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ON SCHUR 3-GROUPS

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Abstract. Let G be a finite group. An S-ring $\mathcal{A}$ over G is a subring of the group ring $\mathbb{Z}G$ that has a linear basis associated with a special partition of G. About 40 years ago R. Pöschel suggested the problem which can be formulated as follows: for which group G every S-ring $\mathcal{A}$ over it is schurian, i.e. the partition of G corresponding to $\mathcal{A}$ consists of the orbits of the one point stabilizer of a permutation group in $Sym(G)$ that contains a regular subgroup isomorphic to G. The main result of the paper says that such G can not be non-abelian p-group, where p is an odd prime. In fact, modulo known results, it was sufficient to show that for every $n \geq 3$ there exists a non-schurian S-ring over the group $M_{3^n} = \langle a, b \mid a^{3^{n-1}} = b^3 = e, a^b = a^{3^{n-2}+1}\rangle$.

Keywords: Permutation groups, Cayley schemes, S-rings, Schur groups.

1. Introduction

Let G be a finite group, e the identity element of G. Let $\mathbb{Z}G$ be the integer group ring. Given $X \subseteq G$, denote the element $\sum_{x\in X} x$ by $\underline{X}$.

Definition 1. *A subring $\mathcal{A}$ of $\mathbb{Z}G$ is called an* S-ring *over G if there exists a partition $\mathcal{S} = \mathcal{S}(\mathcal{A})$ of G such that:*
(1) $\{e\} \in \mathcal{S}$,
(2) $X \in \mathcal{S} \Rightarrow X^{-1} \in \mathcal{S}$,
(3) $\mathcal{A} = Span_{\mathbb{Z}}\{\underline{X} : X \in \mathcal{S}\}$.

The elements of this partition are called *the basic sets* of the S-ring $\mathcal{A}$.

Let Γ be a subgroup of $Sym(G)$ that contains the subgroup of right shifts $G_{right} = \{x \mapsto xg,\ x \in G : g \in G\}$. Let Γ_e stand for the stabilizer of e in Γ

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and $Orb(\Gamma_e, G)$ stand for the set of all orbits Γ_e on G. As I. Schur proved in [9], the $\mathbb{Z}$-submodule

$$\mathcal{A} = \mathcal{A}(\Gamma,\ G) = Span_{\mathbb{Z}}\left\{\underline{X}:\ X \in Orb(\Gamma_e,\ G)\right\},$$

is an S-ring over G.

Definition 2. *An S-ring $\mathcal{A}$ over G is called* schurian *if $\mathcal{A} = \mathcal{A}(\Gamma,\ G)$ for some Γ with $G_{right} \leq \Gamma \leq Sym(G)$.*

Definition 3. *A finite group G is called a* Schur *group if every S-ring over G is schurian.*

The problem of determining all Schur groups was suggested by R. Pöschel in [8] about 40 years ago. He proved that a p-group, $p > 3$, is Schur if and only if it is cyclic. Using this result R. Pöschel and M. Klin solved the isomorphism problem for circulant graphs with p^n vertices, where p is an odd prime and $n \geq 1$ is an integer [4]. Only 30 years later all cyclic Schur groups were classified in [1]. Strong necessary conditions of schurity for abelian groups were recently proved in [2].

All Schur groups of order ≤ 62 were found by computer calculations [3, 10]. It turned out that there are non-abelian Schur groups. However, except for Pöschel's result about p-groups, there were no general results on non-abelian Schur groups. Recently it was proved [7] that every Schur group G is solvable of derived length at most 2 and the number of distinct prime divisors of the order of G does not exceed 7. In the same article it was proved that only the groups $M_{3^n} = \langle a, b \mid a^{3^{n-1}} = b^3 = e, a^b = a^{3^{n-2}+1}\rangle,\ n \geq 3$, might be non-abelian Schur 3-groups.

The main result of this paper is the following

Theorem 1. *The groups $M_{3^n} = \langle a, b \mid a^{3^{n-1}} = b^3 = e, a^b = a^{3^{n-2}+1}\rangle,\ n \geq 3,$ are not Schur.*

It is worth noting that the case of Schur 2-groups was very recently analyzed in [6] by M. Muzychuk and I. Ponomarenko. The author is grateful to both of them for fruitful discussions on the subject matters.

From the above discussion and Theorem 1, we immediately obtain the following statement.

Corollary 1. *Every Schur p-group is abelian whenever p is an odd prime.*

2. Preliminaries

In this section we recall some definitions and facts about S-rings and Cayley schemes. All of them are taken from [5], so we skip further references here.

Definition 4. *Let G be a finite group, $\mathcal{R}$ a family of binary relations on G. The pair $\mathcal{C} = (G, \mathcal{R})$ is called a* Cayley scheme *over G if the following properties are satisfied:*

(1) *$\mathcal{R}$ forms a partition of the set $G \times G$;*

(2) *$Diag\,(G \times G) \in \mathcal{R}$;*

(3) *$\mathcal{R} = \mathcal{R}^*$, i. e., if $R \in \mathcal{R}$ then $R^* = \{(h, g) \mid (g, h) \in R\} \in \mathcal{R}$;*

(4) *if $R,\ S,\ T \in \mathcal{R}$ and $(f, g) \in T$, then the number $|\{h \in G : (f, h) \in R,\ (h, g) \in S\}|$ does not depend on the choice of (f, g).*

Let $\mathcal{A}$ be an S-ring over G. We associate each basic set $X \in \mathcal{S}(\mathcal{A})$ with the binary relation $\{(a, xa) \mid a \in G, x \in X\} \subseteq G \times G$ and denote it by $R(X)$. The set of all such binary relations forms a partition $\mathcal{R}(\mathcal{S}(\mathcal{A}))$ of $G \times G$.

Lemma 1. *$\mathcal{C}(\mathcal{A}) = (G, \mathcal{R}(\mathcal{S}(\mathcal{A})))$ is a Cayley scheme over G. The map $\mathcal{A} \mapsto \mathcal{C}(\mathcal{A})$ is a bijection between S-rings and Cayley schemes over G.*

The relation $R(X)$ is called *the basic relation* of the scheme $\mathcal{C}(\mathcal{A})$ corresponding to X.

Definition 5. *Cayley schemes $\mathcal{C} = (G, \mathcal{R})$ and $\mathcal{C}' = (G', \mathcal{R}')$ are called* isomorphic *if there exists a bijection $f : G \to G'$ such that $\mathcal{R}' = \mathcal{R}^f$, where $\mathcal{R}^f = \{R^f : \ R \in \mathcal{R}\}$ and $R^f = \{(a^f, \ b^f) : \ (a, \ b) \in R\}$.*

The set of all isomorphisms from $\mathcal{C}$ onto $\mathcal{C}'$ is denoted by $\mathrm{Iso}(\mathcal{C}, \mathcal{C}')$. The group $\mathrm{Iso}(\mathcal{C}) = \mathrm{Iso}(\mathcal{C}, \mathcal{C})$ of all isomorphisms of $\mathcal{C}$ onto itself contains the normal subgroup

$$\mathrm{Aut}(\mathcal{C}) = \{f \in \mathrm{Iso}(\mathcal{C}) : \ R^f = R, \ R \in \mathcal{R}\}$$

called *the automorphism group* of $\mathcal{C}$.

Definition 6. *Two S-rings $\mathcal{A}$ over G and $\mathcal{A}'$ over G' are called* isomorphic *if there exists an isomorphism of the corresponding Cayley schemes $\mathcal{C}(\mathcal{A})$ and $\mathcal{C}'(\mathcal{A}')$ taking the identity element of G to the identity element of G'. It is called* the isomorphism *from $\mathcal{A}$ to $\mathcal{A}'$.*

We denote the set of all isomorphisms from $\mathcal{A}$ onto $\mathcal{A}'$ by $\mathrm{Iso}(\mathcal{A}, \mathcal{A}')$. The group $\mathrm{Iso}(\mathcal{A}) = \mathrm{Iso}(\mathcal{A}, \mathcal{A})$ contains the normal subgroup

$$\mathrm{Aut}(\mathcal{A}) = \{f \in \mathrm{Iso}(\mathcal{A}) : \ R(X)^f = R(X), \ X \in \mathcal{S}(\mathcal{A})\}$$

called *the automorphism group* of $\mathcal{A}$.

Lemma 2. *Suppose that $\mathcal{A}$ is an S-ring, $\mathcal{C}(\mathcal{A})$ is the corresponding Cayley scheme. Then*

$$\mathrm{Iso}(\mathcal{A}) = \mathrm{Iso}(\mathcal{C}(\mathcal{A}))_e, \ \mathrm{Iso}(\mathcal{C}(\mathcal{A})) = G_{right} \, \mathrm{Iso}(\mathcal{A}),$$
$$\mathrm{Aut}(\mathcal{A}) = \mathrm{Aut}(\mathcal{C}(\mathcal{A}))_e, \ \mathrm{Aut}(\mathcal{C}(\mathcal{A})) = G_{right} \, \mathrm{Aut}(\mathcal{A}).$$

Lemma 3. *An S-ring $\mathcal{A}$ over G is schurian if and only if $\mathcal{S}(\mathcal{A}) = Orb(\mathrm{Aut}(\mathcal{A}), G)$.*

Definition 7. *Let $\mathcal{A}$ be an S-ring over G. A subgroup $H \leq G$ is called* an $\mathcal{A}$-subgroup *if $\underline{H} \in \mathcal{A}$.*

Definition 8. *Let L, U be subgroups of a group G and L be normal in U. A section U/L of G is called an* $\mathcal{A}$-section *if U and L are $\mathcal{A}$-subgroups.*

Lemma 4. *Let $D = U/L$ be an $\mathcal{A}$-section. Then the module*

$$\mathcal{A}_D = Span_{\mathbb{Z}} \left\{\underline{X}^{\pi} : \ X \in \mathcal{S}(\mathcal{A}), \ X \subseteq U\right\},$$

where $\pi : U \to U/L$ is the canonical homomorphism, is an S-ring over D.

In addition, if $\mathcal{A}$ is schurian, then $\mathcal{A}_D$ is schurian too.

Lemma 5. *Let $\mathcal{A}$ be a schurian S-ring, $\mathcal{A} = \mathcal{A}(\Gamma, \ G)$. Then Γ is imprimitive if and only if there exists a non-trivial proper $\mathcal{A}$-subgroup H of G. In this case the left cosets of H form the non-trivial block system of Γ.*

3. Non-schurity of M_{27}

Let $G = M_{27} = \langle a, b \mid a^{3^2} = b^3 = e, a^b = a^4\rangle$. Denote the subgroup $\langle a^3 b\rangle = \{e,\ a^3b,\ a^6b^2\}$ by U. Consider the sets

$$Z_0 = \{e\}\,,$$
$$Z_1 = \left\{a^3b,\ a^6b^2\right\} = U \setminus \{e\}\,,$$
$$Z_2 = \left\{a,\ a^3,\ a^6,\ a^8,\ a^4b^2,\ a^7b,\ a^8b,\ a^8b^2\right\},$$
$$Z_3 = G \setminus (Z_0 \cup Z_1 \cup Z_2)\,.$$

The sets $Z_0,\ Z_1,\ Z_2$, and Z_3 form the partition of G, which is denoted by $\mathcal{S}$. Note that $Z_i = Z_i^{-1},\ i = 0, \dots, 3$.

Lemma 6. *The $\mathbb{Z}$-module $\mathcal{A}$ spanned by the elements $\xi_i = \underline{Z_i},\ i = 0, \dots, 3$, is a commutative S-ring over G.*

Proof. The commutativity of $\mathcal{A}$ immediately follows from the fact that each class of the partition is closed with respect to taking inverse. The computations in the group ring of G show that

$$\xi_0\xi_i = \xi_i\xi_0 = \xi_i;$$

$$\xi_1\xi_1 = 2\xi_0 + \xi_1,$$
$$\xi_1\xi_2 = \xi_2\xi_1 = \xi_3,$$
$$\xi_1\xi_3 = \xi_3\xi_1 = \xi_3 + 2\xi_2;$$

$$\xi_2\xi_2 = 8\xi_0 + \xi_2 + 3\xi_3,$$
$$\xi_2\xi_3 = \xi_3\xi_2 = 8\xi_1 + 6\xi_2 + 4\xi_3;$$

$$\xi_3\xi_3 = 16\xi_0 + 8\xi_1 + 8\xi_2 + 10\xi_3.$$

□

Proposition 1. *The S-ring $\mathcal{A}$ is not schurian.*

Proof. Suppose on the contrary that $\mathcal{A}$ is schurian. Then it follows from Lemma 3 that $\mathcal{A} = \mathcal{A}(KG_{right}, G)$, where $K = \mathrm{Aut}(\mathcal{A})$ and the sets $Z_i,\ i = 0, \dots, 3$, are the orbits of K.

Lemma 7. *The subgroup U is an $\mathcal{A}$-subgroup. The left cosets of U are the blocks of K.*

Proof. Note that $U = Z_0 \cup Z_1$. Then $\underline{U} \in \mathcal{A}$. Therefore U is an $\mathcal{A}$-subgroup and we are done by Lemma 5. □

The elements b and a^2 belong to Z_3. Since the latter is a K-orbit, there exists $\alpha \in K$ taking b to a^2.

Let $\mathcal{C}(\mathcal{A})$ be the Cayley scheme corresponding to $\mathcal{A}$. Then α is an automorphism of $\mathcal{C}(\mathcal{A})$. Therefore α takes the neighborhood $\{a^3b^2, a^6\}$ of b in $R(Z_1)$ to the neighborhood $\{a^8b, a^5b^2\}$ of a^2 in $R(Z_1)$. However, the elements a^6 and a^8b lie in Z_2, the elements a^3b^2 and a^5b^2 lie in Z_3. Thus, since α preserves Z_2 and Z_3, we have

$$(a^6)^{\alpha} = a^8b; \tag{1}$$

$$(a^3b^2)^{\alpha} = a^5b^2. \tag{2}$$

It follows from (1) and Lemma 7 that $(a^3b^2U)^{\alpha} = a^5U$. This contradicts (2) because $(a^3b^2)^{\alpha} = a^5b^2 \notin a^5U$. □

4. Non-schurity of M_{3^n}, $n \geq 4$

Let $G = M_{3^n} = \langle a, b \mid a^{3^{n-1}} = b^3 = e, a^b = a^{3^{n-2}+1}\rangle$ and $n \geq 4$. Put $A = \langle a\rangle$, $B = \langle b\rangle$, $c = a^{3^{n-2}}$, $C = \langle c\rangle$, and $H = C \times B$. Obviously, $|A| = 3^{n-1}$, $|B| = 3$, $|C| = 3$, and $|H| = 9$.

Consider the sets

$$Z_0 = \{e\},$$
$$Z_1 = \{b, b^2\},$$
$$Z_2 = \{c, c^2\},$$
$$Z_3 = \{cb, cb^2, c^2b, c^2b^2\} = H \setminus (Z_0 \cup Z_1 \cup Z_2).$$
$$Z_4 = a\{e, cb, c^2b^2\} \cup a^{-1}\{e, c^2b, cb^2\},$$
$$Z_5 = (aH \cup a^{-1}H) \setminus Z_4,$$
$$X_k = a^{3k}C \cup a^{-3k}C,\ Y_k = (a^{3k}H \cup a^{-3k}H) \setminus X_k,\ k = 1, \dots, \frac{3^{n-3}-1}{2},$$
$$T_j = a^jH \cup a^{-j}H,\ j = 2,\ 4,\ 5, \dots, \frac{3^{n-2}-1}{2},\ j \not\equiv 0 \mod 3.$$

Note that the sets Z_i, X_k, Y_k, T_j form the partition of G, denote it by $\mathcal{S}$. It is easy to check that $Z_i = Z_i^{-1}$, $X_k = X_k^{-1}$, $Y_k = Y_k^{-1}$, $T_j = T_j^{-1}$.

Lemma 8. *The $\mathbb{Z}$-module $\mathcal{A}$ spanned by the elements $\xi_i = \underline{Z_i}$, $\theta_k = \underline{X_k}$, $\psi_k = \underline{Y_k}$, $\varphi_j = \underline{T_j}$ is a commutative S-ring over group G.*

Proof. The commutativity of $\mathcal{A}$ immediately follows from the fact that each class of the partition is closed with respect to taking inverse. The computations in the group ring of G show that

$$\xi_0\xi_i = \xi_i,\ \xi_0\theta_k = \theta_k,\ \xi_0\psi_k = \psi_k,\ \xi_0\varphi_j = \varphi_j;$$

$$\xi_1\xi_1 = 2\xi_0 + \xi_1,$$
$$\xi_1\xi_2 = \xi_2\xi_1 = \xi_3,$$
$$\xi_1\xi_3 = \xi_3\xi_1 = \xi_3 + 2\xi_2,$$
$$\xi_1\xi_4 = \xi_4\xi_1 = \xi_5,$$
$$\xi_1\xi_5 = \xi_5\xi_1 = \xi_5 + 2\xi_4,$$
$$\xi_1\theta_k = \theta_k\xi_1 = \psi_k,$$
$$\xi_1\psi_k = \psi_k\xi_1 = \psi_k + 2\theta_k,$$
$$\xi_1\varphi_j = \varphi_j\xi_1 = 2\varphi_j;$$

$$\xi_2\xi_2 = 2\xi_0 + \xi_2,$$
$$\xi_2\xi_3 = \xi_3\xi_2 = \xi_3 + 2\xi_1,$$
$$\xi_2\xi_4 = \xi_4\xi_2 = \xi_5,$$
$$\xi_2\xi_5 = \xi_5\xi_2 = 2\xi_4 + \xi_5,$$
$$\xi_2\theta_k = \theta_k\xi_2 = 2\theta_k,$$

$$\xi_2\psi_k = \psi_k\xi_2 = 2\psi_k,$$
$$\xi_2\varphi_j = \varphi_j\xi_2 = 2\varphi_j;$$

$$\xi_3\xi_3 = \xi_3 + 2\xi_1 + 2\xi_2 + 4\xi_0,$$
$$\xi_3\xi_4 = \xi_4\xi_3 = \xi_5 + 2\xi_4,$$
$$\xi_3\xi_5 = \xi_5\xi_3 = 3\xi_5 + 2\xi_4,$$
$$\xi_3\theta_k = \theta_k\xi_3 = 2\psi_k,$$
$$\xi_3\psi_k = \psi_k\xi_3 = 2\psi_k + 4\theta_k,$$
$$\xi_3\varphi_j = \varphi_j\xi_3 = 4\varphi_j;$$

$$\xi_4\xi_4 = \varphi_2 + 6\xi_0 + 3\xi_3,$$
$$\xi_4\xi_5 = \xi_5\xi_4 = 2\varphi_2 + 6\xi_1 + 6\xi_2 + 3\xi_3,$$
$$\xi_4\theta_k = \theta_k\xi_4 = \varphi_{3k+1} + \varphi_{3k-1},$$
$$\xi_4\psi_k = \psi_k\xi_4 = 2\varphi_{3k+1} + 2\varphi_{3k-1},$$
$$\xi_4\varphi_j = \varphi_j\xi_4 = 3\varphi_{j+1} + 3\theta_l + 3\psi_l,\ j-1=3l,$$
$$\xi_4\varphi_j = \varphi_j\xi_4 = 3\varphi_{j-1} + 3\theta_l + 3\psi_l,\ j+1=3l;$$

$$\xi_5\xi_5 = 4\varphi_2 + 6\xi_1 + 6\xi_2 + 9\xi_3 + 12\xi_0,$$
$$\xi_5\theta_k = \theta_k\xi_5 = 2\varphi_{3k+1} + 2\varphi_{3k-1},$$
$$\xi_5\psi_k = \psi_k\xi_5 = 4\varphi_{3k+1} + 4\varphi_{3k-1},$$
$$\xi_5\varphi_j = \varphi_j\xi_5 = 6\varphi_{j+1} + 6\theta_l + 6\psi_l,\ j-1=3l,$$
$$\xi_5\varphi_j = \varphi_j\xi_5 = 6\varphi_{j-1} + 6\theta_l + 6\psi_l,\ j+1=3l;$$

$$\theta_k\theta_l = \theta_l\theta_k = \theta_{k+l} + \theta_{k-l},\ k \neq l,$$
$$\theta_k\theta_k = 3\theta_{2k} + 6\xi_0 + 6\xi_2,$$
$$\theta_k\psi_l = \psi_l\theta_k = \psi_{k+l} + \psi_{k-l},\ k \neq l,$$
$$\theta_k\psi_k = \psi_k\theta_k = 3\psi_{2k} + 6\xi_1 + 6\xi_3,$$
$$\theta_k\varphi_j = \varphi_j\theta_k = 3\varphi_{3k+j} + 3\varphi_{3k-j};$$

$$\psi_k\psi_l = \psi_l\psi_k = 2\theta_{k+l} + 2\theta_{k-l} + \psi_{k+l} + \psi_{k-l},\ k \neq l,$$
$$\psi_k\psi_k = 3\psi_{2k} + 6\theta_{2k} + 6\xi_1 + 12\xi_2 + 6\xi_3 + 12\xi_0,$$
$$\psi_k\varphi_j = \varphi_j\psi_k = 6\varphi_{3k+j} + 6\varphi_{3k-j};$$

$$\varphi_i\varphi_j = \varphi_j\varphi_i = 9\varphi_{i+j} + 9\theta_k + 9\psi_k,\ i-j=3k,$$
$$\varphi_i\varphi_j = \varphi_j\varphi_i = 9\varphi_{i-j} + 9\theta_k + 9\psi_k,\ i+j=3k,$$
$$\varphi_j\varphi_j = 9\varphi_{2j} + 18\xi_0 + 18\xi_1 + 18\xi_2 + 18\xi_3.$$

Let us, for example, check that $\varphi_j\varphi_j = 9\varphi_{2j} + 18\xi_0 + 18\xi_1 + 18\xi_2 + 18\xi_3$. Since $G^{'} \leq H$, the subgroup H is normal in G and $gH = Hg$ for every $g \in G$. So $\varphi_j\varphi_j = (a^j\underline{H} + a^{-j}\underline{H})^2 = a^{2j}(\underline{H})^2 + a^{-2j}(\underline{H})^2 + 2(\underline{H})^2 = 9a^{2j}\underline{H} + 9a^{-2j}\underline{H} + 18\underline{H} = 9\varphi_{2j} + 18\xi_0 + 18\xi_1 + 18\xi_2 + 18\xi_3$.

□

Proposition 2. *The S-ring $\mathcal{A}$ is not schurian.*

Proof. Assume the contrary. Then it follows from Lemma 3 that $\mathcal{A} = \mathcal{A}(KG_{right}, G)$ where $K = \mathrm{Aut}(\mathcal{A})$ and the sets $Z_i,\ X_k,\ Y_k,\ T_j$ are the orbits of K.

Lemma 9. *The groups C, B, H are $\mathcal{A}$-subgroups. The left cosets of C, B, H are the blocks of K.*

Proof. Note that $B = Z_0 \cup Z_1,\ C = Z_0 \cup Z_2,\ H = Z_0 \cup Z_1 \cup Z_2 \cup Z_3$. Then $\underline{C},\ \underline{B},\ \underline{H} \in \mathcal{A}$. Therefore C, B, H are $\mathcal{A}$-subgroups and it follows from Lemma 5 that the left cosets of C, B, H are the blocks of K. □

Since H is a normal $\mathcal{A}$-subgroup, following Lemma 4 we can form an S-ring over G/H

$$\mathcal{A}_{G/H} = Span_{\mathbb{Z}} \{\underline{X}^{\pi}:\ X \in \mathcal{S}(\mathcal{A})\},$$

where $\pi : G \to G/H$ is a canonical homomorphism.

Lemma 10. *The automorphism group of the S-ring $\mathcal{A}_{G/H}$ is of order 2. Its non-identity element permutes a^iH and $a^{-i}H$.*

Proof. The basic sets of $\mathcal{A}_{G/H}$ are of the form

$$\{H\},\ \left\{a^iH, a^{-i}H\right\},\ i = 1, \dots, \frac{3^{n-2}-1}{2}.$$

Let $\mathcal{C}(\mathcal{A}_{G/H})$ be the Cayley scheme corresponding to $\mathcal{A}_{G/H}$. The basic relation of $\mathcal{C}(\mathcal{A}_{G/H})$ corresponding to the basic set $\left\{aH, a^{-1}H\right\}$ is a cycle of length 3^{n-2}. The conclusion of the lemma is a direct consequence of the fact that the automorphism group of the graph of undirected cycle is dihedral, and the one point stabilizer of it is of order 2. □

Consider the action of the elements from K on the set Z_3. The automorphisms of $\mathcal{A}$ do not contain in their cyclic structure cycles of length 3 and 4 consisting of the elements from Z_3 because by Lemma 9 the left cosets cB, c^2B and bC, b^2C are the blocks of K. Since Z_3 is an orbit of K, it follows that K acts on Z_3 as the Klein four-group K_4. Since Z_1 and Z_2 are K-orbits of length 2, each permutation of K can be written as follows:

$$\gamma, \tag{1}$$
$$(b,\ b^2)(cb,\ cb^2)(c^2b,\ c^2b^2)\gamma, \tag{2}$$
$$(c,\ c^2)(cb,\ c^2b)(cb^2,\ c^2b^2)\gamma, \tag{3}$$
$$(c,\ c^2)(b,\ b^2)(cb,\ c^2b^2)(cb^2,\ c^2b)\gamma, \tag{4}$$

where $\gamma \in K$ acting on H trivially.

Let $\mathcal{C}(\mathcal{A})$ be the Cayley scheme corresponding to $\mathcal{A}$. Below we list the elements adjacent to $e,\ b,\ b^2,\ c,\ c^2$ in $R(Z_4)$:

$$e : a,\ cab,\ c^2ab^2,\ a^{-1},\ ca^{-1}b^2,\ c^2a^{-1}b;$$
$$b : ab,\ cab^2,\ c^2a,\ a^{-1}b,\ ca^{-1},\ c^2a^{-1}b^2;$$
$$b^2 : ab^2,\ ca,\ c^2ab,\ a^{-1}b^2,\ ca^{-1}b,\ c^2a^{-1};$$
$$c : ab^2,\ ca,\ c^2ab,\ a^{-1}b,\ ca^{-1},\ c^2a^{-1}b^2;$$
$$c^2 : ab,\ cab^2,\ c^2a,\ a^{-1}b^2,\ ca^{-1}b,\ c^2a^{-1}.$$

Denote the set of all elements adjacent to $q \in G$ in $R(Z_4)$ by L_q. Let $\alpha \in K$ is of type (4). Then $b^{\alpha} = b^2$ and $(c^2)^{\alpha} = c$. The element ab is adjacent to b and

c^2 in $R(Z_4)$. Therefore $(ab)^\alpha$ is adjacent to b^2 and c in $R(Z_4)$. We conclude that $(ab)^\alpha \in L_{b^2} \cap L_c = \{ab^2,\ ca,\ c^2ab\}$. Similarly an element of type (2) takes ab to one of the elements $ca^{-1}b,\ a^{-1}b^2,\ c^2a^{-1}$ and an element of type (3) takes ab to one of the elements $a^{-1}b,\ ca^{-1},\ c^2a^{-1}b^2$. This means that $(ab)^\alpha \in aH \cap (aH)^\alpha$ for $\alpha \in K$ of types (1) and (4), $(ab)^\alpha \in a^{-1}H \cap (aH)^\alpha$ for $\alpha \in K$ of types (2) and (3). The left cosets of H are the blocks of K, so $aH = (aH)^\alpha$ for every $\alpha \in K$ of types (1) and (4), $a^{-1}H = (aH)^\alpha$ for every $\alpha \in K$ of types (2) and (3). Applying Lemma 10, we conclude that elements of types (1) and (4) fix all left cosets of H while elements of types (2) and (3) interchange cosets a^iH and $a^{-i}H$.

Let $\alpha \in K$ be an automorphism of $\mathcal{A}$ that fixes $a^2 \in T_2$. The elements

$$a^3,\ a,\ a^{3^{n-2}+3}b^2,\ a^{2\cdot 3^{n-2}+3}b,\ ab^2,\ ab$$

are adjacent to a^2 in $R(Z_4)$. The element a is the only one of them from Z_4, the element a^3 is the only one of them from X_1. Therefore a and a^3 are fixed by α.

Further, we prove that α fixes a^i for every $i = 1, \ldots, 3^{n-1} - 1$. We proceed by induction on i. Suppose that α fixes $a^j, j \leq i,$. The elements

$$a^{i+1},\ a^{i-1},\ a^{k_1}b^{l_1},\ a^{k_2}b^{l_2},\ a^{k_3}b^{l_3},\ a^{k_4}b^{l_4},\ l_m \neq 0,\ m = 1, \ldots, 4$$

are adjacent to a^i in $R(Z_4)$. They are permuted by α because α fixes a^i. The elements

$$a^{i+1},\ a^{i+1+3^{n-2}},\ a^{i+1+2\cdot 3^{n-2}},\ a^{i-5+3^{n-2}},\ a^{i-5+2\cdot 3^{n-2}},\ a^{i-5}$$

are adjacent to a^{i-2} in $R(X_1)$. They are also permuted by α because α fixes a^{i-2}. However, α can not take a^{i+1} into a^{i-1} under action of α because α fixes a^{i-1}. Therefore

$$\left(a^{i+1}\right)^\alpha \in C \cap D,$$

where

$$C = \{a^{i+1},\ a^{k_1}b^{l_1},\ a^{k_2}b^{l_2},\ a^{k_3}b^{l_3},\ a^{k_4}b^{l_4},\ l_m \neq 0,\ m = 1, \ldots, 4\},$$

$$D = \{a^{i+1},\ a^{i+1+3^{n-2}},\ a^{i+1+2\cdot 3^{n-2}},\ a^{i-5+3^{n-2}},\ a^{i-5+2\cdot 3^{n-2}},\ a^{i-5}\}.$$

Since $|C \cap D| = 1$, we have $\left(a^{i+1}\right)^\alpha = a^{i+1}$.

The element α can act non-trivially only by permuting a^ib and a^ib^2 for some i because the left cosets of B are the blocks of K. Therefore α fixes each left coset of H as a set (they are the blocks of K) and α is not of type (4) because it fixes the elements $a^{3^{n-2}}$ and $a^{2\cdot 3^{n-2}}$. Therefore α is of type (1). Thus α acts on H trivially, in particular, it fixes the elements b and b^2. Note that α acts trivially on Z_4 and X_k.

The element α can act non-trivially on Z_5 only by permuting ab and ab^2 or $a^{-1}b$ and $a^{-1}b^2$. However, ab and $a^{-1}b$ are adjacent to b in $R(Z_4)$ when the elements ab^2 and $a^{-1}b^2$ are not adjacent to b in $R(Z_4)$. Therefore α acts trivially on Z_5. The elements of the form $a^{3l}b$ from Y_k are adjacent to b in $R(X_k)$, the elements of the form $a^{3l}b^2$ from Y_k are not adjacent to b in $R(X_k)$. Therefore α acts trivially on Y_k.

Two elements from T_i, which can be permuted by α, must be of the form a^ib and a^ib^2, where i is not a multiple of 3. If $i = 3k + 1$, then the element a^ib is adjacent in $R(Z_4)$ to the element $a^{3k}b$, which is fixed by α, the element a^ib^2 is not adjacent to the element $a^{3k}b$ in $R(Z_4)$. The case when $i = 3k - 1$ is similar. This means that α acts on T_i trivially.

Thus, $\alpha = 1$ and the stabilizer of a^2 in K is trivial. Then $|K| = |K_{a^2}| \cdot |a^2 K| = 18$. We have a contradiction because the lengths of the orbits Z_5 and Y_k of K equal to 12 do not divide the order of K equal to 18.

□

The main theorem follows from Proposition 1 and Proposition 2.

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