S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 12, стр. 130–149 (2015) УДК 512.54+519.17
MSC 05C25

LIMIT GRAPHS OF DEGREE LESS THAN 24 FOR MINIMAL VERTEX-PRIMITIVE GRAPHS OF HA-TYPE

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Abstract. A primitive permutation group is called a group of HA-type, if it contains regular abelian normal subgroup. A finite connected graph Γ is called a minimal vertex-primitive graph of HA-type, if there exists a vertex-primitive group G of automorphisms of Γ of HA-type, such that Γ has a minimal degree among all connected graphs Δ, with $V(\Delta) = V(\Gamma)$ and $G \leq \mathrm{Aut}(\Delta)$. For the class of minimal vertex-primitive graphs of HA-type we find all limit graphs of degree less than 24 (it is shown that there are 23 such graphs). In the previous paper the author proved that there are infinitely many such limit graphs of degree 24.

Keywords: vertex-primitive graph and limit graph and Cayley graph and free abelian group.

1. Introduction and main results

Throughout the paper, by a graph we mean an undirected graph without loops or multiple edges. The vertex set and the edge set of a graph Γ are denoted by $V(\Gamma)$ and $E(\Gamma)$ respectively. By an automorphism of graph Γ we mean a permutation on the set $V(\Gamma)$ preserving the adjacency relation.

A graph Γ is called a *vertex-primitive* if it admits a primitive on $V(\Gamma)$ group of automorphisms. It is easy to see that each vertex-primitive graph with non-empty set of edges is connected. We denote the class of connected vertex-primitive graphs by $\mathcal{FP}$ (here and below by a class of graphs we mean the set of isomorphic types of these graphs). The class $\mathcal{FP}$ has an important subclass $\mathcal{FP}^{min}$ consisting of graphs of minimal degree for finite vertex-primitive groups. A finite connected graph Γ is

Kostousov, K.V., Limit graphs of degree less than 24 for minimal vertex-primitive graphs of HA-type.

The work is supported by RFFI (grant 10-01-00349).

Received January, 12, 2015, published February, 26, 2015.

called a graph *of minimal degree for a vertex-primitive group G of automorphisms of* Γ if Γ has a minimal degree among all connected graphs Δ, with $V(\Delta) = V(\Gamma)$ and $G \leq \mathrm{Aut}(\Delta)$. Each primitive permutation group has a graph of minimal degree. Note that vertex-primitive graphs of minimal valency for a given finite primitive permutation group give the most simple its realizations as an automorphism group of a graph.

To investigate the class $\mathcal{FP}$, in [1] and [3] it was proposed to investigate the limit graphs for the class $\mathcal{FP}$. If $\mathcal{C}$ is an arbitrary class of connected vertex-primitive graphs, then an infinite connected graph, each ball of which is isomorphic to a ball of some graph from $\mathcal{C}$, is called a *limit graph for* $\mathcal{C}$. The class of limit graphs for $\mathcal{C}$ is denoted by $\lim(\mathcal{C})$. A description of $\lim(\mathcal{C})$ provides a useful description of the possible local structures of generic graphs from $\mathcal{C}$. The problem of description of $\lim(\mathcal{FP})$ was posed by V. Trofimov in [2, question 12.89]. The investigation of the structure of graphs from $\lim(\mathcal{FP}^{min})$ is of special interest (see the end of the previous paragraph).

The systematic study of class $\lim(\mathcal{FP})$ was started by M. Giudici, C. Li, C. Praeger, A. Seress, and V. Trofimov in [3]. There the importance of following subclasses $\mathcal{FP}_{HA}$, $\mathcal{FP}_{AS}$, $\mathcal{FP}_{PA}$ of class $\mathcal{FP}$ was shown. Let G be a primitive permutation group acting on finite set V. If G has an abelian normal subgroup which acts regularly on V, then G is said to be of *HA-type*. If G is an almost simple group, i.e., there exists a finite nonabelian simple group T such that $\mathrm{Inn}(T) \trianglelefteq G \lesssim \mathrm{Aut}(T)$, then G is said to be of *AS-type*. If G has a minimal normal subgroup $N \cong T^k$ ($k \geq 2$) for some nonabelian simple group T and the stabiliser in N of a point is nontrivial and has no composition factor isomorphic to T, then G is said to be of *PA-type*. For each of these types X, the class of all vertex-primitive graphs with an automorphism group of type X is denoted by $\mathcal{FP}_X$, and the subclass of $\mathcal{FP}_X$ consisting of all graphs of minimal degree for vertex-primitive groups of type X is denoted by $\mathcal{FP}_X^{min}$.

By [3, Theorem 1] we have

$$\lim(\mathcal{FP}) = \lim(\mathcal{FP}_{HA}) \cup \lim(\mathcal{FP}_{AS}) \cup \lim(\mathcal{FP}_{PA}), \text{ and}$$

$$\lim(\mathcal{FP}^{min}) = \lim(\mathcal{FP}_{HA}^{min}) \cup \lim(\mathcal{FP}_{AS}^{min}) \cup \lim(\mathcal{FP}_{PA}^{min}).$$

Therefore, the investigation of the class $\lim(\mathcal{FP}_{HA})$ and the class $\lim(\mathcal{FP}_{HA}^{min})$, which is of independent interest, is a necessary step of the investigation of classes $\lim(\mathcal{FP})$ and $\lim(\mathcal{FP}^{min})$.

In [4], [5], [6], all graphs of degree ≤ 14 from $\lim(\mathcal{FP}_{HA}^{min})$ were found (it was shown that there are 12 such graphs), and also a countable set of pairwise nonisomophic graphs of degree 24 from $\lim(\mathcal{FP}_{HA}^{min})$ was found. In the present paper (see Theorem 1 below), we find all graphs of degree < 24 from $\lim(\mathcal{FP}_{HA}^{min})$ (we will show that there are 23 such graphs).

Our terminology is mostly standard. Let d be a positive integer. Let M be a set of generators of $\mathbb{Z}^d$ such that $M = -M$ and $\boldsymbol{0} \notin M$. A graph $\Gamma_{\mathbb{Z}^d,M}$ with vertex set $\mathbb{Z}^d$ and such that two vertexes are adjacent iff their difference lies in M is the *Cayley graph of the group $\mathbb{Z}^d$ corresponding to the set of generators M*. Let $\mathrm{Aut}(\Gamma_{\mathbb{Z}^d,M})_0$ be the stabilizer of the vertex $\boldsymbol{0}$ in the group of automorphism of $\Gamma_{\mathbb{Z}^d,M}$. We say that $\Gamma_{\mathbb{Z}^d,M}$ is a *minimal* Cayley graph of $\mathbb{Z}^d$ if M is an orbit of minimal cardinality of the group $\mathrm{Aut}(\Gamma_{\mathbb{Z}^d,M})_0$, restricted to $\mathbb{Z}^d \setminus \{\boldsymbol{0}\}$. The class of all Cayley graphs of the group $\mathbb{Z}^d$ is denoted by $\mathrm{Cay}(\mathbb{Z}^d)$, and the class of all minimal

Cayley graphs of $\mathbb{Z}^d$ is denoted by $\mathrm{Cay}^{min}(\mathbb{Z}^d)$. It follows from [3, Theorem 2] that each element of $\lim(\mathcal{FP}_{HA})$ lies in $\mathrm{Cay}(\mathbb{Z}^d)$ for some d. Furthermore, it follows from [3, Theorem 2] and the definition of limit graphs that each element of $\lim(\mathcal{FP}_{HA}^{min})$ lies in $\mathrm{Cay}^{min}(\mathbb{Z}^d)$ for some d, i.e.

$$\lim(\mathcal{FP}_{HA}^{min}) \subseteq \bigcup_{d=1}^{\infty} \mathrm{Cay}^{min}(\mathbb{Z}^d). \tag{1}$$

Thus to describe $\lim(\mathcal{FP}_{HA}^{min})$ it is sufficient to describe classes $\mathrm{Cay}^{min}(\mathbb{Z}^d)$ and pick out of them graphs lying in $\lim(\mathcal{FP}_{HA}^{min})$. We will prove that all graphs of degree < 24 in $\bigcup_{d=1}^{\infty} \mathrm{Cay}^{min}(\mathbb{Z}^d)$ lie also in $\lim(\mathcal{FP}_{HA}^{min})$.

We identify $\mathbb{Z}^d$ with the set of integral row vectors of length d with coordinate-wise addition. By $\mathrm{GL}_d(\mathbb{Z})$ we denot the group of integral $d \times d$-matrices with determinant equal to ± 1. It follows from [4, Theorem 3.(a)] that for any graph Γ from $\mathrm{Cay}(\mathbb{Z}^d)$ the group $\mathrm{Aut}(\Gamma)_0$ (the stabilizer of the vertex $\boldsymbol{0}$ in $\mathrm{Aut}(\Gamma)$) is a subgroup of $\mathrm{GL}_d(\mathbb{Z})$, acting naturally on $\mathbb{Z}^d$. For $i \in \{1, \ldots, d\}$, let $\boldsymbol{e}_i$ be a row-vector of length d having 1 in the position i and 0 in other positions. For a finite subgroup G of $\mathrm{GL}_d(\mathbb{Z})$, let $\mathcal{MO}_{\mathbb{Z}}(G)$ be the set of G-orbits M on $\mathbb{Z}^d \setminus \{\boldsymbol{0}\}$ such that M generates $\mathbb{Z}^d$, $M = -M$ and M has minimal cardinality among all G-orbits on $\mathbb{Z}^d \setminus \{\boldsymbol{0}\}$. Thus, if $\Gamma_{\mathbb{Z}^d,M} \in \mathrm{Cay}^{min}(\mathbb{Z}^d)$, then $M \in \mathcal{MO}_{\mathbb{Z}}(G)$ for some $G \leq \mathrm{GL}_d(\mathbb{Z})$.

For each positive integer $d \geq 1$, we set $M_{d,1} = \{\pm \boldsymbol{e}_i : i \in \{1, \ldots, d\}\}$. Of course the graph $\Gamma_{\mathbb{Z}^d,M_{d,1}}$ is the only graph from $\mathrm{Cay}(\mathbb{Z}^d)$ of degree $2d$, and, by [6, Proposition 1.5], it lies in $\lim(\mathcal{FP}_{HA}^{min})$. We also set $M_{d,2} = M_{d,1} \cup \{\pm \sum_{i=1}^{d} \boldsymbol{e}_i\}$.

Theorem 1. *The class of all graphs from* $\lim(\mathcal{FP}_{HA}^{min})$ *of degree* < 24 *is equal to the class of all graphs from* $\bigcup_{d=1}^{\infty} \mathrm{Cay}^{min}(\mathbb{Z}^d)$ *of degree* < 24 *and consists of the following graphs:*
$\Gamma_{\mathbb{Z},M_{1,1}}$ *of degree* 2*;*
$\Gamma_{\mathbb{Z}^2,M_{2,1}}$ *of degree* 4*;*
$\Gamma_{\mathbb{Z}^2,M_{2,2}}$ *and* $\Gamma_{\mathbb{Z}^3,M_{3,1}}$ *of degree* 6*;*
$\Gamma_{\mathbb{Z}^4,M_{4,1}}$ *of degree* 8*;*
$\Gamma_{\mathbb{Z}^4,M_{4,2}}$ *and* $\Gamma_{\mathbb{Z}^5,M_{5,1}}$ *of degree* 10*;*
$\Gamma_{\mathbb{Z}^4,M_{4,3}}$, $\Gamma_{\mathbb{Z}^5,M_{5,2}}$ *and* $\Gamma_{\mathbb{Z}^6,M_{6,1}}$ *of degree* 12*, where* $M_{4,3} = M_{4,1} \cup \pm\{\boldsymbol{e}_1 + \boldsymbol{e}_2, \boldsymbol{e}_3 + \boldsymbol{e}_4\}$*;*
$\Gamma_{\mathbb{Z}^6,M_{6,2}}$ *and* $\Gamma_{\mathbb{Z}^7,M_{7,1}}$ *of degree* 14*;*
$\Gamma_{\mathbb{Z}^7,M_{7,2}}$ *and* $\Gamma_{\mathbb{Z}^8,M_{8,1}}$ *of degree* 16*;*
$\Gamma_{\mathbb{Z}^6,M_{6,3}}$, $\Gamma_{\mathbb{Z}^8,M_{8,2}}$ *and* $\Gamma_{\mathbb{Z}^9,M_{9,1}}$ *of degree* 18*, where* $M_{6,3} = M_{6,1} \cup \pm\{\boldsymbol{e}_1 + \boldsymbol{e}_2, \boldsymbol{e}_3 + \boldsymbol{e}_4, \boldsymbol{e}_5 + \boldsymbol{e}_6\}$*;*
$\Gamma_{\mathbb{Z}^6,M_{6,4}}$, $\Gamma_{\mathbb{Z}^8,M_{8,3}}$, $\Gamma_{\mathbb{Z}^9,M_{9,2}}$ *and* $\Gamma_{\mathbb{Z}^{10},M_{10,1}}$ *of degree* 20*, where* $M_{6,4} = M_{6,1} \cup \pm\{\boldsymbol{e}_1 - \boldsymbol{e}_2 + \boldsymbol{e}_4, \boldsymbol{e}_1 - \boldsymbol{e}_3 + \boldsymbol{e}_5, \boldsymbol{e}_2 - \boldsymbol{e}_3 + \boldsymbol{e}_6, \boldsymbol{e}_4 - \boldsymbol{e}_5 + \boldsymbol{e}_6\}$*,* $M_{8,3} = M_{8,1} \cup \{\pm(\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3 + \boldsymbol{e}_4), \pm(\boldsymbol{e}_5 + \boldsymbol{e}_6 + \boldsymbol{e}_7 + \boldsymbol{e}_8)\}$*;*
$\Gamma_{\mathbb{Z}^{10},M_{10,2}}$ *and* $\Gamma_{\mathbb{Z}^{11},M_{11,1}}$ *of degree* 22*.*

In the proof of Theorem 1 we use our earlier results about the structure of classes $\mathrm{Cay}^{min}(\mathbb{Z}^d)$ for small d. Below we give a description of graphs from $\mathrm{Cay}^{min}(\mathbb{Z}^d)$ for $d \leq 6$ extracted form [4, 6] (all these graphs are in $\lim(\mathcal{FP}_{HA}^{min})$). We define graphs by corresponding systems of generators of $\mathbb{Z}^d$ (some of which were simplified in comparison to [4, 6]).

$d = 1 : M_{1,1}$.
$d = 2 : M_{2,1},\ M_{2,2}$.
$d = 3 : M_{3,1}$.

$d=4: M_{4,1}$, $M_{4,2}$, $M_{4,3}$, and countable collection $\mathcal{MO}_{\mathbb{Z}}(G_1)\cup\mathcal{MO}_{\mathbb{Z}}(G_2)$ (all of them are of order 24).

$d=5: M_{5,1}$, $M_{5,2}$.

$d=6: M_{6,1}$, $M_{6,2}$, $M_{6,3}$, $M_{6,4}$, $M_{6,5}=\boldsymbol{e}_1F_1$ of order 36, $M_{6,6}=\boldsymbol{e}_1F_2$ of order 42, $M_{6,7}=\boldsymbol{e}_1F_3$ of order 54.

Here $G_1=\langle h_1,h_2\rangle$, $G_2=\langle h_3,h_4,h_5\rangle$, $F_1=\langle f_1,f_2\rangle$, $F_2=\langle f_3,f_4\rangle$, $F_3=\langle f_5,f_6,f_7\rangle$,

$$h_1=\begin{pmatrix} 0&0&-1&-1\\ 0&-1&-1&-1\\ 0&0&0&-1\\ -1&1&1&1 \end{pmatrix},\qquad h_2=\begin{pmatrix} 0&-1&-1&-1\\ 0&-1&0&-1\\ 1&-1&-1&0\\ 0&1&0&0 \end{pmatrix},$$

$$h_3=\begin{pmatrix} -1&1&1&0\\ -1&0&0&-1\\ -1&1&1&1\\ 1&0&-1&0 \end{pmatrix},\qquad h_4=\begin{pmatrix} 0&-1&0&-1\\ 1&-1&-1&-1\\ -1&0&0&-1\\ 0&1&1&1 \end{pmatrix},$$

$$h_5=\begin{pmatrix} -1&0&1&0\\ 0&0&1&1\\ -1&0&0&0\\ 1&-1&-1&-1 \end{pmatrix},$$

$$f_1=\begin{pmatrix} 0&1&0&0&-1&1\\ -1&2&1&0&-1&1\\ 1&0&0&0&-1&0\\ 1&-1&-1&1&0&-1\\ 0&1&1&0&-1&1\\ 0&-1&0&0&0&0 \end{pmatrix}, f_2=\begin{pmatrix} 0&0&0&0&1&0\\ 0&-1&0&0&1&-1\\ -1&1&1&-1&0&1\\ 0&0&0&-1&0&0\\ -1&0&0&0&1&0\\ 1&-1&-2&1&1&0 \end{pmatrix},$$

$$f_3=\begin{pmatrix} 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&1&-1&0&0&-1\\ 0&0&0&0&1&0\\ 1&0&-1&-1&0&0 \end{pmatrix}, f_4=\begin{pmatrix} 0&1&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&1&1&0&0\\ 1&0&-1&-1&0&0\\ 0&0&0&-1&0&1\\ 1&0&-1&-1&-1&1 \end{pmatrix},$$

$$f_5=\begin{pmatrix} 0&0&1&0&0&0\\ 1&0&1&0&1&-1\\ -1&1&1&1&0&0\\ 0&0&-1&0&-1&1\\ 0&0&0&0&1&0\\ -1&1&1&0&0&0 \end{pmatrix}, f_6=\begin{pmatrix} 1&-1&-1&-1&0&0\\ 0&0&-1&-1&-1&0\\ 0&-1&0&-1&0&0\\ 0&0&0&1&1&0\\ 0&0&0&0&-1&0\\ 0&-1&-1&-1&-1&1 \end{pmatrix},$$

$$f_7=\begin{pmatrix} 0&1&1&1&1&0\\ 0&0&1&1&1&0\\ 0&0&-1&0&0&1\\ 0&0&0&-1&-1&0\\ 1&-1&0&0&1&-1\\ 1&0&0&1&1&0 \end{pmatrix}.$$

It is obvious that each graph from $\mathrm{Cay}(\mathbb{Z}^d)$ has an even degree not smaller than $2d$. Also, as it was mentioned above, $\Gamma_{\mathbb{Z}^d,M_{d,1}}$ is the only graph from $\mathrm{Cay}(\mathbb{Z}^d)$ of degree $2d$. Therefore, by (1) the list above contains all graphs from $\lim(\mathcal{FP}_{HA}^{min})$ of degree ≤ 14, excluding $\Gamma_{\mathbb{Z}^7,M_{7,1}}$, and each non-isomorphic to $\Gamma_{\mathbb{Z}^{12},M_{12,1}}$ graph of degree ≤ 24 from $\lim(\mathcal{FP}_{HA}^{min})$ is contained in $\bigcup_{d=1}^{11}\mathrm{Cay}^{min}(\mathbb{Z}^d)$.

In the present work we prove the following Theorem 2, which describes the class $\mathrm{Cay}^{min}(\mathbb{Z}^7)$.

Theorem 2. *$\mathrm{Cay}^{min}(\mathbb{Z}^7)=\{\Gamma_{\mathbb{Z}^7,M_{7,i}}: i=1,2,3\}\subseteq\lim(\mathcal{FP}_{HA}^{min})$, where $M_{7,3}=(1,-1,1,-1,-1,-1,-1)W(E_7)$ ($|M_{7,3}|=56$) and $W(E_7)$ is a finite group of integer matrices, generated by*

$$\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&1&0&0&0&-1\\ 0&0&1&0&0&-1&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0 \end{pmatrix}.$$

This result, which we prove in Sec. 3, gives a list of all graphs from $\lim(\mathcal{FP}_{HA}^{min})$ with degree ≤ 16 (to 12 graphs of degree ≤ 14, listed above, including $\Gamma_{\mathbb{Z}^7,M_{7,1}}$, two new graphs $\Gamma_{\mathbb{Z}^7,M_{7,2}}$ and $\Gamma_{\mathbb{Z}^8,M_{8,1}}$ are added). The remaining part of the proof of Theorem 1 is as follows: we obtain the description of graphs from $\lim(\mathcal{FP}_{HA}^{min})$ of degree 18 in Sec. 5, of degree 20 in Sec. 6, and of degree 22 in Sec. 7. Sec. 2 and Sec. 4 contain some preliminary results.

2. Preliminaries

Let d be a positive integer. By $\mathbb{Q}^d$ we denote the vector space of rational row-vectors of length d. For $M \subseteq \mathbb{Q}^d$, by $\langle M \rangle$ we denote the subgroup of abelian group of vectors of $\mathbb{Q}^d$ generated by M, and by $\langle M \rangle_{\mathbb{Q}}$ we denote the subspace of $\mathbb{Q}^d$ generated by M. If $M = -M$ and $\boldsymbol{0} \notin M$, then by $\Gamma_{\langle M \rangle, M}$ we denote the Caley graph of the group $\langle M \rangle$ corresponding to the set of generators M. By $\mathrm{GL}_d(\mathbb{Q})$ we denote the group of all nondegenerate $d \times d$-matrices over $\mathbb{Q}$. By $\boldsymbol{I}$ we denote the identity $d \times d$-matrix. Subsets M, M' of $\mathbb{Q}^d$ are *linearly equivalent* iff $M' = M\boldsymbol{A}$ for some $\boldsymbol{A} \in \mathrm{GL}_d(\mathbb{Q})$.

For any subgroup G of $\mathrm{GL}_d(\mathbb{Q})$, we define

$$\mathrm{m}(G) := \min\{|\boldsymbol{x}G| : \boldsymbol{x} \in \mathbb{Q}^d \setminus \{\boldsymbol{0}\}\},$$

$$\mathcal{MO}_{\mathbb{Q}}(G) := \{\boldsymbol{x}G : \boldsymbol{x} \in \mathbb{Q}^d \setminus \{\boldsymbol{0}\}, |\boldsymbol{x}G| = \mathrm{m}(G)\},$$

$$\mathcal{O}\Gamma^{min}(G) := \{\Gamma_{\langle M \rangle, M} : M \in \mathcal{MO}_{\mathbb{Q}}(G), M = -M, \dim(\langle M \rangle_{\mathbb{Q}}) = d\}$$

and

$$\mathcal{OS}\Gamma^{min}(G) := \bigcup_{H \leq G} \mathcal{O}\Gamma^{min}(H).$$

Now we formulate few Propositions containing some results from [6].

Proposition 1. [6, Theorem 1.2]. *Let $G_1, \ldots, G_n$ be all, up to conjugation in $\mathrm{GL}_d(\mathbb{Q})$, finite subgroups of $\mathrm{GL}_d(\mathbb{Z})$ containing $-\boldsymbol{I}$. Then*

$$\mathrm{Cay}^{min}(\mathbb{Z}^d) = \bigcup_{i=1}^{n} \mathcal{O}\Gamma^{min}(G_i).$$

The following result emediately follows from Proposition 1.

Proposition 2. *Let $G_1, \ldots, G_n$ be all, up to conjugation in $\mathrm{GL}_d(\mathbb{Q})$, maximal finite subgroups of $\mathrm{GL}_d(\mathbb{Z})$. Then*

$$\mathrm{Cay}^{min}(\mathbb{Z}^d) = \bigcup_{i=1}^{n} \mathcal{OS}\Gamma^{min}(G_i).$$

Proposition 3. [6, Theorem 1.1(a)]. *For $i = 1, 2$, let M_i be a finite subset of $\mathbb{Q}^d$ such that $\boldsymbol{0} \notin M_i$ and $M_i = -M_i$. Then Cayley graphs $\Gamma_{\langle M_1 \rangle, M_1}$ and $\Gamma_{\langle M_2 \rangle, M_2}$ are isomorphic if and only if M_1 and M_2 are linearly equivalent.*

Let p be a prime number. For $\boldsymbol{v} \in \mathbb{Z}^d$ and $\boldsymbol{A} \in \mathrm{GL}_d(\mathbb{Z})$ by $\phi_p(\boldsymbol{v})$ and $\phi_p(\boldsymbol{A})$ we denote the vector and the matrix which are obtained respectively from $\boldsymbol{v}$ and $\boldsymbol{A}$ via substituting their integer entries by residues modulo p. By $\mathbb{Z}_p^d$ we denote the vector space of all row-vectors of length d of residues modulo p from the field $\mathrm{GF}(p)$. For any $G \leq \mathrm{GL}_d(\mathbb{Z})$, we define $\phi_p(G) := \{\phi_p(\boldsymbol{A}) : \boldsymbol{A} \in G\}$.

Proposition 4. [6, Lemma 1.3]. *For any finite group* $G \leq \mathrm{GL}_d(\mathbb{Z})$*, there exists a positive integer* p_0 *such that for any prime* $p > p_0$ *the group* $\phi_p(G)$ *acts exactly on* $\mathbb{Z}_p^d$ *and* $|\boldsymbol{x}\phi_p(G)| \geq \mathrm{m}(G)$ *for any* $\boldsymbol{x} \in \mathbb{Z}_p^d \setminus \{\boldsymbol{0}\}$.

Proposition 5. [6, Proposition 1.4]. *Let* G *be a finite subgroup of* $\mathrm{GL}_d(\mathbb{Z})$*, and* $\Gamma_{\mathbb{Z}^d,M} \in \mathcal{O}\Gamma^{min}(G)$*. If there exists an infinite set* P *of primes such that for any* $p \in P$ *the group* $\phi_p(G)$ *is irreducible over* $\mathrm{GF}(p)$*, then* $\Gamma_{\mathbb{Z}^d,M} \in \lim(\mathcal{FP}^{min}_{HA})$.

Proposition 6. [6, Proposition 1.8]. *Let* H, G *be finite subgroups of* $\mathrm{GL}_d(\mathbb{Z})$*,* $\mathrm{m}(H) = \mathrm{m}(G)$ *and* $H^{\boldsymbol{A}} \leq G$ *for some* $\boldsymbol{A} \in \mathrm{GL}_d(\mathbb{Q})$*. Then* $\mathcal{O}\Gamma^{min}(G) \subseteq \mathcal{O}\Gamma^{min}(H)$.

For any positive integer d we define

$$\mathrm{m}(d) = \{|M| : M \in \mathcal{MO}_{\mathbb{Q}}(G), G \leq \mathrm{GL}_d(\mathbb{Z}), M = -M, \dim(\langle M\rangle_{\mathbb{Q}}) = d\}.$$

Remark. 1. In the definition of $\mathrm{m}(d)$, the group $\mathrm{GL}_d(\mathbb{Z})$ may be replaced by $\mathrm{GL}_d(\mathbb{Q})$, since each finite subgroup of $\mathrm{GL}_d(\mathbb{Q})$ is conjugate in $\mathrm{GL}_d(\mathbb{Q})$ to a subgroup of $\mathrm{GL}_d(\mathbb{Z})$.

Remark. 2. It follows from the description of $\mathrm{Cay}^{min}(\mathbb{Z}^d)$ for $d \leq 6$ in Introduction that $\mathrm{m}(1) = \{2\}, \mathrm{m}(2) = \{4, 6\}, \mathrm{m}(3) = \{6\}, \mathrm{m}(4) = \{8, 10, 12, 24\}, \mathrm{m}(5) = \{10, 12\}$ and $\mathrm{m}(6) = \{12, 14, 18, 20, 36, 42, 54\}$.

Proposition 7. *Let* $G \leq \mathrm{GL}_d(\mathbb{Z})$ *be a reducible on* $\mathbb{Q}^d$ *group. If* $\mathcal{O}\Gamma^{min}(G) \neq \emptyset$ *then* $\mathrm{m}(G) \in \mathrm{m}(d')$ *for some* $d' \leq d/2$.

Proof. By Maschke's theorem there exists a decomposition of $\mathbb{Q}^d$ into a direct sum $V_1 \oplus V_2 \oplus ... \oplus V_k$ of G-invariant subspaces $V_1, V_2, ..., V_k$ such that G acts irreducibly on each of them. Let $d_i = \dim(V_i)$ for $i = 1, ..., k$. Fix some basis $\boldsymbol{v}_1^i, \boldsymbol{v}_2^i, ..., \boldsymbol{v}_{d_i}^i$ of V_i for each $i = 1, ..., k$. Then, in the basis $\boldsymbol{v}_1^1, ..., \boldsymbol{v}_{d_1}^1, \boldsymbol{v}_1^2, ..., \boldsymbol{v}_{d_2}^2, ..., \boldsymbol{v}_1^k, ..., \boldsymbol{v}_{d_k}^k$ of V, each element $\boldsymbol{g} \in G$ is represented by a matrix

$$\begin{pmatrix} \boldsymbol{g}_1 & 0 & \dots & 0 \\ 0 & \boldsymbol{g}_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & 0 & \dots & \boldsymbol{g}_k \end{pmatrix},$$

where $\boldsymbol{g}_i \in \mathrm{GL}_{d_i}(\mathbb{Q})$. By $\tilde{G} \leq \mathrm{GL}_d(\mathbb{Q})$ we denote a subgroup of $\mathrm{GL}_d(\mathbb{Q})$ consisting of these matrices when $\boldsymbol{g}$ runs through G. By G_i denote a subgroup of $\mathrm{GL}_{d_i}(\mathbb{Q})$ consisting of matrices $\boldsymbol{g}_i$ when $\boldsymbol{g}$ runs through G.

Let $\Gamma_{\langle \boldsymbol{x}\tilde{G}\rangle, \boldsymbol{x}\tilde{G}} \in \mathcal{O}\Gamma^{min}(\tilde{G})$, where $\boldsymbol{x} \in \mathbb{Q}^d$. The vector $\boldsymbol{x}$ is decomposed into $\boldsymbol{x}_1 + ... + \boldsymbol{x}_k$, where $\boldsymbol{x}_i \in V_i$. Since $\dim(\langle \boldsymbol{x}\tilde{G}\rangle_{\mathbb{Q}}) = d$, we have $\langle \boldsymbol{x}_i G_i\rangle_{\mathbb{Q}} = V_i$ for $i \in \{1, ..., k\}$. Since $|\boldsymbol{x}\tilde{G}| = \mathrm{m}(\tilde{G})$, by definition of $\mathrm{m}(\tilde{G})$ we get $|\boldsymbol{x}_i G_i| = |\boldsymbol{x}\tilde{G}|$ and $|\boldsymbol{x}_i G_i| = \mathrm{m}(G_i)$ for $i \in \{1, ..., k\}$.

Since $k \geq 2$, there exists a number $l \in \{1, ..., k\}$ such that $d_l \leq d/2$. By Remark 1 we have $\mathrm{m}(G_l) \in \mathrm{m}(d_l)$. Now since $\mathrm{m}(G) = \mathrm{m}(\tilde{G}) = \mathrm{m}(G_l)$ it follows that $\mathrm{m}(G) \in \mathrm{m}(d_l)$. □

For some finite subgroups G of $\mathrm{GL}_d(\mathbb{Z})$ the classes $\mathcal{O}\Gamma^{min}(G)$ and $\mathcal{OS}\Gamma^{min}(G)$ can be explicitly found using the following Algorithm 1 and Algorithm 2, which require some definition. For a finite subgroup G of $\mathrm{GL}_d(\mathbb{Z})$ we put

$$\mathcal{MF}(G) := \{\mathrm{fix}(L) : L \subseteq G, |L| = s\} \setminus \{\{\emptyset\}\},$$

where $s = |G|/\mathrm{m}(G)$ and $\mathrm{fix}(L) = \{\boldsymbol{x} \in \mathbb{Q}^d : \boldsymbol{x}g = \boldsymbol{x} \text{ for } g \in L\}$. It is clear that

$$\mathcal{MO}_{\mathbb{Q}}(G) = \{xG : x \in V \setminus \{\boldsymbol{0}\}, V \in \mathcal{MF}(G)\}.$$

Algorithm 1. *Calculating* $\mathrm{m}(G)$ *and* $\mathcal{MO}_{\mathbb{Q}}(G)$
Input: G, a finite subgroup $\mathrm{GL}_d(\mathbb{Z})$;
Output: $\mathrm{m}(G)$ and $\mathcal{MF}(G)$.
Description of Algorithm 1 is given in [6].

We will apply Algorithm 1 together with the following proposition.

Proposition 8. *Let Algorithm 1 applied to the group* $G \leq \mathrm{GL}_d(\mathbb{Z})$ *give* $\mathcal{MF}(G) = \{\langle \boldsymbol{x}_1\rangle_{\mathbb{Q}}, \langle \boldsymbol{x}_2\rangle_{\mathbb{Q}}, ..., \langle \boldsymbol{x}_n\rangle_{\mathbb{Q}}\}$; (*thus all subspaces from* $\mathcal{MF}(G)$ *are one-dimensional*). *Assume that all orbits* $\boldsymbol{x}_iG$ *are pairwise linearly equivalent. Then*

$$\mathcal{O}\Gamma^{min}(G) = \begin{cases} \Gamma_{\langle \boldsymbol{x}_1G\rangle, \boldsymbol{x}_1G}, \text{if } \boldsymbol{x}_1G = -\boldsymbol{x}_1G \text{ and } \dim(\langle \boldsymbol{x}_1G\rangle_{\mathbb{Q}}) = d \\ \emptyset, \ \text{overwise.} \end{cases}$$

Proof. If there exists $\Gamma_{\langle \boldsymbol{x}G\rangle, \boldsymbol{x}G} \in \mathcal{O}\Gamma^{min}(G)$ for some $\boldsymbol{x} \in \mathbb{Q}^d$, then it follow from the definition of $\mathcal{MF}(G)$ that $\boldsymbol{x} = c\boldsymbol{x}_i$ for some $c \in \mathbb{Q}$ and $i \in \{1, ..., n\}$. Since $\boldsymbol{x}G = c\boldsymbol{x}_iG$ is linearly equivalent to $\boldsymbol{x}_1G$, we have $\boldsymbol{x}_1G = -\boldsymbol{x}_1G$ and $\dim(\langle \boldsymbol{x}_1G\rangle_{\mathbb{Q}}) = d$. Therefore by Proposition 3 we get $\Gamma_{\langle \boldsymbol{x}G\rangle, \boldsymbol{x}G} \cong \Gamma_{\langle \boldsymbol{x}_1G\rangle, \boldsymbol{x}_1G}$.

If $\mathcal{O}\Gamma^{min}(G) = \emptyset$, then $\boldsymbol{x}_1G \neq -\boldsymbol{x}_1G$ or $\dim(\langle \boldsymbol{x}_1G\rangle_{\mathbb{Q}}) \neq d$. In all cases the statement of Proposition 8 holds. □

Algorithm 2. *Growing of subgroup tree of* $G < \mathrm{GL}_d(\mathbb{Z})$
Input: G, a finite subgroup of $\mathrm{GL}_d(\mathbb{Z})$;
Output: *The List of subgroups* $G_1, G_2, ..., G_n$ for G.
Description. We start the growing subgroup tree by setting G as a root vertex. For the group G, using the *GAP*-procedure MaximalSubgroupClassReps, we find a list containing all maximal subgroups of G up to conjugation in $\mathrm{GL}_d(\mathbb{Q})$. All groups from this list are added into the tree as child vertices of the root vertex G. For each added vertex H we find a list, containing all maximal subgroups of H up to conjugation in $\mathrm{GL}_d(\mathbb{Q})$. All groups from this list are added into the tree as child vertices of H. We proceed in the same manner and stop the growing of subgroup tree when for each leave H of the tree either $-\boldsymbol{I} \notin H$ or $\mathrm{m}(H) \leq 2d$ holds.

For each non-leave vertex (subgroup) H, we put H into the List of subgroups, if H has not descendant vertex (subgroup) H_1 with $\mathrm{m}(H) = \mathrm{m}(H_1)$ and $-\boldsymbol{I} \in H_1$.

Proposition 9. *Let* $G_1, G_2, ..., G_n$ *be the List of subgroups of a group* $G \leq \mathrm{GL}_d(\mathbb{Z})$ *obtained after applying Algorithm 2. Then*

$$\mathcal{OS}\Gamma^{min}(G) \setminus \{\Gamma_{\mathbb{Z}^d, M_{d,1}}\} = \bigcup_{i=1}^{n} \mathcal{O}\Gamma^{min}(G_i).$$

Proof. Let $\Gamma \in \mathcal{OS}\Gamma^{min}(G) \setminus \{\Gamma_{\mathbb{Z}^d, M_{d,1}}\}$ and $H = \mathrm{Aut}(\Gamma)_0$. Then $\deg(\Gamma) = \mathrm{m}(H) > 2d$ and $-\boldsymbol{I} \in H$. It follows from the description of Algorithm 2 that there exists $\boldsymbol{A} \in \mathrm{GL}_d(\mathbb{Q})$ such that the group $H^{\boldsymbol{A}}$ is a vertex of the subgroup tree. It is easy to see that there exists $i \in \{1, ..., n\}$ such that $G_i \leq H^{\boldsymbol{A}}$ and $\mathrm{m}(G_i) = \mathrm{m}(H^{\boldsymbol{A}})$. By Proposition 6, it follows $\Gamma \in \mathcal{O}\Gamma^{min}(G_i)$ as required. □

For $d \geq 1$, let K_d be the subgroup of $\mathrm{GL}_d(\mathbb{Z})$ consisting of all elements $\boldsymbol{g}$ of $\mathrm{GL}_d(\mathbb{Z})$, such that $(x_1, \ldots, x_d)\boldsymbol{g} = (\varepsilon_1 x_{\tau(1)}, \ldots, \varepsilon_d x_{\tau(d)})$ for some permutation τ

on $\{1,\cdots,d\}$ and some $\varepsilon_i \in \{-1,1\}$, $i \in \{1,\dots,d\}$. Obviously, $\mathcal{OS}\Gamma^{min}(K_d) = \{\Gamma_{\mathbb{Z}^d,M_{d,1}}\}$.

For $d \geq 1$, let $S_{d+1} \leq \mathrm{GL}_{d+1}(\mathbb{Z})$ be the symmetric group of degree $d+1$ acting in the natural way on the basis $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, ..., $\boldsymbol{e}_{d+1}$. Next, by $\hat{S}_{d+1}$ we denote the extension of S_{d+1} via $-\boldsymbol{I}$. The group $\hat{S}_{d+1}$ has an invariant subspace $\langle \boldsymbol{e}_1+\boldsymbol{e}_2+...+\boldsymbol{e}_{d+1}\rangle_{\mathbb{Q}}$ of $\mathbb{Q}^{d+1}$. Let $P : \mathbb{Q}^{d+1} \mapsto \mathbb{Q}^d$ be defined by $(x_1, x_2, ..., x_d, x_{d+1})P = (x_1 - x_{d+1}, x_2 - x_{d+1}, ..., x_d - x_{d+1})$ for all $x_1, ..., x_{d+1} \in \mathbb{Q}$. The mapping $\pi : \hat{S}_{d+1} \mapsto \mathrm{GL}_d(\mathbb{Z})$ is defined by $(x_1, ..., x_d)\boldsymbol{g}\pi = (x_1, ..., x_d, 0)\boldsymbol{g}P$ for all $\boldsymbol{g} \in \hat{S}_d$ and $x_1, ..., x_d \in \mathbb{Q}$. We put $\widetilde{S}_{d+1} := \hat{S}_{d+1}\pi$.

It is easy to see that π is an isomorphism of $\hat{S}_{d+1}$ onto $\widetilde{S}_{d+1}$.

Proposition 10. $\mathrm{m}(\widetilde{S}_{d+1}) = 2(d+1)$ *and* $\mathcal{MO}_{\mathbb{Q}}(\widetilde{S}_{d+1}) = \{\{\pm c\boldsymbol{e}_1, ..., \pm c\boldsymbol{e}_d, \pm c\sum_{i=1}^{d} \boldsymbol{e}_i : c \in \mathbb{Q}^*\}$ *for any* $d \geq 6$.

Proof. First we will show that $|P(X)| \geq |X|/2$ holds for any $\hat{S}_{d+1}$-orbit X. Assume the contrary. Then there exist some distinct elements $\boldsymbol{x} = (x_1, ..., x_{d+1})$, $\boldsymbol{y} = (y_1, ..., y_{d+1})$, $\boldsymbol{z} = (z_1, ..., z_{d+1})$ of some $\hat{S}_{d+1}$-orbit on $\mathbb{Q}^{d+1}$ such that $\boldsymbol{x}P = \boldsymbol{y}P = \boldsymbol{z}P$. Hence

$$\boldsymbol{y} - \boldsymbol{x} = t_1 \sum_{i=1}^{d+1} \boldsymbol{e}_i, \qquad \boldsymbol{z} - \boldsymbol{x} = t_2 \sum_{i=1}^{d+1} \boldsymbol{e}_i. \tag{2}$$

Since elements of $\hat{S}_{d+1}$ preserve the absolute value of the sum of vector coordinates, we have $\sum_{i=1}^{d+1} y_i = \varepsilon_1 \sum_{i=1}^{d+1} x_i$ and $\sum_{i=1}^{d+1} z_i = \varepsilon_2 \sum_{i=1}^{d+1} x_i$ for some $\varepsilon_1, \varepsilon_2 \in \{1, -1\}$. By (2), it follows $(\varepsilon_1 - 1)\sum_{i=1}^{d+1} x_i = t_1(d+1)$ and $(\varepsilon_2 - 1)\sum_{i=1}^{d+1} x_i = t_2(d+1)$. Since t_1, t_2 are non-zero, we get $\varepsilon_1 = -1$ and $\varepsilon_2 = -1$. Thus $t_1 = t_2$. Hence $\boldsymbol{y} = \boldsymbol{z}$ according to (2). This contradicts to the choice of $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$.

There are the following three types of $\hat{S}_{d+1}$-orbits on $\mathbb{Q}^{d+1}$.

Type 1: orbits of the form $(a, a, a, ..., a)\hat{S}_{d+1}$ for some $a \in \mathbb{Q}$.

Type 2: orbits of the form $(a, b, b, ..., b)\hat{S}_{d+1}$ for some $a, b \in \mathbb{Q}$, $a \neq b$.

Type 3: all other $\hat{S}_{d+1}$-orbits on $\mathbb{Q}^{d+1}$.

It is easy to see that P-projection of any orbit of type 1 is 0 and P-projection of any orbit of type 2 is $\{\{\pm c\boldsymbol{e}_1, ..., \pm c\boldsymbol{e}_d, \pm c\sum_{i=1}^{d} \boldsymbol{e}_i : c \in \mathbb{Q}^*\}$. Since any $\widetilde{S}_{d+1}$-orbit on $\mathbb{Q}^d$ is a P-image of some $\hat{S}_{d+1}$-orbit on $\mathbb{Q}^{d+1}$, to complete the proof it is sufficient to show that the order of any type 3 orbit of $\hat{S}_d$ is greater that $4(d+1)$.

Let $\boldsymbol{x}\widetilde{S}_{d+1}$, $\boldsymbol{x} = (x_1, ..., x_{d+1}) \in \mathbb{Q}^{d+1}$, be an arbitrary orbit of type 3, $\{|x_i| : i \in \{1, ..., d+1\}\} = \{v_1, ..., v_k\}$, $n_i := |\{j : |x_j| = v_i\}|$ for $i = 1, ..., k$. Without loss of generality we can assume that $n_1 \leq n_2 \leq ... \leq n_k$. For each $i = 1, ..., k$ let $n_i = n_{i,1} + n_{i,2}$, where integers $n_{i,1}$, $n_{i,2}$ are defined as follows. If $v_i \neq 0$, then $n_{i,1}$ and $n_{i,2}$ are respectively the greatest and the smallest of numbers $|\{j : x_j = v_i\}|$ and $|\{j : x_j = -v_i\}|$. If $v_i = 0$, then $n_{i,1} = n_i$ and $n_{i,2} = 0$.

Assume that $k \geq 2$. Then $n_1 \leq (d+1)/2$. If $n_1 \geq 3$, then $|\boldsymbol{x}\hat{S}_{d+1}| \geq C_{d+1}^{n_1} \geq C_{d+1}^{3} > 4(d+1)$. If $n_1 = 2$, then in the both cases $n_{1,1} = 1$ and $n_{1,1} = 2$ we have $|\boldsymbol{x}\hat{S}_{d+1}| \geq 2C_{d+1}^{2} = (d+1)d > 4(d+1)$. If, lastly, $n_1 = 1$, then in the case of $n_2 = d$ we have $n_{2,1} \leq d-1$ (overwise the orbit $\boldsymbol{x}\hat{S}_{d+1}$ would be of type 2) and $|\boldsymbol{x}\hat{S}_{d+1}| \geq d + 1C_d^{n_{2,1}} \geq (d+1)d > 4(d+1)$, and in the case of $n_2 < d$ we have $|\boldsymbol{x}\hat{S}_{d+1}| \geq (d+1)C_d^{n_2} \geq (d+1)d > 4(d+1)$.

Now assume that $k = 1$, and therefore $n_1 = d + 1$. Then $n_{1,2} \geq 2$. In the case $n_{1,1} \neq n_{1,2}$ we have $|\boldsymbol{x}\hat{S}_{d+1}| \geq 2C_{d+1}^{n_{1,2}} \geq (d+1)d > 4(d+1)$, and in the case $n_{1,1} = n_{1,2} = (d+1)/2$ we have $|\boldsymbol{x}\hat{S}_{d+1}| = C_{d+1}^{(d+1)/2} > C_{d+1}^3 > 4(d+1)$.

Hence in all cases the order of third-type orbit is greater than $4(d+1)$. □

Proposition 11. $\Gamma_{\mathbb{Z}^d, M_{d,2}} \in \lim(\mathcal{F}P_{HA}^{min})$ *for each* $d \in \{2, 4, 5, 6, ...\}$.

Proof. For $d \in \{2, 4, 5\}$ it is mentioned in Introduction that $\Gamma_{\mathbb{Z}^d, M_{d,2}} \in \lim(\mathcal{F}P_{HA}^{min})$. Now suppose $d \geq 6$. It is clear that $M_{d,2}$ is the $\widetilde{S}_{d+1}$-orbit. By Proposition 10 it follows that $\Gamma_{\mathbb{Z}^d, M_{d,2}} \in \mathcal{O}\Gamma^{min}(\widetilde{S}_{d+1})$. It is easy to see that for any prime $p > d+1$ the group $\phi_p(\widetilde{S}_{d+1})$ is irreducible on $\mathbb{Z}_p^d$. Therefore by Proposition 5 we have $\Gamma_{\mathbb{Z}^d, M_{d,2}} \in \lim(\mathcal{F}P_{HA}^{min})$. □

3. Proof of Theorem 2

To prove Theorem 2 we use Proposition 2 to get the description of $\text{Cay}^{min}(\mathbb{Z}^7)$, and then we show that each graph from $\text{Cay}^{min}(\mathbb{Z}^7)$ lies in $\lim(\mathcal{F}\mathcal{P}_{HA}^{min})$.

The list of all maximal irreducible over $\mathbb{Q}$ subgroups of $\text{GL}_7(\mathbb{Z})$, up to conjugation in $\text{GL}_7(\mathbb{Q})$, is known and is available via GAP-function ImfMatrixGroup [7]. It consists of three groups: K_7, $\widetilde{S}_8$ (defined in previous section) and $W(E_7)$ from Theorem 2. At the same time, for each reducible over $\mathbb{Q}$ subgroup H of $\text{GL}_7(\mathbb{Z})$, Proposition 7 and Remark 2 imply that the degree of any graph from $\mathcal{OS}\Gamma^{min}(H)$ is not greater than 6; but the degree of any graphs from $\mathcal{OS}\Gamma^{min}(H)$ is greater that 14. Therefore $\mathcal{OS}\Gamma^{min}(H) = \emptyset$. Now Proposition 2 implies $\text{Cay}^{min}(\mathbb{Z}^7) = \mathcal{OS}\Gamma^{min}(K_7) \cup \mathcal{OS}\Gamma^{min}(\widetilde{S}_8) \cup \mathcal{OS}\Gamma^{min}(W(E_7))$. To get the description of $\text{Cay}^{min}(\mathbb{Z}^7)$ we find classes $\mathcal{OS}\Gamma^{min}(K_7)$, $\mathcal{OS}\Gamma^{min}(\widetilde{S}_8)$, and $\mathcal{OS}\Gamma^{min}(W(E_7))$.

As mentioned before, $\mathcal{OS}\Gamma^{min}(K_7) = \{\Gamma_{\mathbb{Z}^7, M_{7,1}}\}$.

Now we find $\mathcal{OS}\Gamma^{min}(\widetilde{S}_8)$. Applying Algorithm 2 to $\widetilde{S}_8$, we get the following List of its subgroups:

$$G_{7,1} = \langle \boldsymbol{P}_1, \boldsymbol{P}_1' \rangle \cong C_2 \times (PSL_3(2) : C_2),$$

$$G_{7,2} = \langle \boldsymbol{P}_2, -\boldsymbol{I}, \boldsymbol{R}_2, \boldsymbol{Q}_2 \rangle \cong C_2 \times (C_2^3 : PSL_3(2)), \text{ where}$$

$$\boldsymbol{P}_1 = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \boldsymbol{P}_1' = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\boldsymbol{P}_2 = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{R}_2 : \boldsymbol{e}_1 \mapsto \boldsymbol{e}_7 \mapsto \boldsymbol{e}_2 \mapsto \boldsymbol{e}_6 \mapsto \boldsymbol{e}_3 \mapsto \boldsymbol{e}_4 \mapsto \boldsymbol{e}_5 \mapsto \boldsymbol{e}_1,$$

$$\boldsymbol{Q}_2 : \boldsymbol{e}_1 \mapsto \boldsymbol{e}_7 \mapsto \boldsymbol{e}_4 \mapsto \boldsymbol{e}_1, \boldsymbol{e}_2 \mapsto \boldsymbol{e}_6 \mapsto \boldsymbol{e}_3 \mapsto \boldsymbol{e}_2, \boldsymbol{e}_5 \mapsto \boldsymbol{e}_5.$$

It is easy to see that $M_{7,2}$ is both a $G_{7,1}$-orbit and a $G_{7,2}$-orbit. We will show that $\mathcal{MO}_{\mathbb{Q}}(G_{7,1}) = \mathcal{MO}_{\mathbb{Q}}(G_{7,2}) = \{cM_{7,2} : c \in \mathbb{Q}^*\}$. For this we will show that $|\boldsymbol{x}G_{7,k}| > 16$ for $k \in \{1, 2\}$ and any nonzero $\boldsymbol{x} \in \mathbb{Q}^7$ such that $\boldsymbol{x} \notin cM_{7,2}$ for any $c \in \mathbb{Q}$.

For $k \in \{1, 2\}$, let H_k be the stabilizer of the set $\{\boldsymbol{e}_1, \boldsymbol{e}_2, ..., \boldsymbol{e}_7\}$ in the group $G_{7,k}$. It is easy to see that $H_k = \langle \boldsymbol{R}_k, \boldsymbol{Q}_k \rangle$ for $k \in \{1, 2\}$, where

$$\boldsymbol{R}_1 : \boldsymbol{e}_1 \mapsto \boldsymbol{e}_7 \mapsto \boldsymbol{e}_4 \mapsto \boldsymbol{e}_2 \mapsto \boldsymbol{e}_6 \mapsto \boldsymbol{e}_3 \mapsto \boldsymbol{e}_5 \mapsto \boldsymbol{e}_1,$$

$$\boldsymbol{Q}_1 : \boldsymbol{e}_1 \mapsto \boldsymbol{e}_7 \mapsto \boldsymbol{e}_5 \mapsto \boldsymbol{e}_2 \mapsto \boldsymbol{e}_4 \mapsto \boldsymbol{e}_6 \mapsto \boldsymbol{e}_1, \boldsymbol{e}_3 \mapsto \boldsymbol{e}_3$$

$(H_1 \cong (C_7 : C_3) : C_2,\ H_2 \cong PSL_3(2))$. It is also easy to see that $\boldsymbol{P}'_2 \in H_2$, where

$$\boldsymbol{P}'_2 : \boldsymbol{e}_1 \mapsto \boldsymbol{e}_6 \mapsto \boldsymbol{e}_1, \boldsymbol{e}_2 \mapsto \boldsymbol{e}_2, \boldsymbol{e}_3 \mapsto \boldsymbol{e}_4 \mapsto \boldsymbol{e}_3, \boldsymbol{e}_5 \mapsto \boldsymbol{e}_5, \boldsymbol{e}_7 \mapsto \boldsymbol{e}_7.$$

It is easy to see that H_k is transitive on 2-element subsets of $\{\boldsymbol{e}_1, \boldsymbol{e}_2, ..., \boldsymbol{e}_7\}$ for $k \in \{1, 2\}$. Also it is obvious that the length of H_k-orbit on 3-element subsets of $\{\boldsymbol{e}_1, \boldsymbol{e}_2, ..., \boldsymbol{e}_7\}$ is not less than 7 for $k \in \{1, 2\}$.

Let $\boldsymbol{x} = (x_1, ..., x_7)$ be a nonzero element of $\mathbb{Q}^7$, such that $\boldsymbol{x} \notin cM_{7,2}$ for any $c \in \mathbb{Q}$. By X we denote the multiset $X = \{x_1, ..., x_7\}$. Then one of the following cases is realized.

1. X contains two elements that differ from all other elements ($\{x_i, x_j\} \cap \{x_l : l \neq i, j\} = \emptyset$ for some i, j). In this case $|\boldsymbol{x}H_k| \geq C_7^2 = 21$ for $k \in \{1, 2\}$.
2. $X = \{a, a, a, a, b, b, b\}$ for some $a, b \in \mathbb{Q}, a \neq b$. In this case $\boldsymbol{x}H_k \geq 7$, and for some $i \in \{0, ..., 6\}$ the row-vector $\boldsymbol{y} = \boldsymbol{x}\boldsymbol{R}_k^i\boldsymbol{P}_k$ has the multiset of coordinates $\{-a, 0, 0, 0, b-a, b-a, b-a\} \neq X$. Therefore, for $k \in \{1, 2\}$ it holds $|\boldsymbol{x}G_{7,k}| \geq |\boldsymbol{x}H_k \cup (-\boldsymbol{x})H_k \cup \boldsymbol{y}H_k \cup (-\boldsymbol{y})H_k| \geq 28$.
3. $X = \{a, a, a, b, b, b, c\}$ for some differet $a, b, c \in \mathbb{Q}$, $a \neq 0$. In this case $\boldsymbol{x}\boldsymbol{R}_k^i\boldsymbol{P}_k$ have the multiset of coordinates $\{-a, 0, 0, b-a, b-a, b-a, c-a\}$. Therefore, $|\boldsymbol{x}H_k| \geq 21$ for $k \in \{1, 2\}$ as in the case 1.
4. $X = \{a, a, a, a, a, a, b\}$ for some $a \in \mathbb{Q}$, $a \neq 0$. In this case $\boldsymbol{x}\boldsymbol{R}_k^i\boldsymbol{P}_k$ have the multiset of coordinates equal to $\{-a, b-a, 0, 0, 0, 0, 0\}$. Therefore, $|\boldsymbol{x}H_k| \geq 21$ for $k \in \{1, 2\}$ as in the case 1.

Thus in all cases $|\boldsymbol{x}H_k| > 16$, and we have shown that $\mathcal{MO}_{\mathbb{Q}}(G_{7,k}) = \{cM_{7,2} : c \in \mathbb{Q}^*\}$ for $k \in \{1, 2\}$.

Now we have $\mathcal{O}\Gamma^{min}(G_{7,k}) = \{\Gamma_{\mathbb{Z}^7, M_{7,2}}\}$ for $k \in \{1, 2\}$. By Proposition 9 we have $\mathcal{O}S\Gamma^{min}(\widetilde{S}_8) \setminus \{\Gamma_{\mathbb{Z}^7, M_{7,1}}\} = \{\Gamma_{\mathbb{Z}^7, M_{7,2}}\}$.

Now we find $\mathcal{O}S\Gamma^{min}(W(E_7))$. Applying Algorithm 2 to $W(E_7)$, we get the following List of its subgroups: just defined groups $G_{7,1}, G_{7,2}$,

$$G_{7,3} = \langle \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & -1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 1 \end{pmatrix} \rangle,$$

$$G_{7,3} \cong C_2 \times (PSL_3(2) : C_2),$$

$$G_{7,4} = \langle \begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ -1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & 0 \end{pmatrix} \rangle,$$

$$G_{7,4} \cong C_2 \times PSU(3, 3),$$

$$G_{7,5} = \langle \begin{pmatrix} 0 & 1 & 0 & 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & -1 & 1 \\ -1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 1 & -1 & 1 \\ -1 & 0 & -1 & 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & -1 & 1 \\ 1 & 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix} \rangle,$$

$$G_{7,5} \cong C_2 \times PSL_2(8).$$

Applying Algorithm 1 to the groups $G_{7,k}$ for $k \in \{3,4,5\}$ gives: $\mathcal{MF}(G_{7,3}) = \{\langle \boldsymbol{v}_1 \rangle_{\mathbb{Q}}\}$ where $\boldsymbol{v}_1 = (1,-3,-1,-1,1,1,-1)$, $\mathrm{m}(G_{7,3}) = 16$; $\mathcal{MF}(G_{7,4}) = \{\langle \boldsymbol{v}_2 \rangle_{\mathbb{Q}}\}$ where $\boldsymbol{v}_2 = (1,-1,-1,-1,-1,1,-1)$, $\mathrm{m}(G_{7,4}) = 56$; $\mathcal{MF}(G_{7,5}) = \{\langle \boldsymbol{v}_3 \rangle_{\mathbb{Q}}\}$ where $\boldsymbol{v}_3 = (1,-1,1,-1,-1,-1,-1)$, $\mathrm{m}(G_{7,5}) = 56$. By Proposition 8 we get $\mathcal{O}\mathit{\Gamma}^{min}(G_{7,i}) \subseteq \{\Gamma_{\langle \boldsymbol{v}_{i-2} G_{7,i}\rangle, \boldsymbol{v}_{i-2} G_{7,i}}\}$ for $i = 3,4,5$. Since $\boldsymbol{v}_1 G_{7,3} = M_{7,2}\boldsymbol{g}_3\boldsymbol{g}_1$, $\boldsymbol{v}_2 G_{7,4} = M_{7,3}\boldsymbol{g}_2$, $\boldsymbol{v}_3 G_{7,5} = M_{7,3}$, where

$$\boldsymbol{g}_1 = \begin{pmatrix} 1 & -1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & -1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 \end{pmatrix}, \boldsymbol{g}_2 = \begin{pmatrix} 0 & 1 & -1 & 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{g}_3 = \begin{pmatrix} 1 & -1 & -1 & 3 & -1 & 3 & -3 \\ 1 & 3 & -1 & -1 & -1 & -1 & 1 \\ 1 & -1 & 3 & -1 & 3 & -1 & 1 \\ 1 & -1 & -1 & -1 & -1 & 3 & 1 \\ 1 & -1 & 3 & -1 & -1 & -1 & -3 \\ -3 & -1 & -1 & -1 & 3 & -1 & 1 \\ 1 & -1 & -1 & -1 & -1 & -1 & 1 \end{pmatrix},$$

by Proposition 3 we get $\mathcal{O}\mathit{\Gamma}^{min}(G_{7,3}) \subseteq \{\Gamma_{\mathbb{Z}^7, M_{7,2}}\}$ and $\mathcal{O}\mathit{\Gamma}^{min}(G_{7,i}) \subseteq \{\Gamma_{\mathbb{Z}^7, M_{7,3}}\}$ for $i = 4,5$. By Proposition 9 we have $\mathcal{OS}\mathit{\Gamma}^{min}(W(E_7)) \setminus \{\Gamma_{\mathbb{Z}^7, M_{7,1}}\} = \bigcup_{i=1}^{5} \mathcal{O}\mathit{\Gamma}^{min}(G_{7,i}) = \{\Gamma_{\mathbb{Z}^7, M_{7,2}}, \Gamma_{\mathbb{Z}^7, M_{7,3}}\}$.

Now we have $\mathrm{Cay}^{min}(\mathbb{Z}^7) = \mathcal{OS}\mathit{\Gamma}^{min}(K_7) \cup \mathcal{OS}\mathit{\Gamma}^{min}(\tilde{S}_8) \cup \mathcal{OS}\mathit{\Gamma}^{min}(W(E_7)) = \{\Gamma_{\mathbb{Z}^7, M_{7,i}} : i = 1,2,3\}$. As we noticed in Introduction, $\Gamma_{\mathbb{Z}^7, M_{7,1}} \in \lim(\mathcal{FP}_{HA}^{min})$. By Proposition 11, $\Gamma_{\mathbb{Z}^7, M_{7,2}} \in \lim(\mathcal{FP}_{HA}^{min})$. Thus to complete the proof of Theorem 2 it is sufficient to prove that $\Gamma_{\mathbb{Z}^7, M_{7,3}} \in \lim(\mathcal{F}P_{HA}^{min})$.

Since $\boldsymbol{v}_3 G_{7,5} = M_{7,3}$, we have $M_{7,3} \in \mathcal{MO}_{\mathbb{Z}}(G_{7,5})$. Since $|G_{7,5}| = 1008$ and Proposition 4, where exists a positive integer p_0, such that for each prime $p > p_0$ and $\boldsymbol{x} \in \mathbb{Z}_p^7 \setminus \{\boldsymbol{0}\}$ the condition $|\boldsymbol{x}\phi_p(G)| \geq 56$ holds. Let $P = \{p \mid p \text{ is a prime number}, p > p_0, p > 1008, p \equiv 2 (\mathrm{mod}\ 9) \text{ and } p \equiv 3 (\mathrm{mod}\ 7)\}$. Now we show that the group $\phi_p(G_{7,5})$ is irreducible on $\mathbb{Z}_p^d$ for each $p \in P$. Assume, contrary, that, for some $p \in P$ there exists a proper $\phi_p(G_{7,5})$-invariant subspace U of $\mathbb{Z}_p^d$. Since $p > |G_{7,5}|$, by and Maschke's theorem, we can assume without loss of generality that $\dim(U) \leq 3$. By the choice of p_0, we have $|\phi_p(G_{7,5})^U| \geq 56$, where $\phi_p(G_{7,5})^U$ is the restriction of the group $\phi_p(G_{7,5})$ to the subspace U. Therefore the condition $|G_{7,5}| = 1008 = 2^4 3^2 7$ implies that either 7 or 9 divides $|\phi_p(G_{7,5})^U|$. Since $\phi_p(G_{7,5})^U \lesssim \mathrm{GL}_3(p)$, it follows that 7 or 9 divides $|\mathrm{GL}_3(p)| = (p^3-1)(p^2-1)(p-1)p^3$. But this contradicts to the condition $p \equiv 3 (\mathrm{mod}\ 7)$ and $p \equiv 2 (\mathrm{mod}\ 9)$. Therefore, $\phi_p(G_{7,5})$ is irreducible on $\mathbb{Z}_p^d$ for each $p \in P$, and by Proposition 5 we get $\Gamma_{\mathbb{Z}^7, M_{7,3}} \in \lim(\mathcal{F}P_{HA}^{min})$.

The proof of Theorem 2 is complete.

4. Proof of Theorem 1: the beginning

We started the proof of Theorem 1 in Introduction by giving the description of graphs of degree ≤ 16 lying in $\lim(\mathcal{FP}_{HA}^{min})$. The present section results will be used to get the description of graphs of degrees 18 and 20.

For each positive integer s, we denote by $\mathcal{T}_s$ the class of all (up to permutation isomorphism) minimal transitive permutation groups on the set $\{1,2,...,s\}$. Next we will show, how classes $\mathcal{T}_s$ can be used to find minimal Cayley graphs of groups $\mathbb{Z}^d$. For each positive integers s and d we denote by $\mathrm{Cay}_{2s}^{min}(\mathbb{Z}^d)$ the subclass of graphs of degree $2s$ from $\mathrm{Cay}^{min}(\mathbb{Z}^d)$. We define a binary relation $\tau_{d,s}$ between $\mathrm{Cay}_{2s}^{min}(\mathbb{Z}^d)$ and $\mathcal{T}_s$ as follows. Let $\Gamma_{\mathbb{Z}^d,M} \in \mathrm{Cay}_{2s}^{min}(\mathbb{Z}^d)$. Set $H = \mathrm{Aut}(\Gamma_{\mathbb{Z}^d,M})_0$. We consider H as a subgroup of $\mathrm{GL}_d(\mathbb{Z})$, as described in Introduction. Let $\overline{M}$ be the set of two-element sets of mutually inverse elements of M (for $\boldsymbol{x} \in M$ the corresponding element $\{-\boldsymbol{x}, \boldsymbol{x}\}$ of $\overline{M}$ we will denote by $\overline{\boldsymbol{x}}$). Since H is transitive on M, there is at least one group $T \in \mathcal{T}_s$, such that permutation group $H^{\overline{M}}$ contains a subgroup, which is permutationally isomorphic to T. For all such groups T we put the pair $(\Gamma_{\mathbb{Z}^d,M}, T)$ into $\tau_{d,s}$. The condition that some soubgroup of $H^{\overline{M}}$ is permutationally isomorphic to T implies that there exists a bijection $\psi : \overline{M} \mapsto \{1, 2, ..., s\}$ such that $T \leq \psi H^{\overline{M}} \psi^{-1}$. By χ we denote the homomorphism of H to symmetrical group on $\{1, ..., s\}$, defined by $\boldsymbol{g} \mapsto \psi \boldsymbol{g}^{\overline{M}} \psi^{-1}$. We get the whole $\tau_{d,s}$, when $\Gamma_{\mathbb{Z}^d,M}$ runs through all graphs from $\mathrm{Cay}_{2s}^{min}(\mathbb{Z}^d)$. Clearly, the following proposition holds.

Proposition 12. *For any positive integers d and s*

$$\mathrm{Cay}_{2s}^{min}(\mathbb{Z}^d) = \bigcup_{T \in \mathcal{T}_s} \tau_{d,s}^{-1}(T).$$

In the next two sections we also need the following Propositions 13, 14 and 15.

Proposition 13. *Let $\boldsymbol{A} \in GL_d(\mathbb{Z})$ and $M = \{\boldsymbol{x}_1, -\boldsymbol{x}_1, \boldsymbol{x}_2, -\boldsymbol{x}_2, ..., \boldsymbol{x}_{d+1}, -\boldsymbol{x}_{d+1}\}$ be a $\boldsymbol{A}$-invariant system of generators of $\mathbb{Z}^d$, where $\boldsymbol{x}_i \in \mathbb{Z}^d$ for $i = 1, ..., d+1$. And let $\boldsymbol{A}^{\overline{M}}$ be a cyclic permutation, such that $\boldsymbol{x}_i \boldsymbol{A} = \boldsymbol{x}_{i+1}$ for $i = 1, ..., d$ and $\overline{\boldsymbol{x}_{d+1}} \boldsymbol{A} = \overline{\boldsymbol{x}_1}$. Then the following holds.*
1a) If d is even then $\boldsymbol{x}_{d+1} = -\boldsymbol{x}_1 - \varepsilon \boldsymbol{x}_2 - \boldsymbol{x}_3 - \varepsilon \boldsymbol{x}_4 - ... - \boldsymbol{x}_{d-1} - \varepsilon \boldsymbol{x}_d$ for some $\varepsilon \in \{1, -1\}$.
1b) If d is odd then $\boldsymbol{x}_{d+1} = \boldsymbol{x}_1 - \boldsymbol{x}_2 + \boldsymbol{x}_3 - ... + \boldsymbol{x}_d$.
2) $\tau_{d,d+1}^{-1}(C_{d+1}) = \{\Gamma_{\mathbb{Z}^d,M_{d,2}}\}$, where C_{d+1} is a cyclic permutation group of order $d+1$ on the set $\{1, 2, ..., d+1\}$.

Proof. Substituting, if need, some of $\boldsymbol{x}_i$ $(i = 2, ..., d+1)$ by their inverses, we get $\boldsymbol{x}_i \boldsymbol{A} = \boldsymbol{x}_{i+1}$ for $i = 1, ..., d$ and $\boldsymbol{x}_{d+1} \boldsymbol{A} = \varepsilon \boldsymbol{x}_1$ for some $\varepsilon \in \{1, -1\}$. Since $\langle \boldsymbol{x}_1, ..., \boldsymbol{x}_d \rangle_{\mathbb{Q}} \boldsymbol{A}^i = \langle \{\boldsymbol{x}_1, ..., \boldsymbol{x}_{d+1}\} \setminus \{\boldsymbol{x}_i\} \rangle_{\mathbb{Q}}$ for $i = 1, ..., d$ and the rank of vector system $\boldsymbol{x}_1, ..., \boldsymbol{x}_d, \boldsymbol{x}_{d+1}$ over $\mathbb{Q}$ is equal to d, the rank of system $\boldsymbol{x}_1, ..., \boldsymbol{x}_d$ is also equal to d. Let $\boldsymbol{x}_{d+1} = \sum_{i=1}^d \alpha_i \boldsymbol{x}_i$ be a decomposition of vector $\boldsymbol{x}_{d+1}$ in the basis $\boldsymbol{x}_1, ..., \boldsymbol{x}_d$. The equality $\boldsymbol{x}_{d+1} \boldsymbol{A} = \varepsilon \boldsymbol{x}_1$ gives $\alpha_1 \alpha_d \boldsymbol{x}_1 + (\alpha_1 + \alpha_2 \alpha_d) \boldsymbol{x}_2 + (\alpha_2 + \alpha_3 \alpha_d) \boldsymbol{x}_3 + ... + (\alpha_{d-2} + \alpha_{d-1} \alpha_d) \boldsymbol{x}_{d-1} + (\alpha_{d-1} + \alpha_d^2) \boldsymbol{x}_d = \varepsilon \boldsymbol{x}_1$. It follows $\alpha_{d-1} = -\alpha_d^2$, $\alpha_{d-2} = \alpha_d^3$, $\alpha_{d-3} = -\alpha_d^4$, ..., $\alpha_1 = (-1)^{d-1} \alpha_d^d$, $(-1)^{d-1} \alpha_d^{d+1} = \varepsilon$. If d is even, we get $\alpha_d = -\varepsilon$, $\alpha_k = -\varepsilon \varepsilon^k$ for $k = 1, ..., d$ and assertion 1a) is proved; if d is odd, we get $\varepsilon = 1$, $\alpha_d \in \{1, -1\}$ and $\alpha_k = -(-\alpha_d)^k$ for $k = 1, ..., d$, and assertion 1b) is proved.

Let $\Gamma_{\mathbb{Z}^d,M} \in \tau_{d,d+1}^{-1}(C_{d+1})$. Then by 1$a$) and 1$b$) M is linearly equivalent to $M_{d,2}$ and therefore $\Gamma_{\mathbb{Z}^d,M} \cong \Gamma_{\mathbb{Z}^d,M_{d,2}}$. To prove $\Gamma_{\mathbb{Z}^d,M_{d,2}} \in \tau_{d,d+1}^{-1}(C_{d+1})$ we choose $\boldsymbol{B} \in \mathrm{GL}_d(\mathbb{Z})$ such that $e_i \boldsymbol{B} = e_{i+1}$ for $i \in \{1, ..., d-1\}$ and $e_d \boldsymbol{B} = -e_1 - ... - e_d$. It is easy to see that $M_{d,2}$ is an orbit of group $\langle \boldsymbol{B} \rangle$, $\boldsymbol{B} \in \mathrm{Aut}(\Gamma_{\mathbb{Z}^d,M})_0$ and $(\Gamma_{\mathbb{Z}^d,M_{d,2}}, C_{d+1}) \in \tau_{d,d+1}$. Therefore, $\Gamma_{\mathbb{Z}^d,M_{d,2}} \in \tau_{d,d+1}^{-1}(C_{d+1})$ and assertion 2) holds. □

Proposition 14. *Let $T = \langle a, b \rangle$ be a permutation group of degree 10, where $a = (1, 2, 3, 4, 5)(6, 7, 8, 9, 10)$ and b interchanges $\{1, 2, 3, 4, 5\}$ and $\{6, 7, 8, 9, 10\}$. Then $\tau_{8,10}^{-1}(T) = \{\Gamma_{\mathbb{Z}^8,M_{8,3}}\}$.*

Proof. First we proove $\tau_{8,10}^{-1}(T) \subseteq \{\Gamma_{\mathbb{Z}^8, M_{8,3}}\}$. Let $(\Gamma_{\mathbb{Z}^8,M}, T) \in \tau_{8,10}$, $\boldsymbol{A} \in \chi^{-1}(a)$ and $\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_{10} \in M$ be vectors such that $\overline{\boldsymbol{x}_i} = \psi^{-1}(i)$ $(i = 1, ..., 10)$. Let $k = \dim(\langle \boldsymbol{x}_1, ..., \boldsymbol{x}_5 \rangle_{\mathbb{Q}})$. Since b interchanges $\{1,2,3,4,5\}$ and $\{6,7,8,9,10\}$, $\dim(\langle \boldsymbol{x}_6, ..., \boldsymbol{x}_{10} \rangle_{\mathbb{Q}}) = k$.

By $\dim(\langle \boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_{10} \rangle) \leq \dim(\langle \boldsymbol{x}_1, ..., \boldsymbol{x}_5 \rangle_{\mathbb{Q}}) + \dim(\langle \boldsymbol{x}_6, ..., \boldsymbol{x}_{10} \rangle_{\mathbb{Q}})$ we have $k \geq 4$. If $k = 5$, then there exist $\boldsymbol{y}_1 \in \mathbb{Q}^8, \boldsymbol{y}_2 \in \mathbb{Q}^8 \setminus \langle \boldsymbol{y}_1 \rangle_{\mathbb{Q}}$, such that $\langle \boldsymbol{y}_1, \boldsymbol{y}_2 \rangle = \langle \boldsymbol{x}_1, ..., \boldsymbol{x}_5 \rangle_{\mathbb{Q}} \cap \langle \boldsymbol{x}_6, ..., \boldsymbol{x}_{10} \rangle_{\mathbb{Q}}$, and, therefore, the two-dimensional subspace $\langle \boldsymbol{y}_1, \boldsymbol{y}_2 \rangle_{\mathbb{Q}}$ is $\boldsymbol{A}$-invariant. By Maschke's theorem we can complement the pair $\boldsymbol{y}_1, \boldsymbol{y}_2$ of vectors of subspace $\langle \boldsymbol{x}_1, ..., \boldsymbol{x}_5 \rangle_{\mathbb{Q}}$ by some $\boldsymbol{y}_3, \boldsymbol{y}_4, \boldsymbol{y}_5$ to the basis of this space, such that $\langle \boldsymbol{y}_3, \boldsymbol{y}_4, \boldsymbol{y}_5 \rangle_{\mathbb{Q}}$ is $\boldsymbol{A}$-invariant. The mapping $\boldsymbol{A}$ in the basis $\boldsymbol{y}_1, ..., \boldsymbol{y}_5$ will be specified by the matrix

$$\begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & 0 & 0 & 0 \\ \alpha_{2,1} & \alpha_{2,2} & 0 & 0 & 0 \\ 0 & 0 & \beta_{1,1} & \beta_{1,2} & \beta_{1,3} \\ 0 & 0 & \beta_{2,1} & \beta_{2,2} & \beta_{2,3} \\ 0 & 0 & \beta_{3,1} & \beta_{3,2} & \beta_{3,3} \end{pmatrix},$$

where $\alpha_{i,j} \in \mathbb{Q}$ for $i, j \in \{1,2\}$ and $\beta_{i,j} \in \mathbb{Q}$ for $i, j \in \{1,2,3\}$. Since $|x_1 A| \in \{5, 10\}$, the order of the matrix $(\alpha_{i,j})_{i=1,j=1}^{2,2}$ or the order of the matrix $(\beta_{i,j})_{i=1,j=1}^{3,3}$ is a multiple of 5. But Minkowski's theorem implies that the least common multiple of the orders of finite subgroups of $\mathrm{GL}_n(\mathbb{Q})$ is equal to $\prod p^{\lfloor \frac{n}{p-1} \rfloor + \lfloor \frac{n}{p(p-1)} \rfloor + \lfloor \frac{n}{p^2(p-1)} \rfloor + \cdots}$, where the product is taken over all primes p. In both cases $n = 2$ and $n = 3$ the exponent index corresponding to $p = 5$ is equal to zero, and, therefore, $\mathrm{GL}_2(\mathbb{Q})$ and $\mathrm{GL}_3(\mathbb{Q})$ has no subgroups of order 5. Thus the case $k = 5$ is impossible and therefore $k = 4$.

By Proposition 13.1*a*), after replacing if need, some of $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4$ by their inverses, we can assume without loss of generality that $\boldsymbol{x}_5 = -\boldsymbol{x}_1 - \varepsilon_1 \boldsymbol{x}_2 - \boldsymbol{x}_3 - \varepsilon_1 \boldsymbol{x}_4$ for some $\varepsilon_1 \in \{1, -1\}$. Similarily, we can assume that $\boldsymbol{x}_{10} = -\boldsymbol{x}_6 - \varepsilon_2 \boldsymbol{x}_7 - \boldsymbol{x}_8 - \varepsilon_2 \boldsymbol{x}_9$ for some $\varepsilon_2 \in \{1, -1\}$. Therefore, M is linearly equivalent to $M_{8,3}$, $\Gamma_{\mathbb{Z}^8,M} \cong \Gamma_{\mathbb{Z}^8,M_{8,3}}$, and $\tau_{8,10}^{-1}(T) \subseteq \{\Gamma_{\mathbb{Z}^8,M_{8,3}}\}$ is proved.

Now we proove $\{\Gamma_{\mathbb{Z}^8,M_{8,3}}\} \subseteq \tau_{8,10}^{-1}(T)$. Let ω be a mapping of the set $\{1, ..., 10\}$ into $\mathbb{Z}^8$, defined by

$$\omega(i) = \begin{cases} \boldsymbol{e}_i \text{ for } i \in \{1, ..., 4\}, \\ -\boldsymbol{e}_1 - \boldsymbol{e}_2 - \boldsymbol{e}_3 - \boldsymbol{e}_4 \text{ for } i = 5, \\ \boldsymbol{e}_{i-1} \text{ for } i \in \{6, ..., 9\}, \\ -\boldsymbol{e}_5 - \boldsymbol{e}_6 - \boldsymbol{e}_7 - \boldsymbol{e}_8 \text{ for } i = 10. \end{cases}$$

Let $\boldsymbol{A}, \boldsymbol{B} \in \mathrm{GL}_8(\mathbb{Z})$ be such that $e_i \boldsymbol{A} = \omega(a(i))$, $e_i \boldsymbol{B} = \omega(b(i))$ for $i \in \{1, ..., 8\}$. It is easy to see that $M_{8,3}$ is the orbit of the group $G := \langle \boldsymbol{A}, \boldsymbol{B}, -\boldsymbol{I} \rangle$, $G \leq \mathrm{Aut}(\Gamma_{\mathbb{Z}^8,M_{8,3}})_0$ and $G_{8,3}^M$ is permutationally isomorphic to T. Therefore $\Gamma_{\mathbb{Z}^8,M_{8,3}} \in \tau_{8,10}^{-1}(T)$. □

Proposition 15. $\Gamma_{\mathbb{Z}^8,M_{8,3}} \in \lim(\mathcal{FP}_{HA}^{min})$.

Proof. It is easy to see that $M_{8,3} = \boldsymbol{e}_1 G$ for $G = \langle \boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{B} \rangle$, where $\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{B} \in \mathrm{GL}_8(\mathbb{Z})$, $\boldsymbol{e}_1 \boldsymbol{A}_1 = \boldsymbol{e}_2$, $\boldsymbol{e}_2 \boldsymbol{A}_1 = \boldsymbol{e}_3$, $\boldsymbol{e}_3 \boldsymbol{A}_1 = \boldsymbol{e}_4$, $\boldsymbol{e}_4 \boldsymbol{A}_1 = -\boldsymbol{e}_1 - \boldsymbol{e}_2 - \boldsymbol{e}_3 - \boldsymbol{e}_4$, $\boldsymbol{e}_i \boldsymbol{A}_1 = \boldsymbol{e}_i$ for $i = 5, 6, 7, 8$, $\boldsymbol{e}_i \boldsymbol{B} = \boldsymbol{e}_{i+4}$, $\boldsymbol{e}_{i+4} \boldsymbol{B} = \boldsymbol{e}_i$ for $i = 1, 2, 3, 4$, $\boldsymbol{A}_2 = \boldsymbol{B}^{-1} \boldsymbol{A}_1 \boldsymbol{B}$.

Set $P = \{p \mid p\text{-prime}, p \equiv 2 (\mathrm{mod}\ 5) \text{ or } p \equiv 3 (\mathrm{mod}\ 5)\}$. By Proposition 5, it is sufficient to prove that $\phi_p(G)$ is irreducible over $\mathrm{GF}(p)$ for each $p \in P$. Set $\boldsymbol{A}_1' = \boldsymbol{A}_1^{\langle \boldsymbol{e}_1, ..., \boldsymbol{e}_4 \rangle}$. For each $p \in P$ the group $\phi_p(\langle \boldsymbol{A}_1' \rangle)$ is irreducible over $\mathrm{GF}(p)$, because the characteristic polynomial $\det(\boldsymbol{A}_1' - \lambda \boldsymbol{I}) = \lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1$ has no nontrivial factors. To prove the irreducibility of $\phi_p(G)$ for each $p \in P$, we will show that $\langle \boldsymbol{v}\phi_p(G)\rangle = \mathbb{Z}_p^8$ for each nonzero element $\boldsymbol{v}$ from $\mathbb{Z}_p^8$. Let $\boldsymbol{v} \in \mathbb{Z}_p^8 \setminus \{\boldsymbol{0}\}$.

If $\boldsymbol{v} \in \langle \phi_p(\boldsymbol{e}_1), ..., \phi_p(\boldsymbol{e}_4)\rangle$, then the irreducibility of $\phi_p(\langle \boldsymbol{A}_1' \rangle)$ implies that $\boldsymbol{v}\langle \boldsymbol{A}_1'\rangle$ generates $\langle \phi_p(\boldsymbol{e}_1), ..., \phi_p(\boldsymbol{e}_4)\rangle$. At the same time, $\boldsymbol{v}\phi_p(\boldsymbol{B}) \in \langle \phi_p(\boldsymbol{e}_5), ..., \phi_p(\boldsymbol{e}_8)\rangle$ and $\boldsymbol{v}\phi_p(\boldsymbol{B})\langle \phi_p(\boldsymbol{A}_2^{\langle \boldsymbol{e}_5, ..., \boldsymbol{e}_8\rangle})\rangle$ generates $\langle \phi_p(\boldsymbol{e}_5), ..., \phi_p(\boldsymbol{e}_8)\rangle$. Therefore, $\boldsymbol{v}\phi_p(G)$ generates $\mathbb{Z}_p^8$. The case $\boldsymbol{v} \in \langle \phi_p(\boldsymbol{e}_5), ..., \phi_p(\boldsymbol{e}_8)\rangle$ is similar.

If $\boldsymbol{v} \notin \langle \phi_p(\boldsymbol{e}_1), ..., \phi_p(\boldsymbol{e}_4)\rangle$ and $\boldsymbol{v} \notin \langle \phi_p(\boldsymbol{e}_5), ..., \phi_p(\boldsymbol{e}_8)\rangle$, set $\boldsymbol{v}_1 = \boldsymbol{v}\phi_p(\boldsymbol{A}_1) - \boldsymbol{v}$. We have $\boldsymbol{v}_1 \in \langle \phi_p(\boldsymbol{e}_1), ..., \phi_p(\boldsymbol{e}_4)\rangle$ and $\boldsymbol{v}_1 \neq \boldsymbol{0}$. Therefore, $\boldsymbol{v}_1\phi_p(G)$ generates $\mathbb{Z}_p^8$. But $\boldsymbol{v}_1\phi_p(G) \subseteq \langle \boldsymbol{v}\phi_p(G)\rangle$ and, therefore, $\boldsymbol{v}\phi_p(G)$ generates $\mathbb{Z}_p^8$.

In all cases $\phi_p(G)$ is irreducible over GF(p) for each $p \in P$. □

5. Proof of Theorem 1: the case of graphs of degree 18

The class of graphs of degree 18 from $\lim(\mathcal{FP}_{HA}^{min})$ is a subclass of $\bigcup_{d=1}^{9} \mathrm{Cay}_{18}^{min}(\mathbb{Z}^d)$ (see the inclusion (1) in Introduction). The description of $\mathrm{Cay}^{min}(\mathbb{Z}^d)$ for $d \leq 6$ (see Introduction) and Theorem 2 imply that $\mathrm{Cay}_{18}^{min}(\mathbb{Z}^d) = \emptyset$ for $d = 1, 2, 3, 4, 5, 7$ and that $\mathrm{Cay}_{18}^{min}(\mathbb{Z}^6) = \{\Gamma_{\mathbb{Z}^6, M_{6,3}}\}$. In addition, $\mathrm{Cay}_{18}^{min}(\mathbb{Z}^9) = \{\Gamma_{\mathbb{Z}^9, M_{9,1}}\}$. To describe the remained class $\mathrm{Cay}_{18}^{min}(\mathbb{Z}^8)$, we will use Proposition 12.

Up to permutation isomorphism, there exist 37 transitive permutation groups of degree 9 (see [7]), and two of them are minimal. Their set we denote by $\mathcal{T}_9 = \{T_{9,1}, T_{9,2}\}$, where $T_{9,1} = \langle (1, 2, ..., 9)\rangle$, $T_{9,2} = \langle a, b\rangle$, $a = (1, 2, 3)(4, 5, 6)(7, 8, 9)$, $b = (1, 4, 7)(2, 5, 8)(3, 6, 9)$.

By Proposition 13, $\tau_{8,9}^{-1}(T_{9,1}) = \{\Gamma_{\mathbb{Z}^8, M_{8,2}}\}$.

Now we will find all graphs from the class $\tau_{8,9}^{-1}(T_{9,2})$. Let $(\Gamma_{\mathbb{Z}^8, M}, T_{9,2}) \in \tau_{8,9}$. Let $\boldsymbol{A} \in \chi^{-1}(a)$, $\boldsymbol{B} \in \chi^{-1}(b)$ and $\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_9 \in M$ with $\overline{\boldsymbol{x}_i} = \psi^{-1}(i)$ $(i = 1, ..., 9)$. Replacing, if need, some of $\boldsymbol{x}_2, \boldsymbol{x}_1, \boldsymbol{x}_5, \boldsymbol{x}_4, \boldsymbol{x}_8, \boldsymbol{x}_9, \boldsymbol{x}_7, \boldsymbol{x}_6, \boldsymbol{x}_3$ by their inverses we can assume without loss of generality that $\boldsymbol{x}_1\boldsymbol{A} = \boldsymbol{x}_2$, $\boldsymbol{x}_2\boldsymbol{A} = \boldsymbol{x}_3$, $\boldsymbol{x}_4\boldsymbol{A} = \boldsymbol{x}_5$, $\boldsymbol{x}_5\boldsymbol{A} = \boldsymbol{x}_6$, $\boldsymbol{x}_7\boldsymbol{A} = \boldsymbol{x}_8$, $\boldsymbol{x}_8\boldsymbol{A} = \boldsymbol{x}_9$, $\boldsymbol{x}_4\boldsymbol{B} = \boldsymbol{x}_7$, $\boldsymbol{x}_3\boldsymbol{B} = \boldsymbol{x}_6$, $\boldsymbol{x}_6\boldsymbol{B} = \boldsymbol{x}_9$. Since the group $\langle \boldsymbol{A}, \boldsymbol{B}\rangle$ is transitive on $\overline{M}$, any 8 vectors of $\boldsymbol{x}_1, ..., \boldsymbol{x}_9$ form a basis of $\mathbb{Q}^8$. In particular, $\boldsymbol{x}_9 = \sum_{j=1}^{8} \alpha_j \boldsymbol{x}_j$ for some $\alpha_1, ..., \alpha_8 \in \mathbb{Q}$. Thus there exist $\varepsilon_3, \varepsilon_6, \varepsilon \in \{-1, 1\}$ such that in the basis $\boldsymbol{x}_1, ..., \boldsymbol{x}_8$ the matrix of $\boldsymbol{A}$ is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \varepsilon_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 \end{pmatrix}$$

and $\boldsymbol{x}_9\boldsymbol{A} = \varepsilon\boldsymbol{x}_7$. Also there exist $\varepsilon_1, \varepsilon_2, \varepsilon_5, \varepsilon_7, \varepsilon_8, \varepsilon_9 \in \{-1, 1\}$ such that in the basis $\boldsymbol{x}_1, ..., \boldsymbol{x}_8$ the matrix of $\boldsymbol{B}$ is

$$\begin{pmatrix} 0 & 0 & 0 & \varepsilon_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon_5 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 \\ \varepsilon_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \varepsilon_8 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $\boldsymbol{x}_9\boldsymbol{B} = \varepsilon_9\boldsymbol{x}_3$. The equation $\boldsymbol{x}_9\boldsymbol{A} = \varepsilon\boldsymbol{x}_7$ is equvivalent to $(\varepsilon_3\alpha_3 + \alpha_1\alpha_8, \alpha_1 + \alpha_2\alpha_8, \alpha_2 + \alpha_3\alpha_8, \varepsilon_6\alpha_6 + \alpha_4\alpha_8, \alpha_4 + \alpha_5\alpha_8, \alpha_5 + \alpha_6\alpha_8, \alpha_7\alpha_8, \alpha_7 + \alpha_8^2) = (0, 0, 0, 0, 0, 0, \varepsilon, 0)$. The equations in two last coordinates give $\alpha_7 = -\alpha_8^2$, $-\alpha_8^3 = \varepsilon$ and, therefore, $\alpha_8 = -\varepsilon$, $\alpha_7 = -1$; equations in coordinates 2,3,5,6,1,4 give $\alpha_1 = \varepsilon\alpha_2$, $\alpha_3 = \varepsilon\alpha_2$, $\alpha_4 = \varepsilon\alpha_5$, $\alpha_6 = \varepsilon\alpha_5$, $(\varepsilon_3\varepsilon - 1)\alpha_2 = 0$, $(\varepsilon_6\varepsilon - 1)\alpha_5 = 0$. The equation $\boldsymbol{x}_9\boldsymbol{B} = \varepsilon_9\boldsymbol{x}_3$ is now equivalent to $(\alpha_2\alpha_5 - \varepsilon_7, \varepsilon\alpha_2\alpha_5 - \varepsilon_8\varepsilon, \alpha_2\alpha_5, \varepsilon_1\varepsilon\alpha_2 + \alpha_5^2, \varepsilon_2\alpha_2 + \varepsilon\alpha_5^2, \varepsilon\alpha_2 + \alpha_5^2, 0, \varepsilon_5\alpha_5 - \alpha_5) = (0, 0, \varepsilon_9, 0, 0, 0, 0, 0)$. Equations in coordinates 6,3 give $\alpha_2 = -\varepsilon\alpha_5^2$, $-\varepsilon\alpha_5^3 = \varepsilon_9$ and, therefore, $\alpha_5 = -\varepsilon\varepsilon_9$, $\alpha_2 = -\varepsilon$; equations in the coordinates 1,2,4, correspondingly, give $\varepsilon_7 = \varepsilon_9$, $\varepsilon_8 = \varepsilon_9$, $\varepsilon_1 = 1$; equations in the coordinates 5,8 correspondingly give $\varepsilon_2 = 1$, $\varepsilon_5 = 1$. The equations $(\varepsilon_3\varepsilon - 1)\alpha_2 = 0$, $(\varepsilon_6\varepsilon - 1)\alpha_5 = 0$ now give $\varepsilon_3 = \varepsilon_6 = \varepsilon$. Since $\boldsymbol{x}_9 = -\boldsymbol{x}_1 - \varepsilon\boldsymbol{x}_2 - \boldsymbol{x}_3 - \varepsilon_9\boldsymbol{x}_4 - \varepsilon\varepsilon_9\boldsymbol{x}_5 - \varepsilon_9\boldsymbol{x}_6 - \boldsymbol{x}_7 - \varepsilon\boldsymbol{x}_8$, M is linearly equivalent to $M_{8,2}$. Therefore, $\Gamma_{\mathbb{Z}^8,M} \cong \Gamma_{\mathbb{Z}^8,M_{8,2}}$ and $\tau_{8,9}^{-1}(T_{9,2}) \subseteq \{\Gamma_{\mathbb{Z}^8,M_{8,2}}\}$.

Now we will prove that $\{\Gamma_{\mathbb{Z}^8,M_{8,2}}\} \subseteq \tau_{8,9}^{-1}(T_{9,2})$. It is easy to see that $M_{8,2}$ is an orbit of group $G := \langle \boldsymbol{A}, \boldsymbol{B}, -\boldsymbol{I}\rangle$, where matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are defined as above with $\varepsilon = \varepsilon_9 = 1$. It is clear that $G \leq \mathrm{Aut}(\Gamma_{\mathbb{Z}^8,M_{8,2}})_0$ and $\Gamma_{\mathbb{Z}^d,M_{8,2}} \in \tau_{8,9}^{-1}(T_{9,2})$.

Since $\tau_{8,9}^{-1}(T_{9,1}) = \tau_{8,9}^{-1}(T_{9,2}) = \{\Gamma_{\mathbb{Z}^8,M_{8,2}}\}$, it follows from Proposition 12 that $\mathrm{Cay}_{18}^{min}(\mathbb{Z}^8) = \{\Gamma_{\mathbb{Z}^8,M_{8,2}}\}$. Therefore, $\bigcup_{d=1}^{9}\mathrm{Cay}_{18}^{min}(\mathbb{Z}^d) = \{\Gamma_{\mathbb{Z}^6,M_{6,3}}, \Gamma_{\mathbb{Z}^8,M_{8,2}}, \Gamma_{\mathbb{Z}^9,M_{9,1}}\}$. Since $\Gamma_{\mathbb{Z}^6,M_{6,3}}, \Gamma_{\mathbb{Z}^9,M_{9,1}} \in \lim(\mathcal{FP}_{HA}^{min})$ (see Introduction) and $\Gamma_{\mathbb{Z}^8,M_{8,2}} \in \lim(\mathcal{FP}_{HA}^{min})$ by Proposition 11, the assertion of Theorem 1 for degree 18 is proved.

6. Proof of Theorem 1: the case of graphs of degree 20

The class ofl graphs of degree 20 from $\lim(\mathcal{FP}_{HA}^{min})$ is a subclass of $\bigcup_{d=1}^{10}\mathrm{Cay}_{20}^{min}(\mathbb{Z}^d)$ (see equation (1) in Introduction). It follows from the description of $\mathrm{Cay}^{min}(\mathbb{Z}^d)$ for $d \leq 6$ and Theorem 2 (see Introduction) that $\mathrm{Cay}_{20}^{min}(\mathbb{Z}^d) = \emptyset$ for $d = 1, 2, 3, 4, 5, 7$ and $\mathrm{Cay}_{20}^{min}(\mathbb{Z}^6) = \{\Gamma_{\mathbb{Z}^6,M_{6,4}}\}$. Since $\mathrm{Cay}_{20}^{min}(\mathbb{Z}^{10}) = \{\Gamma_{\mathbb{Z}^{10},M_{10,1}}\}$ to get the description of $\bigcup_{d=1}^{10}\mathrm{Cay}_{20}^{min}(\mathbb{Z}^d)$ it remaines to find all graphs from $\mathrm{Cay}_{20}^{min}(\mathbb{Z}^8)$ and $\mathrm{Cay}_{20}^{min}(\mathbb{Z}^9)$. We will use Proposition 12 for this purpose.

Up to permutation isomorphism, there exist 45 transitive permutation groups of degree 10 (see [7]), and only six of them are minimal. Their set we denote by $\mathcal{T}_{10} = \{T_{10,1}, T_{10,2}, T_{10,3}, T_{10,4}, T_{10,5}, T_{10,6}\}$. For each $i = 1, ..., 6$ we now define permutation group $T_{10,i}$ and find all graphs from $\tau_{8,10}^{-1}(T_{10,i})$ and $\tau_{9,10}^{-1}(T_{10,i})$.

1. $T_{10,1} = \langle a, b\rangle$, where $a = (1, 2, 3, 4, 5)(6, 7, 8, 9, 10)$, $b = (1, 5)(2, 7)(3, 10)(4, 8)$, thus $T_{10,1} \cong A_5$.

1.1. We find all graphs from the class $\tau_{8,10}^{-1}(T_{10,1})$. Let $(\Gamma_{\mathbb{Z}^8,M}, T_{9,2}) \in \tau_{8,10}$, $\boldsymbol{A} \in \chi^{-1}(a)$, $\boldsymbol{B} \in \chi^{-1}(b)$. For $i = 1, ..., 10$ chose $\boldsymbol{x}_i \in \psi^{-1}(i)$, such that $\boldsymbol{x}_1 A = \boldsymbol{x}_{i+1}$ for $i \in \{1, 2, 3, 4, 6, 7, 8, 9\}$. If $\dim(\langle \boldsymbol{x}_1, ..., \boldsymbol{x}_5 \rangle_{\mathbb{Q}}) = \dim(\langle \boldsymbol{x}_6, ..., \boldsymbol{x}_{10} \rangle_{\mathbb{Q}}) = 4$, then Proposition 13 implies $\boldsymbol{x}_5 = -\boldsymbol{x}_1 - \varepsilon_1 \boldsymbol{x}_2 - \boldsymbol{x}_3 - \varepsilon_1 \boldsymbol{x}_4$ and $\boldsymbol{x}_{10} = -\boldsymbol{x}_6 - \varepsilon_2 \boldsymbol{x}_7 - \boldsymbol{x}_8 - \varepsilon_2 \boldsymbol{x}_9$ for some $\varepsilon_1, \varepsilon_2 \in \{1, -1\}$. Now the condition $\overline{\boldsymbol{x}_5} \boldsymbol{B} = \overline{\boldsymbol{x}_1}$ contradicts to the linear independence of $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4, \boldsymbol{x}_6, \boldsymbol{x}_7, \boldsymbol{x}_8, \boldsymbol{x}_9$. Therefore, $\dim(\langle \boldsymbol{x}_1, ..., \boldsymbol{x}_5 \rangle) = 5$ or $\dim(\langle \boldsymbol{x}_6, ..., \boldsymbol{x}_{10} \rangle_{\mathbb{Q}}) = 5$. Suppose first that $\dim(\langle \boldsymbol{x}_1, ..., \boldsymbol{x}_5 \rangle_{\mathbb{Q}}) = 5$. The 5-dimensional subspace $\langle \boldsymbol{x}_1, ..., \boldsymbol{x}_5 \rangle_{\mathbb{Q}}$ of $\mathbb{Q}^8$ is $\boldsymbol{A}$-invariant, and hence by Maschke's theorem it has $\boldsymbol{A}$-invariant complement V in $\mathbb{Q}^8$. We have found an element of order 5 in $\mathrm{GL}_3(\mathbb{Q})$ (say $\boldsymbol{A}^2$) with nontrivial action on V (if the action would be trivial, then the transitivity of action of $\boldsymbol{A}$ and $\boldsymbol{A}^2$ on $\{\langle \boldsymbol{x}_6 \rangle_{\mathbb{Q}}, ..., \langle \boldsymbol{x}_{10} \rangle_{\mathbb{Q}}\}$ would imply $8 = \dim(\langle \boldsymbol{x}_1, ..., \boldsymbol{x}_{10} \rangle_{\mathbb{Q}}) \leq 6$). But this contradicts to Minkowski's theorem mentioned in the proof of Proposition 14.

The case $\dim(\langle \boldsymbol{x}_6, ..., \boldsymbol{x}_{10} \rangle_{\mathbb{Q}}) = 5$ is impossible by analogous arguments. Therefore, $\tau_{8,10}^{-1}(T_{10,1}) = \emptyset$.

1.2. We find all graphs from the class $\tau_{9,10}^{-1}(T_{10,1})$. Let $(\Gamma_{\mathbb{Z}^9,M}, T_{10,1}) \in \tau_{9,10})$ and $\boldsymbol{A} \in \chi^{-1}(a)$, $\boldsymbol{B} \in \chi^{-1}(b)$. Choose $\boldsymbol{x}_i \in \psi^{-1}(i)$ $(i = 1, ..., 10)$ such that $\boldsymbol{x}_1 \boldsymbol{A} = \boldsymbol{x}_2$, $\boldsymbol{x}_2 \boldsymbol{A} = \boldsymbol{x}_3$, $\boldsymbol{x}_3 \boldsymbol{A} = \boldsymbol{x}_4$, $\boldsymbol{x}_4 \boldsymbol{A} = \boldsymbol{x}_5$, $\boldsymbol{x}_5 \boldsymbol{A} = \varepsilon \boldsymbol{x}_1$, $\boldsymbol{x}_6 \boldsymbol{A} = \boldsymbol{x}_7$, $\boldsymbol{x}_7 \boldsymbol{A} = \boldsymbol{x}_8$, $\boldsymbol{x}_8 \boldsymbol{A} = \boldsymbol{x}_9$, $\boldsymbol{x}_9 \boldsymbol{A} = \boldsymbol{x}_{10}$, $\boldsymbol{x}_{10} \boldsymbol{A} = \varepsilon' \boldsymbol{x}_6$ for some $\varepsilon, \varepsilon' \in \{1, -1\}$. Since $\dim(\langle \boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_{10} \rangle_{\mathbb{Q}}) = 9$, the transitivity of the group $T_{10,1}$ implies that vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_9$ form a basis of $\mathbb{Q}^9$. Let $\sum_{i=1}^{9} \alpha_i \boldsymbol{x}_i$ be a decomposition of $\boldsymbol{x}_{10}$ in this basis. The equation $\boldsymbol{x}_{10} \boldsymbol{A} = \varepsilon' \boldsymbol{x}_6$ gives $(\sum_{i=1}^{9} \alpha_i \boldsymbol{x}_i) \boldsymbol{A} = \varepsilon' \boldsymbol{x}_6$, or $(\varepsilon \alpha_5 + \alpha_9 \alpha_1) \boldsymbol{x}_1 + (\alpha_1 + \alpha_9 \alpha_2) \boldsymbol{x}_2 + (\alpha_2 + \alpha_9 \alpha_3) \boldsymbol{x}_3 + (\alpha_3 + \alpha_9 \alpha_4) \boldsymbol{x}_4 + (\alpha_4 + \alpha_9 \alpha_5) \boldsymbol{x}_5 + (\alpha_9 \alpha_6 - \varepsilon') \boldsymbol{x}_6 + (\alpha_6 + \alpha_9 \alpha_7) \boldsymbol{x}_7 + (\alpha_7 + \alpha_9 \alpha_8) \boldsymbol{x}_8 + (\alpha_8 + \alpha_9^2) \boldsymbol{x}_9 = 0$. Since $\boldsymbol{x}_1, ..., \boldsymbol{x}_9$ are linearly independent, we get: $\alpha_6 = \varepsilon' \alpha_9^{-1}$; $\alpha_7 = -\alpha_6 \alpha_9^{-1} = -\varepsilon' \alpha_9^{-2}$; $\alpha_8 = \varepsilon' \alpha_9^{-3}$; $\alpha_9^2 + \alpha_8 = 0$ and, therefore, $\alpha_9^5 = -\varepsilon'$, $\alpha_9 = -\varepsilon'$, $\alpha_6 = -1$, $\alpha_7 = -\varepsilon'$ and $\alpha_8 = -1$; $\alpha_4 = \varepsilon' \alpha_5$; $\alpha_3 = \varepsilon' \alpha_4 = \varepsilon'^2 \alpha_5 = \alpha_5$; $\alpha_2 = \varepsilon' \alpha_5$; $\alpha_1 = \alpha_5$; $\alpha_5 \varepsilon + \alpha_9 \alpha_1 = 0$, or

$$\alpha_5(\varepsilon - \varepsilon') = 0. \tag{3}$$

Next, we have $\boldsymbol{x}_1 \boldsymbol{B} = \delta_1 \boldsymbol{x}_5$, $\boldsymbol{x}_2 \boldsymbol{B} = \delta_2 \boldsymbol{x}_7$, $\boldsymbol{x}_3 \boldsymbol{B} = \delta_3 \boldsymbol{x}_{10}$, $\boldsymbol{x}_4 \boldsymbol{B} = \delta_4 \boldsymbol{x}_8$, $\boldsymbol{x}_5 \boldsymbol{B} = \delta_5 \boldsymbol{x}_1$, $\boldsymbol{x}_6 \boldsymbol{B} = \delta_6 \boldsymbol{x}_6$, $\boldsymbol{x}_7 \boldsymbol{B} = \delta_7 \boldsymbol{x}_2$, $\boldsymbol{x}_8 \boldsymbol{B} = \delta_8 \boldsymbol{x}_4$, $\boldsymbol{x}_9 \boldsymbol{B} = \delta_9 \boldsymbol{x}_9$, $\boldsymbol{x}_{10} \boldsymbol{B} = \delta_{10} \boldsymbol{x}_3$ for some $\delta_1, ..., \delta_{10} \in \{1, -1\}$. Since $\boldsymbol{x}_{10} = \sum_{i=1}^{9} \alpha_i \boldsymbol{x}_i = \alpha_5 \boldsymbol{x}_1 + \varepsilon' \alpha_5 \boldsymbol{x}_2 + \alpha_5 \boldsymbol{x}_3 + \varepsilon' \alpha_5 \boldsymbol{x}_4 + \alpha_5 \boldsymbol{x}_5 - \boldsymbol{x}_6 - \varepsilon' \boldsymbol{x}_7 - \boldsymbol{x}_8 - \varepsilon' \boldsymbol{x}_9$, the equation $\boldsymbol{x}_{10} \boldsymbol{B} = \delta_{10} \boldsymbol{x}_3$ is equivalent to $\alpha_5 \delta_1 \boldsymbol{x}_5 + \varepsilon' \alpha_5 \delta_2 \boldsymbol{x}_7 + \alpha_5 \delta_3 (\alpha_5 \boldsymbol{x}_1 + \varepsilon' \alpha_5 \boldsymbol{x}_2 + \alpha_5 \boldsymbol{x}_3 + \varepsilon' \alpha_5 \boldsymbol{x}_4 + \alpha_5 \boldsymbol{x}_5 - \boldsymbol{x}_6 - \varepsilon' \boldsymbol{x}_7 - \boldsymbol{x}_8 - \varepsilon' \boldsymbol{x}_9) + \varepsilon' \alpha_5 \delta_4 \boldsymbol{x}_8 + \alpha_5 \delta_5 \boldsymbol{x}_1 - \delta_6 \boldsymbol{x}_6 - \varepsilon_2 \delta_7 \boldsymbol{x}_2 - \delta_8 \boldsymbol{x}_4 - \varepsilon' \delta_9 \boldsymbol{x}_9 - \delta_{10} \boldsymbol{x}_3 = 0$. The coefficient of $\boldsymbol{x}_2$ at left hand side of the equation is $\alpha^5 \delta_3 \varepsilon' - \varepsilon' \delta_7$; therefore, we get $\alpha_5^2 = \delta_3 \delta_7$; hence, $\delta_3 = \delta_7$, $\alpha_5^2 = 1$ and (3) implies $\varepsilon' = \varepsilon$. The coefficient of $\boldsymbol{x}_1$ $\alpha_5^2 \delta_3 + \alpha_5 \delta_5$, and, therefore, $\alpha_5 = -\delta_5 \delta_7$.

Now since $\boldsymbol{x}_{10} = \alpha_5 \boldsymbol{x}_1 + \varepsilon \alpha_5 \boldsymbol{x}_2 + \alpha_5 \boldsymbol{x}_3 + \varepsilon \alpha_5 \boldsymbol{x}_4 + \alpha_5 \boldsymbol{x}_5 - \boldsymbol{x}_6 - \varepsilon \boldsymbol{x}_7 - \boldsymbol{x}_8 - \varepsilon \boldsymbol{x}_9$, M is linearly equivalent to $M_{9,2}$ and $\Gamma_{\mathbb{Z}^9,M} \cong \Gamma_{\mathbb{Z}^9,M_{9,2}}$. Next, we have $\Gamma_{\mathbb{Z}^9,M_{9,2}} \in \tau_{9,10}^{-1}(T_{10,1})$, because $\Gamma_{\mathbb{Z}^9,M_{9,2}} \in \mathcal{MO}_{\mathbb{Q}}(\widetilde{S}_{10})$ by Proposition 10 and it is easy to chose a subgroup G of $\widetilde{S}_{10}$, such that $M_{9,2}$ is an orbit of G, $G \leq \mathrm{Aut}(\Gamma_{\mathbb{Z}^9,M_{9,2}})_0$ and $G^{\overline{M}_{9,2}}$ is permutationally isomorphic to $T_{10,1}$. Therefore $\tau_{9,10}^{-1}(T_{10,1}) = \{\Gamma_{\mathbb{Z}^9,M_{9,2}}\}$.

2. $T_{10,2} = \langle a, b \rangle$, where $a = (1,2,3,4,5)(6,7,8,9,10)$, $b = (1,10)(2,9)(3,8)(4,7)(5,6)$, thus $T_{10,2} \cong D_{10}$.

2.1. It follows from Proposition 14 that $\tau_{8,10}^{-1}(T_{10,2}) = \{\Gamma_{\mathbb{Z}^8,M_{8,3}}\}$.

2.2. We find all graphs from the class $\tau_{9,10}^{-1}(T_{10,2})$. Let $(\Gamma_{\mathbb{Z}^9,M}, T_{10,2}) \in \tau_{9,10})$ and $\boldsymbol{A} \in \chi^{-1}(a)$, $\boldsymbol{B} \in \chi^{-1}(b)$. Using arguments similar to the arguments for the group $T_{10,1}$, choose vectors $\boldsymbol{x}_i \in \psi^{-1}(i)$ $(i = 1, ..., 10)$ such that $\boldsymbol{x}_{10} = \alpha_5\boldsymbol{x}_1 + \varepsilon'\alpha_5\boldsymbol{x}_2 + \alpha_5\boldsymbol{x}_3 + \varepsilon'\alpha_5\boldsymbol{x}_4 + \alpha_5\boldsymbol{x}_5 - \boldsymbol{x}_6 - \varepsilon'\boldsymbol{x}_7 - \boldsymbol{x}_8 - \varepsilon'\boldsymbol{x}_9$ and the condition (3) holds for some $\varepsilon, \varepsilon', \alpha_5 \in \{1, -1\}$. Next, we have $\boldsymbol{x}_1\boldsymbol{B} = \delta_1\boldsymbol{x}_{10}$, $\boldsymbol{x}_2\boldsymbol{B} = \delta_2\boldsymbol{x}_9$, $\boldsymbol{x}_3\boldsymbol{B} = \delta_3\boldsymbol{x}_8$, $\boldsymbol{x}_4\boldsymbol{B} = \delta_4\boldsymbol{x}_7$, $\boldsymbol{x}_5\boldsymbol{B} = \delta_5\boldsymbol{x}_6$, $\boldsymbol{x}_6\boldsymbol{B} = \delta_6\boldsymbol{x}_5$, $\boldsymbol{x}_7\boldsymbol{B} = \delta_7\boldsymbol{x}_4$, $\boldsymbol{x}_8\boldsymbol{B} = \delta_8\boldsymbol{x}_3$, $\boldsymbol{x}_9\boldsymbol{B} = \delta_9\boldsymbol{x}_2$, $\boldsymbol{x}_{10}\boldsymbol{B} = \delta_{10}\boldsymbol{x}_1$ for some $\delta_1, ..., \delta_{10} \in \{1, -1\}$. The equation $\boldsymbol{x}_{10}\boldsymbol{B} = \delta_{10}\boldsymbol{x}_1$ is equivalent to $\alpha_5\delta_1(\alpha_5\boldsymbol{x}_1 + \varepsilon'\alpha_5\boldsymbol{x}_2 + \alpha_5\boldsymbol{x}_3 + \varepsilon'\alpha_5\boldsymbol{x}_4 + \alpha_5\boldsymbol{x}_5 - \boldsymbol{x}_6 - \varepsilon'\boldsymbol{x}_7 - \boldsymbol{x}_8 - \varepsilon'\boldsymbol{x}_9) + \varepsilon'\alpha_5\delta_2\boldsymbol{x}_9 + \alpha_5\delta_3\boldsymbol{x}_8 + \varepsilon'\alpha_5\delta_4\boldsymbol{x}_7 + \alpha_5\delta_5\boldsymbol{x}_6 - \delta_6\boldsymbol{x}_5 - \varepsilon'\delta_7\boldsymbol{x}_4 - \delta_8\boldsymbol{x}_3 - \varepsilon'\delta_9\boldsymbol{x}_2 = \delta_{10}\boldsymbol{x}_1$. Equating the coefficients of $\boldsymbol{x}_1$, we get $\alpha_5^2\delta_1 = \delta_{10}$. Therefore, $\alpha_5 \in \{1, -1\}$, $\delta_{10} = \delta_1$, and by (3) we get $\varepsilon' = \varepsilon$.

Now since $\boldsymbol{x}_{10} = \alpha_5\boldsymbol{x}_1 + \varepsilon\alpha_5\boldsymbol{x}_2 + \alpha_5\boldsymbol{x}_3 + \varepsilon\alpha_5\boldsymbol{x}_4 + \alpha_5\boldsymbol{x}_5 - \boldsymbol{x}_6 - \varepsilon\boldsymbol{x}_7 - \boldsymbol{x}_8 - \varepsilon\boldsymbol{x}_9$, M is linearly equivalent to $M_{9,2}$. Therefore, $\Gamma_{\mathbb{Z}^9,M} \cong \Gamma_{\mathbb{Z}^9,M_{9,2}}$. Next, we have $\Gamma_{\mathbb{Z}^9,M_{9,2}} \in \tau_{9,10}^{-1}(T_{10,2})$, because $\Gamma_{\mathbb{Z}^9,M_{9,2}} \in \mathcal{MO}_{\mathbb{Q}}(\widetilde{S}_{10})$ by proposition 10 and it is easy to chose a subgroup G of $\widetilde{S}_{10}$, such that $M_{9,2}$ is an orbit of G, $G \leq \mathrm{Aut}(\Gamma_{\mathbb{Z}^9,M_{9,2}})_0$ and $G^{\overline{M}_{9,2}}$ is permutationally isomorphic to $T_{10,2}$. Therefore $\tau_{9,10}^{-1}(T_{10,2}) = \{\Gamma_{\mathbb{Z}^9,M_{9,2}}\}$.

3. $T_{10,3} = \langle a, b\rangle$, where $a = (1,2,3,4,5)(6,7,8,9,10)$, $b = (1,7,4,8)(2,9,3,6)(5,10)$, thus $T_{10,3} \cong C_5 : C_4$.

3.1. It follows from Proposition 14 that $\tau_{8,10}^{-1}(T_{10,3}) = \{\Gamma_{\mathbb{Z}^8,M_{8,3}}\}$.

3.2. We find all graphs from the class $\tau_{9,10}^{-1}(T_{10,3})$. Let $(\Gamma_{\mathbb{Z}^9,M}, T_{10,3}) \in \tau_{9,10})$ and $\boldsymbol{A} \in \chi^{-1}(a)$, $\boldsymbol{B} \in \chi^{-1}(b)$. Using arguments similar to the arguments for the group $T_{10,1}$, choose vectors $\boldsymbol{x}_i \in \psi^{-1}(i)$ $(i = 1, ..., 10)$ such that $\boldsymbol{x}_{10} = \alpha_5\boldsymbol{x}_1 + \varepsilon'\alpha_5\boldsymbol{x}_2 + \alpha_5\boldsymbol{x}_3 + \varepsilon'\alpha_5\boldsymbol{x}_4 + \alpha_5\boldsymbol{x}_5 - \boldsymbol{x}_6 - \varepsilon'\boldsymbol{x}_7 - \boldsymbol{x}_8 - \varepsilon'\boldsymbol{x}_9$ and the condition (3) holds for some $\varepsilon, \varepsilon', \alpha_5 \in \{1, -1\}$. Next, we have $\boldsymbol{x}_1\boldsymbol{B} = \delta_1\boldsymbol{x}_7$, $\boldsymbol{x}_2\boldsymbol{B} = \delta_2\boldsymbol{x}_9$, $\boldsymbol{x}_3\boldsymbol{B} = \delta_3\boldsymbol{x}_6$, $\boldsymbol{x}_4\boldsymbol{B} = \delta_4\boldsymbol{x}_8$, $\boldsymbol{x}_5\boldsymbol{B} = \delta_5\boldsymbol{x}_{10}$, $\boldsymbol{x}_6\boldsymbol{B} = \delta_6\boldsymbol{x}_2$, $\boldsymbol{x}_7\boldsymbol{B} = \delta_7\boldsymbol{x}_4$, $\boldsymbol{x}_8\boldsymbol{B} = \delta_8\boldsymbol{x}_1$, $\boldsymbol{x}_9\boldsymbol{B} = \delta_9\boldsymbol{x}_3$, $\boldsymbol{x}_{10}\boldsymbol{B} = \delta_{10}\boldsymbol{x}_5$ for some $\delta_1, ..., \delta_{10} \in \{1, -1\}$. The equation $\boldsymbol{x}_{10}\boldsymbol{B} = \delta_{10}\boldsymbol{x}_5$ is equivalent to $\alpha_5\delta_1\boldsymbol{x}_7 + \varepsilon'\alpha_5\delta_2\boldsymbol{x}_9 + \alpha_5\delta_3\boldsymbol{x}_6 + \varepsilon'\alpha_5\delta_4\boldsymbol{x}_8 + \alpha_5\delta_5(\alpha_5\boldsymbol{x}_1 + \varepsilon'\alpha_5\boldsymbol{x}_2 + \alpha_5\boldsymbol{x}_3 + \varepsilon'\alpha_5\boldsymbol{x}_4 + \alpha_5\boldsymbol{x}_5 - \boldsymbol{x}_6 - \varepsilon'\boldsymbol{x}_7 - \boldsymbol{x}_8 - \varepsilon'\boldsymbol{x}_9) - \delta_6\boldsymbol{x}_5 - \varepsilon'\delta_7\boldsymbol{x}_4 - \delta_8\boldsymbol{x}_1 - \varepsilon'\delta_9\boldsymbol{x}_3 = \delta_{10}\boldsymbol{x}_5$. The coefficient of $\boldsymbol{x}_1$ at left hand side of the equation is $\alpha_5^2\delta_5 - \delta_8$. Therefore, $\alpha_5 \in \{1, -1\}$, $\delta_8 = \delta_5$, and so by (3) we get $\varepsilon' = \varepsilon$.

Now since $\boldsymbol{x}_{10} = \alpha_5\boldsymbol{x}_1 + \varepsilon\alpha_5\boldsymbol{x}_2 + \alpha_5\boldsymbol{x}_3 + \varepsilon\alpha_5\boldsymbol{x}_4 + \alpha_5\boldsymbol{x}_5 - \boldsymbol{x}_6 - \varepsilon\boldsymbol{x}_7 - \boldsymbol{x}_8 - \varepsilon\boldsymbol{x}_9$, M is linearly equivalent to the system $M_{9,2}$. Therefore, $\Gamma_{\mathbb{Z}^9,M} \cong \Gamma_{\mathbb{Z}^9,M_{9,2}}$. Next, we have $\Gamma_{\mathbb{Z}^9,M_{9,2}} \in \tau_{9,10}^{-1}(T_{10,3})$, because $\Gamma_{\mathbb{Z}^9,M_{9,2}} \in \mathcal{MO}_{\mathbb{Q}}(\widetilde{S}_{10})$ by proposition 10 and it is easy to chose a subgroup G of $\widetilde{S}_{10}$, such that $M_{9,2}$ is an orbit of G, $G \leq \mathrm{Aut}(\Gamma_{\mathbb{Z}^9,M_{9,2}})_0$ and $G^{\overline{M}_{9,2}}$ is permutationally isomorphic to $T_{10,3}$. Therefore $\mathcal{O}\mathit{\Gamma}_9(T_{10,3}) = \{\Gamma_{\mathbb{Z}^9,M_{9,2}}\}$.

4. $T_{10,4} = \langle a, b\rangle$, where $a = (1,2,3,4,5)(6,7,8,9,10)$, $b = (4,9)(5,10)$, thus $T_{10,4} \cong C_2^4 : C_5$.

4.1. We find all graphs from $\tau_{8,10}^{-1}(T_{10,4})$. Let $(\Gamma_{\mathbb{Z}^8,M}, T_{10,4}) \in \tau_{8,10})$. Let $\boldsymbol{A} \in \chi^{-1}(a)$, $\boldsymbol{B} \in \chi^{-1}(b)$, $\boldsymbol{C} \in \chi^{-1}(c)$, where $c = a^2ba^{-2}b = (2,7)(3,8)(4,9)(5,10)$. Let $k = \dim(\langle \boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_5\rangle)$, $l = \dim(\langle \boldsymbol{x}_6, \boldsymbol{x}_7, ..., \boldsymbol{x}_{10}\rangle)$. We have $l \leq 1 + \dim(\langle \boldsymbol{x}_7, \boldsymbol{x}_8, \boldsymbol{x}_9, \boldsymbol{x}_{10}\rangle) = 1 + \dim(\langle \boldsymbol{x}_7, \boldsymbol{x}_8, \boldsymbol{x}_9, \boldsymbol{x}_{10}\rangle\boldsymbol{C}) = 1 + \dim(\langle \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4, \boldsymbol{x}_5\rangle) \leq 1 + k$ and $k + l \geq 8$. Therefore, $k \geq 4$; similarly, $l \geq 4$. Let $k = 5$. The case $l = 5$ is impossible (by the same arguments as in Proposition 14 proof) and, therefore, $l = 4$. Now by Proposition 13.1a) we have $\boldsymbol{x}_{10} = -\boldsymbol{x}_6 - \varepsilon\boldsymbol{x}_7 - \boldsymbol{x}_8 - \varepsilon\boldsymbol{x}_9$ for some $\varepsilon \in \{1, -1\}$. Choose $\boldsymbol{x} \in \mathbb{Q}^8$ such that $\langle \boldsymbol{x}\rangle = \langle \boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_5\rangle \cap \langle \boldsymbol{x}_6, \boldsymbol{x}_7, ..., \boldsymbol{x}_{10}\rangle$.

We have $x\boldsymbol{A} = \varepsilon'\boldsymbol{x}$ for some $\varepsilon' \in \{1,-1\}$. Since the group $\langle a\rangle$ is transitive on $\{6,7,8,9,10\}$, vectors $\boldsymbol{x}_6,\boldsymbol{x}_7,\boldsymbol{x}_8,\boldsymbol{x}_9$ form a basis of $\langle \boldsymbol{x}_6,\boldsymbol{x}_7,...,\boldsymbol{x}_{10}\rangle_{\mathbb{Q}}$. Therefore $\boldsymbol{x} = \alpha_1\boldsymbol{x}_6 + \alpha_2\boldsymbol{x}_7 + \alpha_3\boldsymbol{x}_8 + \alpha_4\boldsymbol{x}_9$. The condition $xA = \varepsilon'\boldsymbol{x}$ is equivalent to $\alpha_1\boldsymbol{x}_7+\alpha_2\boldsymbol{x}_8+\alpha_3\boldsymbol{x}_9+\alpha_4(-\boldsymbol{x}_6-\varepsilon\boldsymbol{x}_7-\varepsilon\boldsymbol{x}_8-\boldsymbol{x}_9) = \varepsilon'(\alpha_1\boldsymbol{x}_6+\alpha_2\boldsymbol{x}_7+\alpha_3\boldsymbol{x}_8+\alpha_4\boldsymbol{x}_9)$, or $(-\alpha_4-\varepsilon'\alpha_1)\boldsymbol{x}_6+(\alpha_1-\alpha_4\varepsilon-\varepsilon'\alpha_2)\boldsymbol{x}_7+(\alpha_2-\alpha_4-\varepsilon'\alpha_3)\boldsymbol{x}_8+(\alpha_3-\alpha_4\varepsilon-\varepsilon'\alpha_4)\boldsymbol{x}_9 = 0$. Since $\boldsymbol{x}_6,\boldsymbol{x}_7,\boldsymbol{x}_8$ and $\boldsymbol{x}_9$ are linearly independent, we get: $-\alpha_4 = \varepsilon'\alpha_1$ and, therefore, $\alpha_4 = -\varepsilon'\alpha_1$; $\alpha_1-\alpha_4\varepsilon = \varepsilon'\alpha_2$ and, therefore, $\alpha_2 = \varepsilon'(1+\varepsilon\varepsilon')\alpha_1$; $\alpha_2-\alpha_4 = \varepsilon'\alpha_3$ and, therefore, $\alpha_3 = \varepsilon'(2+\varepsilon\varepsilon')\alpha_1$; $\alpha_3-\alpha_4\varepsilon = \varepsilon'\alpha_4$, or $\alpha_1\varepsilon'(2+\varepsilon\varepsilon')+\varepsilon\varepsilon'\alpha_1 = -\varepsilon'\alpha_1$, or $\varepsilon'(3+2\varepsilon\varepsilon')\alpha_1 = 0$. Therefore, $\alpha_1 = 0$ and $\boldsymbol{x} = 0$. We get a contradiction and, therefore, the case $k = 5$ is impossible. Impossibility of the case $l = 5$ can be show analogously. Therefore, $k = l = 4$. Using Proposition 13.1a) twice, we get $\boldsymbol{x}_5 = -\boldsymbol{x}_1-\varepsilon_1\boldsymbol{x}_2-\boldsymbol{x}_3-\varepsilon_1\boldsymbol{x}_4$ and $\boldsymbol{x}_{10} = -\boldsymbol{x}_6-\varepsilon_2\boldsymbol{x}_7-\boldsymbol{x}_8-\varepsilon_2\boldsymbol{x}_9$ for some $\varepsilon_1,\varepsilon_2 \in \{1,-1\}$. Therefore $\dim(\langle\boldsymbol{x}_1,...,\boldsymbol{x}_5\rangle)_{\mathbb{Q}} = 4$ and $\dim(\langle\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3,\boldsymbol{x}_9,\boldsymbol{x}_{10}\rangle)_{\mathbb{Q}} = 5$. This contradicts to $\overline{\boldsymbol{x}_1}\boldsymbol{B} = \overline{\boldsymbol{x}_1}$, $\overline{\boldsymbol{x}_2}\boldsymbol{B} = \overline{\boldsymbol{x}_2}$, $\overline{\boldsymbol{x}_3}\boldsymbol{B} = \overline{\boldsymbol{x}_3}$, $\overline{\boldsymbol{x}_4}\boldsymbol{B} = \overline{\boldsymbol{x}_9}$, $\overline{\boldsymbol{x}_5}\boldsymbol{B} = \overline{\boldsymbol{x}_{10}}$, and we get $\tau^{-1}_{8,10}(T_{10,4}) = \emptyset$.

4.2. We find all graphs from the class $\tau^{-1}_{9,10}(T_{10,4})$. Let $(\Gamma_{\mathbb{Z}^9,M}, T_{10,4}) \in \tau_{9,10}$ and $\boldsymbol{A} \in \chi^{-1}(a)$, $\boldsymbol{B} \in \chi^{-1}(b)$. Using arguments similar to the arguments for the group $T_{10,1}$, choose vectors $\boldsymbol{x}_i \in \psi^{-1}(i)$ $(i = 1,...,10)$ such that $\boldsymbol{x}_1\boldsymbol{A} = \boldsymbol{x}_2$, $\boldsymbol{x}_2\boldsymbol{A} = \boldsymbol{x}_3$, $\boldsymbol{x}_3\boldsymbol{A} = \boldsymbol{x}_4$, $\boldsymbol{x}_4\boldsymbol{A} = \boldsymbol{x}_5$, $\boldsymbol{x}_5\boldsymbol{A} = \varepsilon\boldsymbol{x}_1$, $\boldsymbol{x}_6\boldsymbol{A} = \boldsymbol{x}_7$, $\boldsymbol{x}_7\boldsymbol{A} = \boldsymbol{x}_8$, $\boldsymbol{x}_8\boldsymbol{A} = \boldsymbol{x}_9$, $\boldsymbol{x}_9\boldsymbol{A} = \boldsymbol{x}_{10}$, $\boldsymbol{x}_{10}\boldsymbol{A} = \varepsilon'\boldsymbol{x}_6$ for some $\varepsilon,\varepsilon' \in \{1,-1\}$. We have $\boldsymbol{x}_{10} = \alpha_5\boldsymbol{x}_1+\varepsilon'\alpha_5\boldsymbol{x}_2+\alpha_5\boldsymbol{x}_3+\varepsilon'\alpha_5\boldsymbol{x}_4+\alpha_5\boldsymbol{x}_5-\boldsymbol{x}_6-\varepsilon'\boldsymbol{x}_7-\boldsymbol{x}_8-\varepsilon'\boldsymbol{x}_9$ and the condition (3) holds. Next, we have $\boldsymbol{x}_1\boldsymbol{B} = \delta_1\boldsymbol{x}_1$, $\boldsymbol{x}_2\boldsymbol{B} = \delta_2\boldsymbol{x}_2$, $\boldsymbol{x}_3\boldsymbol{B} = \delta_3\boldsymbol{x}_3$, $\boldsymbol{x}_4\boldsymbol{B} = \delta_4\boldsymbol{x}_9$, $\boldsymbol{x}_5\boldsymbol{B} = \delta_5\boldsymbol{x}_{10}$, $\boldsymbol{x}_6\boldsymbol{B} = \delta_6\boldsymbol{x}_6$, $\boldsymbol{x}_7\boldsymbol{B} = \delta_7\boldsymbol{x}_7$, $\boldsymbol{x}_8\boldsymbol{B} = \delta_8\boldsymbol{x}_8$, $\boldsymbol{x}_9\boldsymbol{B} = \delta_9\boldsymbol{x}_4$, $\boldsymbol{x}_{10}\boldsymbol{B} = \delta_{10}\boldsymbol{x}_5$ for some $\delta_1,...,\delta_{10} \in \{1,-1\}$. The equation $\boldsymbol{x}_{10}\boldsymbol{B} = \delta_{10}\boldsymbol{x}_5$ is equivalent to $\alpha_5\delta_1\boldsymbol{x}_1+\varepsilon'\alpha_5\delta_2\boldsymbol{x}_2+\alpha_5\delta_3\boldsymbol{x}_3+\varepsilon'\alpha_5\delta_4\boldsymbol{x}_9+\alpha_5\delta_5(\alpha_5\boldsymbol{x}_1+\varepsilon'\alpha_5\boldsymbol{x}_2+\alpha_5\boldsymbol{x}_3+\varepsilon'\alpha_5\boldsymbol{x}_4+\alpha_5\boldsymbol{x}_5-\boldsymbol{x}_6-\varepsilon'\boldsymbol{x}_7-\boldsymbol{x}_8-\varepsilon'\boldsymbol{x}_9)-\delta_6\boldsymbol{x}_6-\varepsilon'\delta_7\boldsymbol{x}_7-\delta_8\boldsymbol{x}_8-\varepsilon'\delta_9\boldsymbol{x}_4 = \delta_{10}\boldsymbol{x}_5$. Equating the coefficients of $\boldsymbol{x}_5$ in left and right hand sides of this equation, we get $\alpha_5^2\delta_5 = \delta_{10}$. Hence, $\alpha_5 \in \{1,-1\}$, $\delta_{10} = \delta_5$, and (3) implies $\varepsilon' = \varepsilon$.

Now since $\boldsymbol{x}_{10} = \alpha_5\boldsymbol{x}_1+\varepsilon\alpha_5\boldsymbol{x}_2+\alpha_5\boldsymbol{x}_3+\varepsilon\alpha_5\boldsymbol{x}_4+\alpha_5\boldsymbol{x}_5-\boldsymbol{x}_6-\varepsilon\boldsymbol{x}_7-\boldsymbol{x}_8-\varepsilon\boldsymbol{x}_9$, M is linearly equivalent to $M_{9,2}$. Therefore, $\Gamma_{\mathbb{Z}^9,M} \cong \Gamma_{\mathbb{Z}^9,M_{9,2}}$. Next, we have $\Gamma_{\mathbb{Z}^9,M_{9,2}} \in \tau^{-1}_{9,10}(T_{10,4})$, because $\Gamma_{\mathbb{Z}^9,M_{9,2}} \in \mathcal{MO}_{\mathbb{Q}}(\widetilde{S}_{10})$ by proposition 10 and it is easy to chose a subgroup G of $\widetilde{S}_{10}$, such that $M_{9,2}$ is an orbit of G, $G \leq \mathrm{Aut}(\Gamma_{\mathbb{Z}^9,M_{9,2}})_0$ and $G^{\overline{M}_{9,2}}$ is permutationally isomorphic to $T_{10,4}$. Therefore $\tau_{9,10}(T_{10,4}) = \{\Gamma_{\mathbb{Z}^9,M_{9,2}}\}$.

5. $T_{10,5} = \langle a,b\rangle$, where $a = (1,2,3,4,5)$, $b = (1,6,3,8,4,9,2,7)(5,10)$, thus $T_{10,5} \cong (C_5\times C_5) : C_8$.

5.1. It is easy to see that $T_{10,5} = \langle a',b\rangle$, where $a' = (1,2,3,4,5)(6,7,8,9,10)$. Therefore by Proposition 14 we get $\tau_{8,10}(T_{10,5}) = \{\Gamma_{\mathbb{Z}^8,M_{8,3}}\}$.

5.2. We find all graphs from the class $\tau_{9,10}(T_{10,5})$. Let $(\Gamma_{\mathbb{Z}^9,M}, T_{10,5}) \in \tau_{9,10}$ and $\boldsymbol{A} \in \chi^{-1}(a)$, $\boldsymbol{B} \in \chi^{-1}(b)$. Choose vectors $\boldsymbol{x}_i \in \psi^{-1}(i)$ $(i = 1,...,10)$ such that $\boldsymbol{x}_1\boldsymbol{A} = \boldsymbol{x}_2$, $\boldsymbol{x}_2\boldsymbol{A} = \boldsymbol{x}_3$, $\boldsymbol{x}_3\boldsymbol{A} = \boldsymbol{x}_4$, $\boldsymbol{x}_4\boldsymbol{A} = \boldsymbol{x}_5$, $\boldsymbol{x}_5\boldsymbol{A} = \varepsilon_5\boldsymbol{x}_1$, $\boldsymbol{x}_6\boldsymbol{A} = \varepsilon_6\boldsymbol{x}_6$, $\boldsymbol{x}_7\boldsymbol{A} = \varepsilon_7\boldsymbol{x}_7$, $\boldsymbol{x}_8\boldsymbol{A} = \varepsilon_8\boldsymbol{x}_8$, $\boldsymbol{x}_9\boldsymbol{A} = \varepsilon_9\boldsymbol{x}_9$, $\boldsymbol{x}_{10}\boldsymbol{A} = \varepsilon_{10}\boldsymbol{x}_{10}$ for some $\varepsilon_5,\varepsilon_{10} \in \{1,-1\}$ and $\boldsymbol{x}_1\boldsymbol{B} = \boldsymbol{x}_6$, $\boldsymbol{x}_2\boldsymbol{B} = \boldsymbol{x}_7$, $\boldsymbol{x}_3\boldsymbol{B} = \boldsymbol{x}_8$, $\boldsymbol{x}_4\boldsymbol{B} = \boldsymbol{x}_9$, $\boldsymbol{x}_5\boldsymbol{B} = \boldsymbol{x}_{10}$, $\boldsymbol{x}_6\boldsymbol{B} = \delta_6\boldsymbol{x}_3$, $\boldsymbol{x}_7\boldsymbol{B} = \delta_7\boldsymbol{x}_1$, $\boldsymbol{x}_8\boldsymbol{B} = \delta_8\boldsymbol{x}_4$, $\boldsymbol{x}_9\boldsymbol{B} = \delta_9\boldsymbol{x}_2$, $\boldsymbol{x}_{10}\boldsymbol{B} = \delta_{10}\boldsymbol{x}_5$ for some $\delta_6,...,\delta_{10} \in \{1,-1\}$. We have $\boldsymbol{x}_{10} = \sum_{i=1}^{9}\alpha_i\boldsymbol{x}_i$ for some $\alpha_1,...,\alpha_9 \in \mathbb{Q}$. The equation $\boldsymbol{x}_{10}\boldsymbol{A} = \varepsilon_{10}\boldsymbol{x}_{10}$ is equivalent to $(\alpha_4-\varepsilon_{10}\alpha_5)\boldsymbol{x}_5+(\alpha_3-\varepsilon_{10}\alpha_4)\boldsymbol{x}_4+(\alpha_2-\varepsilon_{10}\alpha_3)\boldsymbol{x}_3+(\alpha_1-\varepsilon_{10}\alpha_2)\boldsymbol{x}_2+(\alpha_5\varepsilon_5-\varepsilon_{10}\alpha_1)\boldsymbol{x}_1+(\alpha_6\varepsilon_6-\varepsilon_{10}\alpha_6)\boldsymbol{x}_6+(\varepsilon_7-\varepsilon_{10}\alpha_7)\boldsymbol{x}_7+(\alpha_8\varepsilon_8-\varepsilon_{10}\alpha_8)\boldsymbol{x}_8+(\alpha_9\varepsilon_9-\varepsilon_{10}\alpha_9)\boldsymbol{x}_9 = 0$. Therefore, we get $\alpha_4 = \varepsilon_{10}\alpha_5$; $\alpha_3 = \varepsilon_{10}\alpha_4$, and $\alpha_3 = \alpha_5$; $\alpha_2 = \varepsilon_{10}\alpha_3$, and $\alpha_2 = \varepsilon_{10}\alpha_5$; $\alpha_1 = \varepsilon_{10}\alpha_2$, and $\alpha_1 = \alpha_5$; $\alpha_5\varepsilon_5 = \varepsilon_{10}\alpha_1$ or, what is the same,

$$\alpha_5(\varepsilon_5 - \varepsilon_{10}) = 0. \tag{4}$$

Next, the equation $\boldsymbol{x}_{10}\boldsymbol{B} = \delta_{10}\boldsymbol{x}_5$ is equivalent to $(\alpha_5^2 - \delta_{10})\boldsymbol{x}_5 + (\alpha_5\alpha_1 + \alpha_7\delta_7)\boldsymbol{x}_1 + (\alpha_2 + \alpha_5\alpha_7)\boldsymbol{x}_7 + (\alpha_5\alpha_2 + \alpha_9\delta_9)\boldsymbol{x}_2 + (\alpha_4 + \alpha_5\alpha_9)\boldsymbol{x}_9 + (\alpha_5\alpha_3 + \alpha_6\delta_6)\boldsymbol{x}_3 + (\alpha_1 + \alpha_5\alpha_6)\boldsymbol{x}_6 + (\alpha_5\alpha_4 + \alpha_8\delta_8)\boldsymbol{x}_4 + (\alpha_3 + \alpha_5\alpha_8)\boldsymbol{x}_8 = 0$. Therefore, we get: $\alpha_5^2 = \delta_{10}$, $\alpha_5 \in \{1, -1\}$, $\delta_{10} = 1$. Now it follows from (4) that $\varepsilon_5 = \varepsilon_{10}$; $\alpha_5\alpha_1 + \alpha_7\delta_7 = 0$. Therefore $\alpha_7 = \alpha_5^2\delta_7 = \delta_7$; $\alpha_2 + \alpha_5\alpha_7 = 0$ or $\varepsilon_{10}\alpha_5 + \alpha_5\delta_7 = 0$, and, hence, $\delta_7 = -\varepsilon_{10}$; $\alpha_5\alpha_2 + \alpha_9\delta_9 = 0$ or $\alpha_5^2\varepsilon_{10} + \alpha_9\delta_9 = 0$, and, hence, $\alpha_9 = -\varepsilon_{10}\delta_9$; $\alpha_4 + \alpha_5\alpha_9 = 0$ or $\varepsilon_{10}\alpha_5 - \alpha_5\varepsilon_{10}\delta_9 = 0$, and, hence, $\delta_9 = 1$; $\alpha_5\alpha_3 + \alpha_6\delta_6 = 0$ or $\alpha_5^2 + \alpha_6\delta_6 = 0$, and, hence, $\alpha_6 = -\delta_6$; $\alpha_1 + \alpha_5\alpha_6 = 0$ or $\alpha_5 - \alpha_5\delta_6 = 0$, and, hence, $\delta_6 = 1$; $\alpha_5\alpha_4 + \alpha_8\delta_8 = 0$ or $\alpha_5\varepsilon_{10}\alpha_5 + \alpha_8\delta_8 = 0$, and, hence, $\alpha_8 = -\varepsilon_{10}\delta_8$; $\alpha_3 + \alpha_5\alpha_8 = 0$ or $\alpha_5 - \alpha_5\varepsilon_{10}\delta_8 = 0$, and, hence, $\delta_8 = \varepsilon_{10}$.

We get $\boldsymbol{x}_{10} = \alpha_5\boldsymbol{x}_1 + \varepsilon_{10}\alpha_5\boldsymbol{x}_2 + \alpha_5\boldsymbol{x}_3 + \varepsilon_{10}\alpha_5\boldsymbol{x}_4 + \alpha_5\boldsymbol{x}_5 - \boldsymbol{x}_6 - \varepsilon_{10}\boldsymbol{x}_7 - \boldsymbol{x}_8 - \varepsilon_{10}\boldsymbol{x}_9$. Therefore, M is linearly equivalent to $M_{9,2}$, and $\Gamma_{\mathbb{Z}^9,M} \cong \Gamma_{\mathbb{Z}^9,M_{9,2}}$. Next, we have $\Gamma_{\mathbb{Z}^9,M_{9,2}} \in \tau_{9,10}^{-1}(T_{10,5})$, because $\Gamma_{\mathbb{Z}^9,M_{9,2}} \in \mathcal{MO}_{\mathbb{Q}}(\widetilde{S}_{10})$ by proposition 10 and it is easy to chose a subgroup G of $\widetilde{S}_{10}$, such that $M_{9,2}$ is an orbit of G, $G \leq \mathrm{Aut}(\Gamma_{\mathbb{Z}^9,M_{9,2}})_0$ and $G^{\overline{M}_{9,2}}$ is permutationally isomorphic to $T_{10,5}$. Therefore $\tau_{9,10}^{-1}(T_{10,5}) = \{\Gamma_{\mathbb{Z}^9,M_{9,2}}\}$.

6. $T_{10,6} = \langle a\rangle$, where $a = (1,2,3,4,5,6,7,8,9,10)$, thus $T_{10,6} \cong C_{10}$.

6.1. Since $a^2 = (1,2,3,4,5)(6,7,8,9,10)$, it follows from Proposition 14 that $\tau_{8,10}^{-1}(T_{10,6}) = \{\Gamma_{\mathbb{Z}^8,M_{8,3}}\}$.

6.2. It follows from Proposition 13 that $\tau_{9,10}^{-1}(T_{10,6}) = \{\Gamma_{\mathbb{Z}^9,M_{9,2}}\}$.

Now it follows from Proposition 12 that $\mathrm{Cay}_{20}^{min}(\mathbb{Z}^8) = \{\Gamma_{\mathbb{Z}^8,M_{8,3}}\}$ and $\mathrm{Cay}_{20}^{min}(\mathbb{Z}^9) = \{\Gamma_{\mathbb{Z}^9,M_{9,2}}\}$. Using the results formulated in the beginning of this section, we get $\bigcup_{d=1}^{10}\mathrm{Cay}_{20}^{min}(\mathbb{Z}^d) = \{\Gamma_{\mathbb{Z}^6,M_{6,4}}, \Gamma_{\mathbb{Z}^8,M_{8,3}}, \Gamma_{\mathbb{Z}^9,M_{9,2}}, \Gamma_{\mathbb{Z}^{10},M_{10,1}}\}$. Since $\Gamma_{\mathbb{Z}^8,M_{8,3}} \in \lim(\mathcal{FP}_{HA}^{min})$ by Proposition 15, $\Gamma_{\mathbb{Z}^9,M_{9,2}} \in \lim(\mathcal{FP}_{HA}^{min})$ by Proposition 11 and, as it was noted in Introduction, $\Gamma_{\mathbb{Z}^6,M_{6,4}}, \Gamma_{\mathbb{Z}^{10},M_{10,1}} \in \lim(\mathcal{FP}_{HA}^{min})$, the assertion of Theorem 1 for degree 20 is proved.

7. Completion of the proof of Theorem 1: the case of graphs of degree 22

To find all minimal Cayley graph of groups $\mathbb{Z}^d$ of degree 22 we will consider two cases. First we will find all minimal Cayley graph of groups $\mathbb{Z}^d$ of degree 22 with irreducible over $\mathbb{Q}$ stabilizer of the vertex 0.

The GAP ([7]) contains a list of all up to conjugation in $\mathrm{GL}_d(\mathbb{Q})$, maximal irreducible over $\mathbb{Q}$ finite subgroups of $\mathrm{GL}_d(\mathbb{Z})$ for each $d \leq 31$ (such subgroups are available via GAP-function ImfMatrixGroup). None of orders of 9 subgroups for $d = 8$ and 2 subgroups for $d = 9$ is divided by 11. Therefore, for $d = 8$ and $d = 9$ there does not exist a minimal Cayley graph of $\mathbb{Z}^d$ of degree 22 with irreducible over $\mathbb{Q}$ stabilizer of the vertex 0. Among 8 subgroups for $d = 10$ only three groups $\widetilde{S}_{11}$, $G_{10,1} \cong C_2 \times PGL_2(11)$, $G_{10,2} \cong C_2 \times PGL_2(11)$ have orders divided by 11. It follows from Proposition 10 that $\mathcal{OS}\mathit{\Gamma}^{min}(\widetilde{S}_{11}) = \{\Gamma_{\mathbb{Z}^{10},M_{10,2}}\}$.

To get a description of classes $\mathcal{OS}\mathit{\Gamma}^{min}(G_{10,i})$ for $i = 1, 2$, we used a modification of Algorithm 2. This modification differs from the original Algorithm 2 by two additional constraints: 1) we add a subgroup of G into the subgroup tree only if its order is a multiple of 22, and 2) we add a subgroup H of G into a resulting list only if $\mathrm{m}(H) = 22$. Applying this modified algorithm to each of groups $G_{10,1}$ and $G_{10,2}$ gives the following. The List of subgroups consist of three cyclic groups, each of which is conjugate in $\mathrm{GL}_{10}(\mathbb{Z})$ to the group $\widehat{C}_{11} = \langle \boldsymbol{g}, -\boldsymbol{I} \rangle$, where the element $\boldsymbol{g} \in \mathrm{GL}_{10}(\mathbb{Z})$ is defined by conditions $\boldsymbol{e}_i \boldsymbol{g} = \boldsymbol{e}_{i+1}$ for $i = 1, ..., 9$ and $\boldsymbol{e}_{10} \boldsymbol{g} = -\sum_{i=1}^{10} \boldsymbol{e}_i$. Therefore, Proposition 9 implies $\mathcal{OS}\Gamma^{min}(G_{10,i}) \setminus \{\Gamma_{\mathbb{Z}^{10}, M_{10,1}}\} = \{\Gamma_{\mathbb{Z}^{10}, M_{10,2}}\}$ for $i = 1, 2$.

Now consider the case of reducible on $\mathbb{Q}^d$ stabilizer of the vertex 0. Proposition 7 and Remark 2 implies that the only possible degree of minimal Cayley graphs of $\mathbb{Z}^d$ for $d = 8, 9, 10$ with such stabilizer is 24.

Now Proposition 2 implies $\mathrm{Cay}_{22}^{min}(\mathbb{Z}^{10}) = \{\Gamma_{\mathbb{Z}^{10}, M_{10,2}}\}$. Since $\Gamma_{\mathbb{Z}^{10}, M_{10,2}} \in \lim(\mathcal{FP}_{HA}^{min})$ by Proposition 11, the assertion of Theorem 1 for degree 22 is proved.

The proof of Theorem 1 is complete.

The author thanks V.I. Trofimov for few discussions on the topic of this paper.

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