $\mathbf{T} \not\models^{+} \varphi^{T}$, otherwise it is equal $\bot$. By Lemma 3.4 $\mathbf{T} \not\models^{+} \varphi^{T}$ iff $\langle \mathbf{H}, \mathbf{T} \rangle, t \not\models^{+} \varphi^{T}$, which implies the desired equivalence. □

Theorem 3.7. *Let* $\Pi \subset L = \{\varphi \to \psi : \varphi, \psi \in For_{\mathcal{L}^-}\}$*, where* $\mathcal{L}^- := \{\wedge, \vee, \sim, \neg, \bot, \top\}$*, and let* $\mathbf{T}$ *be a set of literals.* $\mathbf{T}$ *is a (consistent) answer set of* Π *iff the* $\mathbf{N_9}$*-model (*$\mathbf{N_5}$*-model)* $\langle \mathbf{T}, \mathbf{T} \rangle$ *is an equilibrium model of* Π.

Proof. Let $\mathbf{T}$ be an answer set for Π. By definition of answer set $\mathbf{T} \models \Pi^{\mathbf{T}}$. Let $\mathbf{H} \leq \mathbf{T}$, assume that $\langle \mathbf{H}, \mathbf{T} \rangle \models \Pi$. By Lemma 3.6 we have $\langle \mathbf{H}, \mathbf{T} \rangle \models \Pi^{\mathbf{T}}$, from which we conclude by Lemma 3.5 that $\mathbf{H} \models \Pi^{\mathbf{T}}$. From the definition of answer sets it follows that $\mathbf{H} = \mathbf{T}$. We have thus proved that $\langle \mathbf{T}, \mathbf{T} \rangle$ is an equilibrium model of Π.

If $\langle \mathbf{T}, \mathbf{T} \rangle$ is an equilibrium model of Π, then $\langle \mathbf{T}, \mathbf{T} \rangle \models \Pi^{\mathbf{T}}$ by Lemma 3.6, which implies $\mathbf{T} \models \Pi^{\mathbf{T}}$ by Lemma 3.5. Assume that $\mathbf{T}$ is not an answer set of Π, i.e., there is $\mathbf{H} \leq \mathbf{T}$ such that $\mathbf{H} \neq \mathbf{T}$ and $\mathbf{H} \models \Pi^{\mathbf{T}}$. By Lemma 3.5 we conclude that $\langle \mathbf{H}, \mathbf{T} \rangle \models \Pi^{\mathbf{T}}$, and that $\langle \mathbf{H}, \mathbf{T} \rangle \models \Pi$ by Lemma 3.6, which contradicts the assumption that $\langle \mathbf{T}, \mathbf{T} \rangle$ is an equilibrium model of Π.

□

4. Strong equivalence of logic programs

Let L be a set of formulas in one or another language. By a program over L we mean a subset of L.

Programs Π_1 and Π_2 over L are *(consistently) strongly equivalent* (in L) if for every program Π over L, the programs $\Pi_1 \cup \Pi$ and $\Pi_2 \cup \Pi$ have the same (consistent) answer sets.

The following statement was proved in [8] for consistent strong equivalence, and in [11] for paraconsistent answer set semantics.

Theorem 4.1. *Let* Π_1 *and* Π_2 *be programs over* L*, where* L *contains all normal programs. The programs* Π_1 *and* Π_2 *are (consistently) strongly equivalent in* L*, if and only if they are equivalent in the logic* $\mathbf{N_9}$ *(*$\mathbf{N_5}$*),*

$$\Pi_1 \equiv_{\mathbf{N_9}} \Pi_2 \quad (\Pi_1 \equiv_{\mathbf{N_5}} \Pi_2).$$

Lemma 4.2. *Let* $L = \{\varphi \to \psi : \varphi, \psi \in For_{\mathcal{L}^-}\}$, $\Gamma \subseteq L$, $\chi \in L$*. Then*

$$\Gamma \vdash_{\mathbf{NKC}} \chi \;\Leftrightarrow\; \Gamma \vdash_{\mathbf{N_9}} \chi,$$

$$\Gamma \vdash_{\mathbf{N3KC}} \chi \;\Leftrightarrow\; \Gamma \vdash_{\mathbf{N_5}} \chi.$$

Proof. The implications from left to right are obvious due to inclusions $\mathbf{NKC} \subseteq \mathbf{N_9}$ and $\mathbf{N3KC} \subseteq \mathbf{N_5}$.

Now let $G \nvdash_{\mathbf{NKC}} \varphi \to \psi$. By the completeness theorem for $\mathbf{NKC}$ there is an $\mathbf{NKC}$-model $\mu = \langle W, \leq, v^+, v^-, \rangle$ with the greatest element t and there is a world $h \in W$ such that $h \models^+ \Gamma$, $h \models^+ \varphi$, and $h \not\models^+ \psi$.

Let us consider an $\mathbf{N_9}$-model $\langle \mathbf{H}, \mathbf{T} \rangle = \langle lit_\mu(h), lit_\mu(t) \rangle$.

We prove several statements about an $\mathbf{N_9}$-model $\langle \mathbf{H}, \mathbf{T} \rangle$, which will imply the conclusion of the lemma:

1. For all $\chi \in For_{\mathcal{L}^-}$, we have $\mu, t \models^\sigma \chi \;\Leftrightarrow\; \langle \mathbf{H}, \mathbf{T} \rangle, t \models^\sigma \chi \;\Leftrightarrow\; \mathbf{T}, t \models^\sigma \chi$.
2. For all $\chi \in For_{\mathcal{L}^-}$, we have $\mu, h \models^\sigma \chi \;\Leftrightarrow\; \langle \mathbf{H}, \mathbf{T} \rangle, h \models^\sigma \chi$, where $\sigma \in \{+, -\}$.
3. For all $\chi \in L$, the following implication holds: $\mu, h \models^+ \chi \;\Rightarrow\; \langle \mathbf{H}, \mathbf{T} \rangle \models \chi$.

The first statement follows from the facts that t is the greatest world in both models by Monotonicity lemma.

The second statement follows from the first one and the fact that only two worlds, h and t, are involved in the definition of verification and falsification of formulas in the world h of μ. The definition of the model $\langle \mathbf{H}, \mathbf{T}\rangle$ gives the base of induction, the only non-trivial case of the intuitionistic implication can be treated as follows. For the case of verification we have the following chain of equivalences:

$$\mu, h \models^+ \neg\chi \quad \Leftrightarrow \quad \mu, t \not\models^+ \chi \quad \Leftrightarrow \quad \langle \mathbf{H}, \mathbf{T}\rangle, t \not\models^+ \chi \quad \Leftrightarrow \quad \langle \mathbf{H}, \mathbf{T}\rangle, h \models^+ \neg\chi.$$

Here the second equivalence is due to statement 1. For the case of falsification we have:

$$\mu, h \models^- \neg\chi \quad \Leftrightarrow \quad \mu, h \models^+ \chi \quad \Leftrightarrow \quad \langle \mathbf{H}, \mathbf{T}\rangle, h \models^+ \chi \quad \Leftrightarrow \quad \langle \mathbf{H}, \mathbf{T}\rangle, h \models^- \neg\chi.$$

In this case the first equivalence follows from the definition of intuitionistic negation, $\neg\chi = \chi \to \bot$, where as the second one follows by induction hypothesis.

The last statement easily follows from the previous ones.

According to statement 3 we have $\langle \mathbf{H}, \mathbf{T}\rangle \models \Gamma$, at the same time it follows from statement 2 that $\langle \mathbf{H}, \mathbf{T}\rangle, h \models^+ \varphi$ and $\langle \mathbf{H}, \mathbf{T}\rangle, h \not\models^+ \psi$. We have thus proved the first equivalence.

The second one can be proved in exactly the same way. It is enough to notice that from an **N3KC**-model μ we obtain an $\mathbf{N}_5$-model $\langle \mathbf{H}, \mathbf{T}\rangle$.

□

Theorem 4.3. *Let* Π_1 *and* Π_2 *are programs over* $L = \{\varphi \to \psi : \varphi, \psi \in For_{\mathcal{L}^-}\}$. *Then* Π_1 *and* Π_2 *are (consistently) strongly equivalent in* L, *if and only if* $\Pi_1 \equiv_{\mathbf{NKC}} \Pi_2$ *(*$\Pi_1 \equiv_{\mathbf{N3KC}} \Pi_2$*). Moreover,* **NKC** *(***N3KC***) is the least extension of* $\mathbf{N4}^\perp$, *satisfying the (consistent) strong equivalence theorem for programs over* L.

Proof. The fact that logic **NKC** (**N3KC**) satisfies the (consistent) strong equivalence theorem for logic programs over L follows from the previous lemma, Theorem 4.1, and the fact that L obviously contains normal programs.

It was proved in [4] (see Lemma 24) that the superintuitionistic logic **KC** can be represented as

$$Int + \{(((p \wedge r) \to t) \wedge (\neg p \to q) \wedge (\neg r \to q)) \to (\neg t \to q)\}.$$

This means that the formulas $A := (((p \wedge r) \to t) \wedge (\neg p \to q) \wedge (\neg r \to q)) \to (\neg t \to q)$ and the standard **KC** axiom $\neg p \vee \neg\neg p$ are equivalent over Int. Consequently, these formulas will be equivalent over logics $\mathbf{N4}^\perp$ and **N3** extending Int. So we can represent the logic **NKC** and **N3KC** as $\mathbf{N4}^\perp + \{A\}$ and $\mathbf{N3} + \{A\}$ respectively.

We just noticed that the (consistent) strong equivalence theorem holds for **NKC** (**N3KC**). It follows from the new axiomatization of these logics that the programs $\Pi_1 = \{(p \wedge r) \to t,\ \neg p \to q,\ \neg r \to q,\ \neg t \to q\}$ and $\Pi_2 = \{(p \wedge r) \to t,\ \neg p \to q,\ \neg r \to q\}$ are (consistently) strongly equivalent. This means that Π_1 and Π_2 must be equivalent in every logic $L \in \mathcal{E}\mathbf{N4}^\perp$ satisfying the (consistent) strong equivalence theorem. In other word, every logic L satisfying the (consistent) strong equivalence theorem must contain the logic **NKC**.

Further, since $\sim p \to (p \to q) \in$ **N3KC** and **N3KC** satisfies the consistent strong equivalence theorem, this means that the programs $\{\sim p, p, q\}$ and $\{\sim p, p\}$ are consistently strongly equivalent. Thus, these programs are equivalent in every $\mathbf{N4}^\perp$-extension, satisfying consistent strong equivalence theorem. We have thus proved that every $\mathbf{N4}^\perp$-extension, satisfying consistent strong equivalence theorem must contain the logic **N3KC**.

□

Notice that all programs mentioned in the proof are normal. This allows us to infer the following

Corollary 4.4. *Let* Π_1 *and* Π_2 *are normal programs with strong negation. Then* Π_1 *and* Π_2 *are (consistently) strongly equivalent in the class of normal programs with strong negation, if and only if* $\Pi_1 \equiv_{\mathbf{NKC}} \Pi_2$ *(*$\Pi_1 \equiv_{\mathbf{N3KC}} \Pi_2$*). Moreover,* **NKC** *(***N3KC***) is the least extension of* $\mathbf{N4}^{\perp}$*, satisfying the (consistent) strong equivalence theorem for normal programs with strong negation.*

References

[1] J. Alcântara, C. Damásio, L. M. Pereira, *A declarative characterisation of disjunctive paraconsistent answer sets.* in R. López de Mántaras, L. Saitta (eds.), Proc. of ECAI 2004, IOS Press, 2004, 951–952.

[2] N.D. Belnap, *How a computer should think.* in:, G. Ryle (ed.) Contemporary Aspects of Philosophy, Oriel Press, Stocksfield, 30–56, 1976.

[3] N.D. Belnap, *A Useful Four-Valued Logic.* in: J.M. Dunn, G. Epstein (eds.) Modern Uses of Multiple-Valued Logics, D. Reidel Publishing Company, Dordrecht, 8–37, 1977. MR0485167

[4] D. De Jong, L. Hendriks, *Characterization of strongly equivalent logic programs in intermediate logics.* Theory and Practice of Logic Programing **3** (2003), 259—270. MR1999186

[5] J. Dietrich, Deductive bases of nonmonotonic inference operations. Technical report, NTZ Report, Universität Leipzig, 1994.

[6] J.M. Dunn, *Intuitive semantics for first degree entailments and coupled trees.* Phil. Stud., **29** (1976), 149–168. MR0491026

[7] M. Gelfond, V. Lifschitz, *Classical negation in logic programs and disjunctive databases.* New Generation Computing, **9** (1991), 365–385. Zbl 0735.68012

[8] V. Lifschitz, D. Pearce, A. Valverde, *Strongly equivalent logic programs.* ACM Trans. on Comput. Log., **2**(4) (2001), 526–541. MR1860432

[9] D. Nelson, *Constructible falsity.* J. Symb. Log., **14**:2 (1949), 16–26. MR0029843

[10] S.P. Odintsov, N.V. Mayatskiy, *On deductive base for paraconsistent answer set semantics.* J. Appl. Non-Class. Logic **23**:1–2 (2013), 131–146. MR3196425

[11] S. Odintsov, D. Pearce, *Routley semantics for answer sets.* in: Proc. 8th International Conference on Logic Programming and Nonmonotonic Reasoning, LNCS 3662, 343–355, 2005. Zbl 1152.68416

[12] D. Pearce, *Answer sets and constructive logic. Part II: extended logic programs and related nonmonotonic formalisms.* in: L. M. Periera, A. Nerode (eds), Logic Programming and Nonmonotonic Reasoning, MIT press, 457–475, 1993. MR1301803

[13] D. Pearce, *A new logical characterization of stable models and answer sets.* in: Proc. of NMELP 96, LNCS 1216, 57–70, Springer, 1997. MR1489287

[14] D. Pearce, G. Wagner. *Logic programming with strong negation.* in: Extensions of Logic Programming, LNCS 475, 311–326, Springer, 1991. MR1101610

[15] H. Rasiowa, *N-lattices and constructive logic with strong negation.* Fund. Math., **46** (1958), 61–80. MR0098682

[16] H. Rasiowa, An algebraic approach to non-classical logic, PWN, Warsaw and North-Holland, Amsterdam, 1974. MR0446968

[17] C. Sakama, K. Inoue, *Paraconsistent stable semantics for extended disjunctive programs.* J. Log. Comput. **5** (1995), 265–285. MR1331754

[18] N. N. Vorob'ev, *A constructive calculus of propositions with strong negation (in russian).* Doklady Akademii Nauk SSSR, **85** (1952), 465–468. MR0049836

[19] N. N. Vorob'ev, *The problem of deducibility in constructive propositional calculus with strong negation (in russian).* Doklady Akademii Nauk SSSR, **85** (1952), 689–692. MR0049837

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