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LINEAR APPROXIMATION METHOD PRESERVING k-MONOTONICITY

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Abstract. The paper presents the example of linear finite-dimensional approximation method that preserves k-monotonicity of approximated functions and uses the values of function at equidistant points on [0,1].

Keywords: shape-preserving approximation, linear approximation, degree of approximation

1. Introduction

The theory of shape-preserving approximation is one of the rapidly developing areas of the approximation theory. This theory studies approximation properties of the different methods for approximation of functions preserving its shape properties (monotonicity, convexity). The most significant results were gathered in [14], [7] for polynomial approximation and in [12] for spline interpolation and approximation.

One of the main directions of research in the theory of shape-preserving approximation is the study of shape-preserving properties of Bernstein-type polynomials [7]. It was shown by J. Pál [16] in 1925 that any convex function defined on [0,1] can be uniformly approximated by a sequence of convex algebraic polynomials on [0,1]. Some years later T. Popoviciu [17] proved that if f is k-monotone on [0,1], then Bernstein polynomial $B_nf(x) := \sum_{i=0}^{n} f(\frac{i}{n})C_n^i x^i(1-x)^{n-i}$ also is monotone of order k on [0,1]. The papers [4], [3], [2], [1], [18] investigate the shape preserving and convergence properties of sequences of linear Bernstein-type operators. On the other hand, it is well-known that one of the shortcomings for Bernstein-type approximation is the low order of approximation [6].

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The main goal of the paper is to present the construction of linear finite-dimensional spline operator that preserves k-monotonicity of approximated functions and has the higher order approximation than Bernstein-type operators. The papers [19], [20] present the example of linear operator of finite rank n that preserves k-monotonicity and uses k-th derivative's values of approximated function at equidistant knots on [0,1], with optimal order of approximation n^{-2}. In this paper we construct the linear finite-dimensional operator that preserves k-monotonicity and is based on the values of function at the points $z_j := j/n$, $j = 0, \dots, n$.

2. Definitions and notations

A continuous function $f : [0,1] \to \mathbb{R}$ is said to be k-monotone, $k \geq 1$, on $[0,1]$ if and only if for all choices of $k+1$ distinct $t_0, \dots, t_k$ in $[0,1]$ the inequality $[t_0, \dots, t_k]f \geq 0$ holds, where $[t_0, \dots, t_k]f$ denotes the k-th divided difference of f at $0 \leq t_0 < t_1 < \dots < t_k \leq 1$. Note that 2-monotone functions are just convex functions. Let Δ^k denote the set of all k-monotone functions defined on [0,1]. If f is a real-valued and k-times continuously differentiable function defined on $[0,1]$, then $f \in \Delta^k$ iff $f^{(k)}(t) \geq 0$, $t \in [0,1]$.

Let D^i denote the i-th differential operator, $D^i f(x) = d^i f(x)/dx^i$, and $D^0 = I$ is the identity operator. Denote $e_j(x) = x^j$, $j = 1, 2, \dots$.

Denote by $C^k[0,1]$, $k \geq 0$, the space of all real-valued and k-times continuously differentiable functions defined on $[0,1]$ endowed with the sup-norm

$$\|f\|_{C^k[0,1]} = \sum_{0 \leq i \leq k} \frac{1}{i!} \sup_{x \in [0,1]} |D^i f(x)|,$$

where the derivatives are taken from the right at 0 and from the left at 1.

It is said that a linear operator L of $C^k[0,1]$ into $C^k[0,1]$ preserves k-monotonicity, if $L(\Delta^k) \subset \Delta^k$.

Let $\mathcal{D}_l(f; z, h)$ denote the centered-difference approximation for l-th derivative of f at points $z \pm jh$, $j \in \{0, 1, 2, \dots\}$ with second order accuracy, i.e. $\mathcal{D}_0(f; z, h) = f(z)$, $\mathcal{D}_1(f; z, h) = (f(z+h) - f(z-h))/(2h)$, $\mathcal{D}_2(f; z, h) = (f(z+h) - 2f(z) + f(z-h))/(h^2)$, $\mathcal{D}_3(f; z, h) = (f(z+2h) - 2f(z+h) + 2f(z-h) - f(z-2h))/(2h^3)$, $\mathcal{D}_4(f; z, h) = (f(z+2h) - 4f(z+h) + 6f(z) - 4f(z-h) + f(z-2h))/(h^4)$, etc.

Denote $B^{(k+2)} := \{f \in C^{k+2}[0,1] : \|f\|_{C^{k+2}[0,1]} \leq 1\}$. It is well-known that there exist $a_r > 0$, $r = 0, 1, \dots, k$, such that for every $f \in B^{k+2}$

$$|D^r f(z) - \mathcal{D}_r(f; z, h)| \leq a_r h^2, \ r = 0, 1, \dots, k. \tag{1}$$

3. Linear Shape Preserving Operator

3.1. **The example (k is even).** In this subsection we will suppose that k is even, i.e. $k = 2p$, $p \in \{1, 2, \dots\}$. Let $z_j := j/n$, $j = 0, \dots, n$. Let the linear operator $M_{k,n} : C^k[0,1] \to C^k[0,1]$, $n \geq k+1$, be defined by

1) at point $x = z_p$:

$$D^l M_{k,n} f(z_p) = \mathcal{D}_l(f; z_p, 1/n), \ l = 0, 1, \dots, k-1; \tag{2}$$

2) in steps from left to right:

$$M_{k,n}f(x)=\sum_{l=0}^{k-1}\frac{1}{l!}(x-z_j)^l D^l M_{k,n}f(z_j)+(x-z_j)^k[z_{j-p},\ldots,z_{j+p}]f+ \\ +(x-z_j)^{k+1}[z_{j-p},\ldots,z_{j+p+1}]f,\ x\in(z_j,z_{j+1}],\ j=p,\ldots,n-p-1. \tag{3}$$

$$M_{k,n}f(x)=\sum_{l=0}^{k-1}\frac{1}{l!}(x-z_j)^l D^l M_{k,n}f(z_j)+(x-z_j)^k[z_{n-k-1},z_{n-k},\ldots,z_{n-1}]f+ \\ +(x-z_j)^{k+1}[z_{n-k-1},\ldots,z_n]f,\ x\in(z_j,z_{j+1}],\ j=n-p,\ldots,n-1. \tag{4}$$

3) in steps from right to left:

$$M_{k,n}f(x)=\sum_{l=0}^{k-1}\frac{1}{l!}(x-z_j)^l D^l M_{k,n}f(z_j)+(x-z_j)^k[z_0,z_{j+1},\ldots,z_k]f+ \\ +(x-z_j)^{k+1}[z_0,z_2,\ldots,z_{k+1}]f,\ \text{if } x\in[z_{j-1},z_j),\ j=p,p-1,\ldots,1. \tag{5}$$

3.2. **The example (the case $k=2p+1$).** Let us suppose that k is odd, i.e. $k=2p+1$, $p\in\{1,2,\ldots\}$. Let $z_j:=j/n$, $j=0,\ldots,n$. Denote $x_j=z_j-\frac{1}{2n}$, $j=1,2,\ldots,n$, $x_0=0$, $x_{n+1}=1$. Let the linear operator $M_{k,n}:C^k[0,1]\to C^k[0,1]$, $n\geq k+1$, be defined by

1) at point $x=z_{p+1}$:

$$D^l M_{k,n}f(z_{p+1})=\mathcal{D}_l(f;z_{p+1},1/n),\ l=0,1,\ldots,k-1; \tag{6}$$

2) in steps from left to right:

$$M_{k,n}f(x)=\sum_{l=0}^{k-1}\frac{1}{l!}(x-z_{p+1})^l D^l M_{k,n}f(z_{p+1})+(x-x_{p+1})^k[z_0,\ldots,z_{2p+1}]f+ \\ +(x-x_{p+1})^{k+1}[z_0,\ldots,z_{2p+2}]f,\ \text{if } x\in[x_{p+1},x_{p+2}], \tag{7}$$

$$M_{k,n}f(x)=\sum_{l=0}^{k-1}\frac{1}{l!}(x-x_j)^l D^l M_{k,n}f(x_j)+(x-x_j)^k[z_{j-p-1},\ldots,z_{j+p}]f+ \\ +(x-x_j)^{k+1}[z_{j-p-1},\ldots,z_{j+p+1}]f,\ x\in(x_j,x_{j+1}],\ j=p+2,\ldots,n-p-1. \tag{8}$$

$$M_{k,n}f(x)=\sum_{l=0}^{k-1}\frac{1}{l!}(x-x_j)^l D^l M_{k,n}f(x_j)+(x-x_j)^k[z_{n-k-1},z_{n-k},\ldots,z_{n-1}]f+ \\ +(x-x_j)^{k+1}[z_{n-k-1},\ldots,z_n]f,\ x\in(x_j,x_{j+1}],\ j=n-p,\ldots,n. \tag{9}$$

3) in steps from right to left:

$$M_{k,n}f(x)=\sum_{l=0}^{k-1}\frac{1}{l!}(x-x_j)^l D^l M_{k,n}f(x_j)+(x-x_j)^k[z_0,z_1,\ldots,z_k]f+ \\ +(x-x_j)^{k+1}[z_0,z_2,\ldots,z_{k+1}]f,\ \text{if } x\in[x_{j-1},x_j),\ j=p+1,p,\ldots,1. \tag{10}$$

3.3. **The Approximation Properties of $M_{k,n}$.** In this subsection we derive properties of the operator $M_{k,n} : C^k[0,1] \to C^k[0,1]$ defined by (2), (3), (4), (5) if k is even, and by (6), (7), (8), (9), (10) if k is odd.

Theorem 1. *Let $n-1 > k \geq 2$. Then $M_{k,n}(\Delta^k) \subset \Delta^k$.*

Proof. We present the proof for the case $k = 2p$. The case $k = 2p+1$ can be examined analogously. Denote

$$\phi_c^{k-1}(x) := \begin{cases} 0, \text{ if } x \in [0,c); \\ (x-c)^{k-1}, \text{ if } x \in [c,1] \end{cases}$$

To prove that $M_{k,n}f$ preserves k-monotonicity we will use the well-known result of J. Tzimbalario [21]. It states that if $L : C[0,1] \to C[0,1]$ is a linear continuous operator, then necessary and sufficient conditions for the implication $f \in \Delta^k \Rightarrow Lf \in \Delta^k$ are as follows

(1) if p is polynomial of degree $\leq k-1$ then so is Lp;
(2) $L\phi_c^{k-1} \in \Delta^k[0,1]$ for every $c \in [0,1]$.

It easy to check that $D^l M_{k,n} e_s(z_p) = D^l e_s(z_p)$, $l = 0,1,\dots,k-1$, for all $s = 0,1,\dots,k-1$.

If $x \in [z_p, z_{p+1}]$ then for all $s = 0,1,\dots,k-1$ we get

$$M_{k,n}e_s(x) = \sum_{l=0}^{k-1} \frac{1}{l!}(x-z_p)^l D^l M_{k,n} e_s(z_p) = \sum_{l=0}^{s} \frac{1}{l!}(x-z_p)^l D^l e_s(z_p) = e_s(x),$$

since all divided difference of e_s with order $\geq s+1$ are equal to 0.

Let $s \in \{0,1,\dots,k-1\}$ and suppose that $M_{k,n}e_s \equiv e_s$ on $[z_p, z_{j+1}]$ for some $j \in \{0,\dots,n-1\}$. We have for $x \in (z_{j+1}, z_{j+2}]$ $M_{k,n}e_s(x) = e_s(x)$.

Thus, $M_{k,n}e_s \equiv e_s$ on [0,1] for every $s \in \{0,1,\dots,k-1\}$, and consequently, if p is polynomial of degree $\leq k-1$ then so is $M_{k,n}p$.

Let us show that $M_{k,n}\phi_c^{k-1} \in \Delta^k[0,1]$ for every $c \in [0,1]$. To prove this it is sufficient to show that $D^k M_{k,n}\phi_c^{k-1} \geq 0$ for every $c \in [0,1]$. Note that if $c = 0$ then $\phi_c^{k-1} \equiv e_{k-1}$ on [0,1]. Since $M_{k,n}e_{k-1} \equiv e_{k-1}$ on [0,1], we get $D^k M_{k,n}\phi_c^{k-1} \equiv 0$ on [0,1]. We will consider a few possible cases.

1) Let $x \in [z_j, z_{j+1}]$, for a fixed $j = p,\dots,n-p-1$. We have (from the definition of divided difference)

$$[z_{j-p},\dots,z_{j+p+1}]f = \frac{n}{k+1}([z_{j-p+1},\dots,z_{j+p+1}]f - [z_{j-p},\dots,z_{j+p}]f).$$

Then

$$\begin{aligned} D^k M_{k,n}\phi_c^{k-1}(x) &= k![z_{j-p},\dots,z_{j+p}]\phi_c^{k-1} + \\ &+ k!n\,(x-z_j)\left([z_{j-p+1},\dots,z_{j+p+1}]\phi_c^{k-1} - [z_{j-p},\dots,z_{j+p}]\phi_c^{k-1}\right) = \\ &= k!(n\,(x-z_j)\,[z_{j-p+1},\dots,z_{j+p+1}]\phi_c^{k-1} + (1-n\,(x-z_j))[z_{j-p},\dots,z_{j+p}]\phi_c^{k-1}) \end{aligned}$$

is non-negative, since $0 \leq n\,(x-z_j) \leq 1$ and all the k-th divided differences of ϕ_c^{k-1} are non-negative (see, for example [9]).

3) If $x \in [z_j, z_{j+1}]$, for a fixed $j = n-p,\dots,n-1$, then

$$\begin{aligned} D^k M_{k,n}\phi_c^{k-1}(x) &= k!(n\,(x-z_j)\,[z_{n-k}, z_{n-k+1},\dots,z_n]\phi_c^{k-1} + \\ &+ (1-n\,(x-z_j))[z_{n-k-1}, z_{n-k},\dots,z_{n-1}]\phi_c^{k-1}) \geq 0, \end{aligned}$$

since $0 \le n\,(x - z_j) \le 1$ and all the divided differences of ϕ_c^{k-1} are non-negative.

4) If $x \in [z_{j-1}, z_j]$, for a fixed $j = p, p-1, \dots, 1$, then

$$D^k M_{k,n}\phi_c^{k-1}(x) = k!(n\,(z_j - x)\,[z_1, \dots, z_k]\phi_c^{k-1} + (1 - n\,(z_j - x))[z_0, \dots, z_k]\phi_c^{k-1})$$

is non-negative, since $0 \le n\,(x - z_j) \le 1$ and all the divided differences of ϕ_c^{k-1} are non-negative. □

Theorem 2. *Let $n - 1 > k \ge 2$. There exist $c_i > 0$ such that for all $f \in B^{(k+2)}$*

$$\sup_{x\in[z_p, z_{n-p}]} |D^i f(x) - D^i M_{k,n} f(x)| \le c_i n^{-2},\ i = 0, 1, \dots, k.$$

Proof. We present the proof for the case $k = 2p$. The case $k = 2p+1$ can be proved analogously.

Since $f \in B^{(k+2)}$, the function $D^k f$ can be represented as

$$D^k f(x) = D^k f(a) + D^{k+1} f(a)(x - a) + \int_a^1 (x - t)_+ D^{k+2} f(t)dt,$$

where $x, a \in [0, 1]$, $\|D^{k+2} f\|_{C[0,1]} \le (k + 2)!$ and $(x - t)_+ = \phi_t^1(x)$.

1) If $x \in [z_j, z_{j+1}]$, $j = p, \dots, n - p - 1$, then

$$\begin{aligned} D^k M_{k,n} f(x) - D^k f(x) &= k![z_{j-p}, \dots, z_{j+p}]f+\\ &+ k!n(x - z_j)([z_{j-p+1} \dots, z_{j+p+1}]f - [z_{j-p}, \dots, z_{j+p}]f)-\\ &- D^k f(z_j) - \frac{D^{k+1} f(z_j)}{1!}(x - z_j) - \frac{1}{(k + 1)!}\int_{z_j}^1 (x - t)_+^{k+1} D^{k+2} f(t)\, dt =\\ &= (k![z_{j-p}, \dots, z_{j+p}]f - D^k f(z_j))+\\ &+ (x - z_j)((k + 1)![z_{j-p}, \dots, z_{j+p+1}]f - D^{k+1} f(z_j))-\\ &- \frac{1}{(k + 1)!}\int_{z_j}^1 (x - t)_+^{k+1} D^{k+2} f(t)\, dt. \end{aligned}$$

It is well-known [5] that if $x_0 < x_1 < \ldots < x_k$ are from [0,1] then

$$|k![x_0, \dots, x_k]f - D^k f(\overline{x})| \le \frac{k}{24n^2}\|D^{k+2} f\|_{C[0,1]},\ \overline{x} := \frac{1}{k+1}\sum_{i=0}^{k} x_i$$

and, consequently, we have

$$|k![z_{j-p}, \dots, z_{j+p}]f - D^k f(z_j)| \le \frac{k(k + 2)!}{24n^2}. \tag{11}$$

It follows from the inequalities

$$|(k + 1)![z_{j-p}, \dots, z_{j+p+1}]f - D^{k+1} f(z_j)| \le \frac{(k + 1)(k + 1)!}{n},$$

$$\left|\int_{z_j}^1 (x - t)_+^{k+1} D^{k+2} f(t)\, dt\right| \le \frac{(k + 2)!}{2n^2},\ x \in [z_j, z_{j+1}],$$

and (11) that there exists $c_k > 0$ such that

$$|D^k M_{k,n} f(x) - D^k f(x)| \le c_k n^{-2},\ x \in [z_j, z_{j+1}],\ j = p, \dots, n - p - 1.$$

Therefore, for every $f \in B^{(k+2)}$ and any $x \in [z_p, z_{n-p}]$

$$|D^k f(x) - D^k M_{k,n} f(x)| \le c_k n^{-2}. \tag{12}$$

Suppose (by induction) that $\|D^i f - D^i M_{k,n} f\|_{C[0,1]} \le c_i n^{-2}$. It follows from (1) that for all $x \in [0,1]$

$$|D^i(M_{k,n}f - f)(x)| = \left| D^i(M_{k,n}f - f)(z_p) + \int_{z_p}^{x} D^{i+1}(M_{k,n}f - f)(t)dt \right| \le$$
$$\le a_i n^{-2} + \left| \int_{z_p}^{x} D^{i+1}(M_{k,n}f - f)(t)dt \right|.$$

Then

$$|D^i(M_{k,n}f - f)(x)| \le a_i n^{-2} + \frac{1}{n^2} \frac{x^{k-1-i}}{(k-1-i)!}. \tag{13}$$

We have used the fact that if $f \in L_\infty[0,1]$ and there exists a real $a \in \mathbb{R}$ such that $|f| \le a$ on [0,1], then

$$\left| \int_0^x \int_0^{t_{p-1}} \dots \int_0^{t_1} f(t_1)dt_1 \dots dt_p \right| \le \int_0^x \int_0^{t_{p-1}} \dots \int_0^{t_1} |f(t_1)|dt_1 \dots dt_p \le a\frac{x^p}{p!}$$

for every $p \in \mathbb{N}$.

Theorem follows from (13) if we take $c_{i-1} = a_i + \frac{c_i x^{k-1-i}}{(k-1-i)!}$. □

The paper [19] shows that the order of approximation n^{-2} can not be improved. As it was shown in [11] non-linear approximation methods preserving k-monotonicity are much better in the terms of approximation error than linear ones. On the other hand, for sequences of linear operators preserving k-monotonicity (as well as intersections of cones) there are simple convergence conditions (Korovkin type results) [8], [15]. Besides, linear algorithms are easy to implement and they can be simpler with computational point of view.

While algorithm presented in this paper uses the values of functions at fixed points, the paper presents the method of *approximation* (not interpolation). It should be noted that shape-preserving *interpolation* spline algorithms were developed and examined in the variety of works (see, for example the bibliography of [10] and [13]).

References

[1] B. Barnabas, L. Coroianu, Sorin G. Gal, *Approximation and shape preserving properties of the Bernstein operator of max-product kind*, International Journal of Mathematics and Mathematical Sciences, **2009** (2009), Article ID 590589, 26 pages. doi:10.1155/2009/590589 Zbl 1188.41016

[2] D. Cárdenas-Morales, P. Garrancho, I. Raşa, *Bernstein-type operators which preserve polynomials*, Computers and Mathematics with Applications, **62** (2011), 158–163. Zbl 1228.41019

[3] D. Cárdenas-Morales, F. J. Muñoz-Delgado, *Improving certain Bernstein-type approximation processes*, Mathematics and Computers in Simulation, **77** (2008), 170–178. MR2397891

[4] D. Cárdenas-Morales, F. J. Muñoz-Delgado, P. Garrancho, *Shape preserving approximation by Bernstein-type operators which fix polynomials*, Applied Mathematics and Computation, **182** (2006), 1615–1622. MR2282600

[5] M. S. Floater, *Error formulas for divided difference expansions and numerical differentiation*, J. Approx. Theory, **122** (2003), 1–9. MR1976120

[6] M. S. Floater, *On the convergence of derivatives of Bernstein approximation*, J. Approx. Theory, **134** (2005), 130–135. MR2137560

[7] S. G. Gal, *Shape-Preserving Approximation by Real and Complex Polynomials*, Springer, 2008. MR2444986

[8] H. H. Gonska, *Quantitative Korovkin type theorems on simultaneous approximation*, Mathematische Zeitschrift, **186**:3 (1984), 419–433. MR0744832

[9] S. Karlin, W. Stadden. *Tchebycheff systems: With applications in analysis and statistics.* New York: Interscience Publishers John Wiley & Sons, 1966. Vol. XV of *Pure and applied mathematics.* MR0204922

[10] L.M. Kocić, G.V. Milovanović, *Shape preserving approximations by polynomials and splines*, Computers & Mathematics with Applications, **33**:11 (1997), 59–97. MR1458960

[11] K. Kopotun, A. Shadrin, *On k-monotone approximation by free knot splines*, SIAM J. Math. Anal., **34** (2003), 901–924. MR1969607

[12] B. I. Kvasov, *Methods of shape preserving spline approximation*, Singapore: World Scientific Publ. Co. Pte. Ltd. 2000. MR1784747

[13] B. I. Kvasov, *Monotone and convex interpolation by weighted cubic splines*, Computational Mathematics and Mathematical Physics, **53**:10 (2013), 1428–1439. MR3254884

[14] K. A. Kopotun, D. Leviatan, A. Prymak, I. A. Shevchuk. *Uniform and Pointwise Shape Preserving Approximation by Algebraic Polynomials*, Surveys in Approximation Theory, **6** (2011), 24–74. MR2832606

[15] F. J. Muñoz-Delgado, V. Ramírez-González, D. Cárdenas-Morales, *Qualitative Korovkin-type results on conservative approximation*, J. Approx. Theory, **94** (1998), 144–159. MR1637819

[16] J. Pál, *Approksimation of konvekse funktioner ved konvekse polynomier*, Mat. Tidsskrift, Vol. B (1925), 60–65. JFM 51.0210.02

[17] T. Popoviciu, *About the Best Polynomial Approximation of Continuous Functions. Mathematical Monography*, Sect. Mat. Univ. Cluj, 1937. (In Romanian), fasc. III. JFM 63.0959.03

[18] R. Păltănea, *A generalization of Kantorovich operators and a shape-preserving property of Bernstein operators*, Bulletin of the Transilvania University of Brasov, Series III, **5(54)**:2 (2012), 65–68. MR3035857

[19] S. P. Sidorov, *On the order of approximation by linear shape-preserving operators of finite rank*, East J. on Approx., **7**:1 (2001), 1–8. MR1834152

[20] S. P. Sidorov, *Negative property of shape preserving finite-dimensional linear operators*, Appl. Math. Lett., **16**:2 (2003), 257–261. MR1962324

[21] J. Tzimbalario, *Approximation of functions by convexity preserving continuous linear operators*, Proc. Amer. Math. Soc. **53** (1975), 129–132. MR0382943

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