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DISTRIBUTIONS OF COUNTABLE MODELS OF THEORIES WITH CONTINUUM MANY TYPES

R. A. POPKOV, S. V. SUDOPLATOV

Abstract. We present distributions of countable models and corresponding structural characteristics of complete theories with continuum many types: for prime models over finite sets relative to Rudin–Keisler preorders, for limit models over types and over sequences of types, and for other countable models of theory.

Keywords: countable model, theory with continuum many types, Rudin—Keisler preorder, prime model, limit model, premodel set.

A classification of models of theories is one of main goals in the modern model theory. For uncountable models, the basic achievements are connected with results by S. Shelah [21] and finally represented in the article by B. Hart, E. Hrushovski, and M. S. Laskowski [9].

The class of countable models has been widely investigated by many specialists counting the number of countable models (see for example, [2], [11], [13], [14], [17], [18], [19], [28]), finding basic properties related to countable models ([1], [12], [15], [22], [25], [27]) and countable models with desired model theoretic and computability properties ([7], [16], [23], [29]).

An approach to the classification of countable models using basic links between countable models has been proposed in [22], [23], [24], [25]. However, it was restricted to the class of small theories. In this paper, this approach is extended to the class of countable theories with continuum many types. We describe *distributions* of countable models of theories T, i. e., possibilities for numbers of prime models over finite sets relative Rudin–Keisler preorders, for numbers of limit

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models over types and over sequences of types, and for numbers of other countable models of theory. In particular, every theory T has three characteristics: the number $P(T)$ of prime models over finite sets, the number $L(T)$ of limit models, and the number $\mathrm{NPL}(T)$ of other countable models refining the number $I(T,\omega)$ of pairwise non-isomorphic countable models of T. It allows to classify countable theories with respect to these characteristics. As it was shown in [22], [23], [24], [25], for small theories T, $\mathrm{NPL}(T) = 0$ and using Morley's theorem all possibilities for $(P(T), L(T))$ are described. For theories T with continuum many types, $\mathrm{NPL}(T)$ can vary from 0 to 2^ω. Under the assumption of Continuum Hypothesis we describe all possibilities for triples $(P(T), L(T), \mathrm{NPL}(T))$.

Denote by $\mathcal{T}_c$ the class of all countable complete theories T with sets $S(T)$ having continuum many types. Below, unless otherwise stated, we shall assume that all theories under consideration belong to the class $\mathcal{T}_c$ and these theories will be called *non-small* or *theories with continuum many types*.

In general case for theories in $\mathcal{T}_c$, there is no correspondence between types and prime models over tuples which we observe for small theories (for a given theory in $\mathcal{T}_c$, some prime models over realizations of types may not exist). Besides, there are continuum many pairwise non-isomorphic countable models for each of these theories. However, as we shall show, in this case the structural links for types allow to distribute and to count the number of prime models over finite sets, limit models, and other countable models of a theory similar to small theories [25] and arbitrary countable theories of unary predicates [20].

Now we outline the content of this paper. In Section 1 we describe some basic examples of theories with continuum many types. In Section 2 we recall Rudin–Keisler preorders and formulate some basic properties and examples related to these preorders. The notion of a premodel set compiling basic properties of Rudin–Keisler preorders for types is presented in Section 3. In Section 4 we prove a criterion for a Rudin–Keisler sequence of types forming a countable model (Theorem 4.1) and consider distributions of countable models with respect to these sequences. In Section 5 we define three classes: **P**, **L**, and **NPL**, of prime over tuples, limit, and others countable models, respectively. We describe the possibilities for the numbers of models in this classes assuming the Continuum Hypothesis and the smallness of the theory (Theorem 5.2); we prove a criterion, for the class $\mathcal{T}_c$, that each countable model is either prime over a tuple or limit (Theorem 5.3, Corollary 5.4); and establish some relations between the numbers of countable models in the classes **P**, **L**, and **NPL** (Propositions 5.5, 5.6). In Section 6 we define operators, used below for realizations of possible distributions of countable models. In Sections 7 and 8 we describe distributions of prime and limit models for finite and countable Rudin–Keisler preorders (Theorem 7.5, Proposition 7.6, Theorems 7.7, 8.3, 8.4). In Section 9 we describe links between the classes **P**, **L**, and **NPL** (Theorems 9.1, 9.2), and possible numbers of pairwise non-isomorphic countable models in these classes under the assumption of Continuum Hypothesis (Theorem 9.4).

1. Examples

Recall some basic examples of theories with continuum many types ([5], [6], [10]):

(1) the theory $\mathrm{Th}(\langle \mathbb{N}; +, \cdot \rangle)$ of the standard model of arithmetic on naturals (for any subset A of the set P of all prime numbers, the set $\Phi(x)$ of formulas describing

the divisibility of an element by a number in A and its non-divisibility by each number in $P \setminus A$ is consistent);

(2) the theory $\mathrm{Th}(\langle\mathbb{Z};+,0\rangle)$ (there are continuum many 1-types by the same reason as in the previous example);

(3) a theory $\mathrm{Th}(\langle\mathbb{Q};+,\cdot,\leq\rangle)$ of ordered fields (there are 2^ω cuts for the set of rationals);

(4) the theory T_{sdup} of a countable set of *sequentially divisible unary predicates* $S_\delta^{(1)}$, $\delta \in 2^{<\omega}$, with the following axioms:

$$\exists^{\geqslant\omega} x\,(S_\delta(x) \wedge \neg S_{\delta\hat{\ }0}(x) \wedge \neg S_{\delta\hat{\ }1}(x));$$
$$S_{\delta\hat{\ }\varepsilon}(x) \to S_\delta(x),\ \varepsilon \in \{0,1\};$$
$$\neg\exists x(S_{\delta\hat{\ }0}(x) \wedge S_{\delta\hat{\ }1}(x));$$

(5) the theory T_{iup} of a countable set of *independent unary predicates* $P_k^{(1)}$, $k \in \omega$, axiomatizable by formulas:

$$\exists x\,(P_{i_1}(x) \wedge \ldots \wedge P_{i_m}(x) \wedge \neg P_{j_1}(x) \wedge \ldots \wedge \neg P_{j_n}(x)),$$

$\{i_1,\ldots,i_m\} \cap \{j_1,\ldots,j_n\} = \varnothing$ (one get continuum many 1-types by consistency of any set of formulas $\{P_k^{\delta(k)}(x) \mid k \in \omega\}$, $\delta \in 2^\omega$);

(6) Example suggested by E. A. Palyutin: the theory T_{sipe} (similar to the theory REF_ω [5]) of a countable set of *sequentially independent unary predicates* $P_k^{(1)}$, $k \in \omega$, *with an equivalence relation* $E^{(2)}$, defined by the following conditions:

(a) there are infinitely many E-classes and each E-class is infinite;

(b) for any $k \in \omega$, there is unique E-class X_k containing infinitely many solutions of each formula $P_0^{\delta_0}(x) \wedge \ldots \wedge P_k^{\delta_k}(x)$, $\delta_0,\ldots,\delta_k \in \{0,1\}$, and X_k is disjoint with relations P_i, $i > k$; there is a prime model consisting of E-classes X_k, $k \in \omega$;

one get continuum many 1-types in E-classes having nonempty intersections with each predicate P_k, $k \in \omega$;

(7) the theory T_{sier} (similar to [4, p. 176]) of a countable set of *sequentially independent equivalence relations* $E_n^{(2)}$, $n \in \omega$, with the following conditions:

(a) $\vdash E_{n+1}(x,y) \to E_0(x,y)$, $n \in \omega$;

(b) $\models \forall x,y(E_0(x,y) \to \exists z(E_m(x,z) \wedge E_n(z,y)))$, $m \neq n$;

(c) each E_0-class is infinite and each E_{n+1}-class is a singleton or infinite, $n \in \omega$;

(d) if an E_{n+1}-class X is contained in an E_0-class Y then Y consists of infinitely many E_{n+1}-classes, each of which is a singleton or infinite, $n \in \omega$;

(e) if X_{n+1} is an infinite E_{n+1}-class contained in an E_0-class Y then Y is represented as a union of infinite intersections $X_1 \cap \ldots \cap X_n \cap X_{n+1}$ for E_i-classes X_i, $1 \leq i \leq n$; moreover, for any $\delta_i \in \{0,1\}$ the sets $X_1^{\delta_1} \cap \ldots \cap X_n^{\delta_n} \cap X_{n+1}^{\delta_{n+1}} \cap Y$ are infinite, $n \in \omega$;

(f) for any $n \in \omega$, there is unique E_0-class containing infinite $\bigcap\limits_{i=1}^{n} E_i$-class and singleton E_m-classes, $n < m$; there is a prime model consisting of these E_0-classes;

there are continuum many 2-types in E_0-classes containing infinite E_{n+1}-classes, $n \in \omega$.

The structures $\langle\mathbb{N};+,\cdot\rangle$ and $\langle\mathbb{Q};+,\cdot,\leq\rangle$ are prime (since the universes of there structures equal to $\mathrm{dcl}(\varnothing)$). The structure $\langle\mathbb{Z};+,0\rangle$ is prime over each its nonzero element a (i. e., $\langle \mathbb{Z}; +, 0, a\rangle$ is prime) but it is not prime over $\varnothing$. Moreover, as shown in [3], the theory $\mathrm{Th}(\langle \mathbb{Z}; +, 0\rangle)$ does not have prime models over $\varnothing$.

The theory $T_{\rm sdup}$ has a prime model, this model omits the type $p_\infty(x)$ deduced from the set of formulas describing the unbounded divisibility of $S_\delta(x)$ by $S_{\delta\hat{}\varepsilon}(x)$, and $p_\infty(x)$ has continuum many completions. Besides the theory $T_{\rm sdup}$ has a prime model over every finite set, hence there are continuum many pairwise non-isomorphic prime models over tuples.

The theory $T_{\rm iup}$ does not have prime models over finite sets. The theories $T_{\rm sipe}$ and $T_{\rm sier}$ have prime models over empty set and do not have prime models over non-isolated types.

2. Rudin–Keisler preorders

Consider a theory $T \in \mathcal{T}_c$, a type $p \in S(T)$, and its realization $\bar{a}$. It is known that all prime models over realizations of p are isomorphic. So if there is a *prime model* $\mathcal{M}(\bar{a})$ *over the tuple* $\bar{a}$, this model will be usually denoted by $\mathcal{M}_p$.

Recall [28] that the prime model of T exists if and only if every formula consistent with T belongs to an isolated type.

Note that an expansion of any countable structure $\mathcal{M}$ by constants for each element transforms this structures to a prime one. Hence, the property of absence of a prime model for a theory is not preserved under expansions of a theory. Clearly, this property is not also preserved under restrictions of a theory.

Definition. Let p and q be types in $S(T)$. Following [22], [26], we say that p *is dominated by a type* q, or p *does not exceed* q *under the Rudin–Keisler preorder* (written $p \leq_{\rm RK} q$), if any model $\mathcal{M} \models T$ realizing q realizes p too.

Similar to small theories, the condition $p \leq_{\rm RK} q$ is implied by the following: there is a (q,p)*-formula*, i. e., a formula $\varphi(\bar{x},\bar{y})$ such that the set $q(\bar{y}) \cup \{\varphi(\bar{x},\bar{y})\}$ is consistent and $q(\bar{y}) \cup \{\varphi(\bar{x},\bar{y})\} \vdash p(\bar{x})$. But this syntactic condition may be not implied by $p \leq_{\rm RK} q$ for theories in $\mathcal{T}_c$. If a (q,p)-formula $\varphi(\bar{x},\bar{y})$ exists, we say that $\varphi(\bar{x},\bar{y})$ *witnesses* that q dominates p and denote by $p \leq^{\varphi}_{\rm RK} q$. We write $p \leq'_{\rm RK} q$ if $p \leq^{\varphi}_{\rm RK} q$ for some formula φ.

Notice that in the contrast to small theories, even having a (q,p)-formula, a principal formula $\varphi(\bar{x},\bar{b})$ with the conditions specified, where $\models q(\bar{b})$, may not exist. If a principal formula $\varphi(\bar{x},\bar{b})$ of that form exists, the (q,p)-formula $\varphi(\bar{x},\bar{y})$ is called (q,p)*-principal*.

Definition. If $p \leq_{\rm RK} q$ and the models $\mathcal{M}_p$ and $\mathcal{M}_q$ exist, we say also that $\mathcal{M}_p$ *is dominated by* $\mathcal{M}_q$, or $\mathcal{M}_p$ *does not exceed* $\mathcal{M}_q$ *under the Rudin–Keisler preorder*, and write $\mathcal{M}_p \leq_{\rm RK} \mathcal{M}_q$.

If the models $\mathcal{M}_p$ and $\mathcal{M}_q$ exist, the condition $\mathcal{M}_p \leq_{\rm RK} \mathcal{M}_q$ means that $\mathcal{M}_q \models p$, i. e., some copy $\mathcal{M}'_p$ of $\mathcal{M}_p$ is an elementary submodel of $\mathcal{M}_q$: $\mathcal{M}'_p \preceq \mathcal{M}_q$.

If the model $\mathcal{M}_q$ exists then the condition $p \leq_{\rm RK} q$ implies $p \leq'_{\rm RK} q$ and it is witnessed by some (q,p)-principal formula. At the same time, there is a theory T with types p and q such that $p \leq_{\rm RK} q$, there is a (q,p)-principal formula, and the model $\mathcal{M}_q$ does not exist (it suffices to take the theory $T_{\rm iup}$ and 1-types p and q with $p = q$).

Obviously, no formula $\varphi(\bar{x},\bar{y})$ can be both a (q,p)-formula and a (q,p')-formula for $p \neq p'$. At the same time, a fixed formula can be a (q,p)-formula even for continuum many types q. For instance, any principal formula $\varphi(\bar{x})$ witnesses that corresponding principal type by all types of a given theory. The following example illustrates the mechanism of domination for a type by continuum many types in structures different from the above.

Example 2.1. Consider a structure with countable disjoint unary predicates R_0 and R_1 whose union forms the universe of required structure. Define a coloring Col: $R_0 \cup R_1 \to \omega \cup \{\infty\}$ with infinitely many elements for each color in each predicate R_0, R_1. Thus each predicate R_0, R_1 is divided by disjoint unary predicates Col_n of elements having n in color, $n \in \omega$. Define a bipartite acyclic directed graph with a relation Q connecting parts R_0 and R_1 and satisfying the following conditions:

• every element $a \in R_1$ of color $m \in \omega$ has infinitely many elements $b \in R_0$ of each color $n \geq m$ such that $(a, b) \in Q$ and there are no elements $c \in R_0$ with $(a, c) \in Q$ and $\mathrm{Col}(c) < m$;

• every element $a \in R_0$ of color $m \in \omega$ has infinitely many elements $b \in R_1$ of each color $n \leq m$ such that $(b, a) \in Q$ and there are no elements $c \in R_1$ with $(c, a) \in Q$ and $\mathrm{Col}(c) > m$.

By the construction, for 1-types p_i isolated by sets $\{R_i(x)\} \cup \{\neg\mathrm{Col}_n(x) \mid n \in \omega\}$, $i = 0, 1$, we have $p_0 \leq_{\mathrm{RK}} p_1$ (witnessed by the formula $Q(x, y)$) and $p_1 \not\leq_{\mathrm{RK}} p_0$.

That structure is denoted by $\mathcal{M}_{01}$ and its theory by T_{01}. Expand the structure $\mathcal{M}_{01}$ by independent unary predicates P_k, $k \in \omega$, on each set defined by the formula $R_1(x) \wedge \mathrm{Col}_n(x)$, $n \in \omega$, such that the type p_0 preserves the completeness. Then the type $p_1(x)$ has continuum many completions $q(x)$, each of which dominates the type $p_0(x)$ by the formula $Q(x, y)$.

A modification of the example with the theory T_{sdup} instead of T_{uip} leads to the theory for which the formula $Q(x, y)$ produces the domination of the model $\mathcal{M}_{p_0}$ to continuum many models $\mathcal{M}_q$, where all types q are completions of the type p_0 in $S^1(T_{01})$.

Definition. Recall that types p and q are said to be *domination-equivalent*, *realization-equivalent*, *Rudin–Keisler equivalent*, or RK-*equivalent* (written $p \sim_{\mathrm{RK}} q$) if $p \leq_{\mathrm{RK}} q$ and $q \leq_{\mathrm{RK}} p$. If $p \sim_{\mathrm{RK}} q$ and the models $\mathcal{M}_p$ and $\mathcal{M}_q$ exist then $\mathcal{M}_p$ and $\mathcal{M}_q$ are also said to be *domination-equivalent*, *Rudin–Keisler equivalent*, or RK-*equivalent* (written $\mathcal{M}_p \sim_{\mathrm{RK}} \mathcal{M}_q$).

As in [27], types p and q are said to be *strongly domination-equivalent*, *strongly realization-equivalent*, *strongly Rudin–Keisler equivalent*, or *strongly* RK-*equivalent* (written $p \equiv_{\mathrm{RK}} q$) if for some realizations $\bar{a}$ and $\bar{b}$ of p and q respectively, both $\mathrm{tp}(\bar{b}/\bar{a})$ and $\mathrm{tp}(\bar{a}/\bar{b})$ are principal. Moreover, If the models $\mathcal{M}_p$ and $\mathcal{M}_q$ exist, they are said to be *strongly domination-equivalent*, *strongly Rudin–Keisler equivalent*, or *strongly* RK-*equivalent* (written $\mathcal{M}_p \equiv_{\mathrm{RK}} \mathcal{M}_q$).

Clearly, domination relations form preorders, (strong) domination-equivalence relations are equivalence relations, and $p \equiv_{\mathrm{RK}} q$ implies $p \sim_{\mathrm{RK}} q$.

If $\mathcal{M}_p$ and $\mathcal{M}_q$ are not domination-equivalent then they are non-isomorphic. Moreover, non-isomorphic models may be found among domination-equivalent ones.

Repeating the proof [26, Proposition 1] we get a syntactic characterization for an isomorphism of models $\mathcal{M}_p$ and $\mathcal{M}_q$. It asserts, as for small theories, that an existence of isomorphism between $\mathcal{M}_p$ and $\mathcal{M}_q$ is equivalent to the strong domination-equivalence of these models.

Proposition 2.2. *For any types $p(\bar{x})$ and $q(\bar{y})$ of a theory T having the models $\mathcal{M}_p$ and $\mathcal{M}_q$, the following conditions are equivalent:*

(1) *the models $\mathcal{M}_p$ and $\mathcal{M}_q$ are isomorphic;*

(2) *the models $\mathcal{M}_p$ and $\mathcal{M}_q$ are strongly domination-equivalent;*

(3) *there exist (p,q)- and (q,p)-principal formulas $\varphi_{p,q}(\bar{y},\bar{x})$ and $\varphi_{q,p}(\bar{x},\bar{y})$ respectively, such that the set*

$$p(\bar{x}) \cup q(\bar{y}) \cup \{\varphi_{p,q}(\bar{y},\bar{x}), \varphi_{q,p}(\bar{x},\bar{y})\}$$

is consistent;

(4) *there exists a (p,q)- and (q,p)-principal formula $\varphi(\bar{x},\bar{y})$, such that the set*

$$p(\bar{x}) \cup q(\bar{y}) \cup \{\varphi(\bar{x},\bar{y})\}$$

is consistent.

Denote by $\mathrm{RK}(T)$ the set $\mathbf{P}$ of isomorphism types of models $\mathcal{M}_p$, $p \in S(T)$, with the relation of domination induced by $\leq_{\mathrm{RK}}$ for models: $\mathrm{RK}(T) = \langle \mathbf{P}; \leq_{\mathrm{RK}} \rangle$. We say that isomorphism types $\mathbf{M}_1, \mathbf{M}_2 \in \mathbf{P}$ are *domination-equivalent* (written $\mathbf{M}_1 \sim_{\mathrm{RK}} \mathbf{M}_2$) if so are their representatives.

We consider also the relations $\leq_{\mathrm{RK}}$ and $\leq'_{\mathrm{RK}}$ being defined on the set $S(T)$ of complete types of a theory T. Denote the structures $\langle S(T); \leq_{\mathrm{RK}} \rangle$ and $\langle S(T); \leq'_{\mathrm{RK}} \rangle$ by $\mathrm{RKT}(T)$ and $\mathrm{RKT}'(T)$ respectively. By the definition, $\mathrm{RKT}'(T)$ is a preordered subset of $\mathrm{RKT}(T)$. Recall that $\mathrm{RKT}(T) = \mathrm{RKT}'(T)$ for small theories T and $\mathrm{RKT}'(T)$ can both coincide with $\mathrm{RKT}(T)$ or $\leq'_{\mathrm{RK}}$ be proper in $\leq_{\mathrm{RK}}$ for theories T in $\mathcal{T}_c$.

Below we investigate properties of the preordered sets $\mathrm{RK}(T)$, $\mathrm{RKT}(T)$, and $\mathrm{RKT}'(T)$ as well as relations between these sets and between arbitrary countable models of a theory with continuum many types.

The following assertion proposes criteria for the existence of the least element in $\mathrm{RK}(T)$.

Theorem 2.3. *For a countable complete theory T, the following conditions are equivalent:*

(1) *the theory T has a prime model;*

(2) *the theory T does not have consistent formulas which not belong to isolated types;*

(3) *the structure* $\mathrm{RKT}(T)$ *has the least $\sim_{\mathrm{RK}}$-class, this class consists of isolated types of T and has a nonempty intersection with any nonempty set*

$$[\varphi(\bar{x})] = \{p(\bar{x}) \in S(T) \mid \varphi(\bar{x}) \in p(\bar{x})\}.$$

Proof. The equivalence (1) $\Leftrightarrow$ (2) forms a criterion for the existence of a prime model of a theory [28]. The implications (1) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2) are obvious. □

Since theories with continuum many types may not have prime models over tuples, the limits models may not exist too. Nevertheless the links between countable models can be observed by the following generalization of Rudin–Keisler preorder on isomorphism types of countable models that will be also denoted by $\leq_{\mathrm{RK}}$. This generalization extends the preorder $\leq_{\mathrm{RK}}$ for isomorphism types of prime models over tuples and is based on the inclusion relation for type diagrams $D(\mathcal{M}) = \{p \in S(\varnothing) \mid \mathcal{M} \models p\}$.

Definition. Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be isomorphism types of models $\mathcal{M}_1$ and $\mathcal{M}_2$ (of T) respectively. We say that $\mathbf{M}_1$ *is dominated by* $\mathbf{M}_2$ and write $\mathbf{M}_1 \leq_{\mathrm{RK}} \mathbf{M}_1$ if each type in $S^1(\varnothing)$, being realized in $\mathbf{M}_1$, is realized in $\mathbf{M}_2$: $D(\mathcal{M}_1) \subseteq D(\mathcal{M}_2)$.

Since the relation $\leq_{\mathrm{RK}}$ does not depend on representatives $\mathcal{M}_1$ and $\mathcal{M}_2$ of isomorphism types $\mathbf{M}_1$ and $\mathbf{M}_2$, we shall also write $\mathcal{M}_1 \leq_{\mathrm{RK}} \mathcal{M}_2$ for the representatives $\mathcal{M}_1$ and $\mathcal{M}_2$ if $\mathbf{M}_1 \leq_{\mathrm{RK}} \mathbf{M}_2$.

We denote by $\mathrm{CM}(T)$ the set $\mathbf{CM}$ of isomorphism types of countable models of T, equipped with the preorder $\leq_{\mathrm{RK}}$ of domination on this set: $\mathrm{CM}(T) = \langle \mathbf{CM}; \leq_{\mathrm{RK}} \rangle$.

Clearly, $\mathrm{RK}(T) \subseteq \mathrm{CM}(T)$. Obviously the equality $\mathrm{RK}(T) = \mathrm{CM}(T)$ is equivalent to ω-categoricity of T.

By the definition, a prime model over a type and a limit model over that type [23], being non-isomorphic, are domination-equivalent. Hence, any two limit models over a common type are also domination-equivalent.

The generalized relation of domination leads to a classification of countable models of an arbitrary theory of unary predicates [20].

As we pointed out, a series of examples shows that, unlike small theories, for theories with continuum many types the relations of domination may not induce least elements (being isomorphism types of prime models). Besides, by the following example, isomorphism types of prime models over tuples can quite freely alternate with other isomorphism types of countable models.

Example 2.4. Consider again a structure with countable disjoint unary predicates R_0 and R_1 whose union forms the universe of the required structure. We define a coloring Col: $R_0 \to \omega \cup \{\infty\}$ with infinitely many elements for each color. On the set R_1, we put a structure of independent unary predicates P_k, $k \in \omega$. We denote by T_0 the complete theory of the described structure.

Now we fix a dense (in the natural topology) set $X = \{q_m \mid m \in \omega\}$ of 1-types containing the formula $R_1(y)$. Using binary predicates Q_m, $m \in \omega$, the type $p_\infty(x)$, being isolated by the set $\{R_0(x) \wedge \neg\mathrm{Col}_n(x) \mid n \in \omega\}$, and neighbourhoods $R_0(x) \wedge \bigwedge\limits_{i=0}^{n} \neg\mathrm{Col}_i(x)$ of $p_\infty(x)$, we get, in the expanded language, that all types in X are approximated so that if the type $p_\infty(x)$ is realized in a model $\mathcal{M}$ of expanded theory, then the type $q_m(y)$ is realized in $\mathcal{M}$ by the principal formula $Q_m(a, y)$, where $\models p_\infty(a)$ and $Q_m(a, y) \vdash q_m(y)$, $m \in \omega$, and the realizability in a model of some types in X does not imply the realizability of $p_\infty(x)$ in that model. Thus, a prime model over p_∞ dominates a prime model over a set A, where A consists of realizations of types in X (one realization of each type).

In turn, the model $\mathcal{M}_{p_\infty}$ is dominated by a countable model (being not prime over tuples) which contains a realization of p_∞ (with realizations of types in X) and a countable set of realizations of 1-types consistent with $R_1(x)$ and not belonging to X.

Using the notion of a dense set of types for the theory T_{iup} (without the predicate R_1) one describes (see [20]) the preordered, with respect to $\leq_{\mathrm{RK}}$, set $\mathbf{M}$ of isomorphism types of countable models of T_{iup}. Each countable model is defined by some countable set of realizations of a dense set. A model $\mathcal{M}_1$ is dominated by a model $\mathcal{M}_2$ if and only if each 1-type p, realized in $\mathcal{M}_1$, is realized in $\mathcal{M}_2$ and the number of realizations of p in $\mathcal{M}_1$ does not exceed the number of realizations of

p in $\mathcal{M}_2$. Since density of a set of types is preserved under arbitrary removing or adding of a 1-type, the set $\mathbf{M}$ does not have minimal and maximal elements.

Example 2.4 illustrates that the absence of a prime model of a theory can be combined with the presence of a prime model over a tuple. At the same time, as the following proposition asserts, if a consistent formula, which does not belong to isolated types, exists then no prime model can be dominated by all countable models of the considered theory.

Proposition 2.5. *For any consistent formula $\varphi(\bar{x})$, which does not belong to isolated types, and for any non-isolated type $p(\bar{y}) \in S(T)$, there is a non-isolated type $q(\bar{x}) \in S(T)$ containing the formula $\varphi(\bar{x})$ and not dominating the type $p(\bar{y})$.*

Proof. By Omitting Type Theorem, there is a countable model $\mathcal{M}$ of T omitting the type $p(\bar{y})$. At the same time, by consistency of $\varphi(\bar{x})$, there is a tuple $\bar{a}$ such that $\mathcal{M} \models \varphi(\bar{a})$. The type $q(\bar{x}) = \mathrm{tp}(\bar{a})$ contains the formula $\varphi(\bar{x})$ and, by the definition, does not dominate the type $p(\bar{y})$. □

Since each consistent conjunction of a formula $\varphi(\bar{x})$, which does not belong to isolated types, and a formula $\psi(\bar{x})$ is again a formula, which does not belong to isolated types, there are infinitely many types $q(\bar{x}) \in S(T)$ containing the formula $\varphi(\bar{x})$ and do not dominating the type $p(\bar{y})$. Moreover, in examples of T like above, there are uncountably many these types since otherwise there is a countable expansion T' of T with new predicates $Q_n(\bar{x}, \bar{y})$, $n \in \omega$, producing the isolation of each type $r(\bar{x}) \in S(T')$, containing $\varphi(\bar{x})$, by its restriction to the language of T, and the domination of $p(\bar{y})$ by each type $q(\bar{x})$. Since the formula $\varphi(\bar{x})$ does not belong to isolated types, we get a contradiction by Proposition 2.5.

Note that if a type $p(\bar{y})$ is not dominated by a type $q(\bar{x})$ then, introducing new independent predicates $P_k(\bar{x})$, $k \in \omega$, transforming a neighbourhood of $q(\bar{x})$ to a formula, which does not belong to isolated types, and $q(\bar{x})$ to 2^ω completions, we get a theory such that $p(\bar{x})$ is not dominated by continuum many types. By a similar way, as in Example 2.1, if a type $p(\bar{y})$ is dominated by a type $q(\bar{x})$ then, in an expansion, the type $p(\bar{y})$ is dominated by continuum many completions of $q(\bar{x})$.

Note also that a structure $\mathrm{RKT}(T)$ can have a minimal but not least $\sim_{\mathrm{RK}}$-class. Indeed, expanding the theory T_{iup} by binary predicates, one can obtain a dense set S of 1-types, each of which is domination-equivalent with the other, and the absence of prime model is preserved (it can be done by a countable set of new binary predicates, each of which is responsible for the domination-equivalence of two 1-types in the given dense set, and this domination-equivalence is obtained by approximations for neighbourhoods of given types). The set S and types, domination-equivalent to types in S, form a minimal $\sim_{\mathrm{RK}}$-class. By similar expansions, one get countably many minimal classes.

Together with Example 2.4 and Proposition 2.5, Example 2.1 illustrates a mechanism of domination of a non-principal type by all non-principal types of a theory with continuum many types and without consistent formulas, which does not belong to isolated types.

Having the features, in the following section, we suggest a list of some basic properties of the structures $\mathrm{RKT}'(T)$ for theories T in $\mathcal{T}_c$ (recall that for the countable structures $\mathrm{RKT}(T)$, we have $\mathrm{RKT}(T) = \mathrm{RKT}'(T)$ and the basic properties: the countable cardinality, the upward direction, the countability of $\sim_{\mathrm{RK}}$-classes, and the existence of least $\sim_{\mathrm{RK}}$-classes are presented in [26]).

3. Premodel sets

Definition. A *height* (*width*) of a preordered set $\langle X;\leq\rangle$ is the supremum of cardinalities for its $\leq$-(anti)chains consisting of pairwise non-$\sim$-equivalent elements, where $\sim\,=(\leq\cap\geq)$. If $a\in X$ then the set $\triangle(a)$ (respectively $\triangledown(a)$) of all elements x in X, for which $x\leq a$ ($a\leq x$), is the *lower* (*upper*) *cone* of a.

A preordered upward directed set $\langle X;\leq\rangle$ with $|X|=2^\omega$ is called *premodel* if it has:

- countably many elements under each element $a\in X$: $|\triangle(a)|=\omega$;
- only countable $\sim$-classes: $|\triangle(a)\cap\triangledown(a)|=\omega$ for any $a\in X$;
- over any elements $a_1,\ldots,a_n\in X$ there are countably many or continuum many common elements; if there are continuum many these common elements then the set of these elements is equal to X or contains co-countably many or co-continuum many elements of X: $|\triangledown(a_1)\cap\ldots\cap\triangledown(a_n)|=\omega$, or $|\triangledown(a_1)\cap\ldots\cap\triangledown(a_n)|=2^\omega$ and $\triangledown(a_1)\cap\ldots\cap\triangledown(a_n)=X$, $|X\setminus(\triangledown(a_1)\cap\ldots\cap\triangledown(a_n))|=\omega$, or
$$|X\setminus(\triangledown(a_1)\cap\ldots\cap\triangledown(a_n))|=2^\omega;$$
- the countable height.

Proposition 3.1. *If* $|S(T)|=2^\omega$ *then the structure* $\mathrm{RKT}'(T)$ *is premodel.*

Proof. The structure $\mathrm{RKT}'(T)$ is upward directed since types $p(\bar{x}),\,q(\bar{y})\in S(T)$, where $\bar{x}$ and $\bar{y}$ are disjoint, are dominated by any type $r(\bar{x},\bar{y})\supset p(\bar{x})\cup q(\bar{y})$ in $S(T)$.

As T is countable, the set of formulas of T is also countable and each type dominates at most countably many types. Having countably many types, being domination-equivalent with a given type (for instance, a type $\mathrm{tp}(\bar{a})$ is domination-equivalent with types $\mathrm{tp}(\bar{a}\hat{\ }\bar{a})$, $\mathrm{tp}(\bar{a}\hat{\ }\bar{a}\hat{\ }\bar{a}),\ldots)$, we get that any type is domination-equivalent with countably many types of T.

Since each formula witnesses domination of a type to at most countably many, or continuum and co-continuum many types, and there are countably many formulas of T, then any types $p_1,\ldots,p_n$ lay under countably many, or continuum many and coinciding with $S(T)$, co-countably many, or co-continuum many types.

As each type dominates countably many types, the height of $\mathrm{RKT}'(T)$ is at most countable. At the same time the height can not be finite since its finiteness, the upward direction of $\mathrm{RKT}'(T)$, and the countable domination imply that $\mathrm{RKT}'(T)$ is countable in spite of $|S(T)|=2^\omega$. □

Since each $\sim_{\mathrm{RK}}$-class of a countable theory T is countable and each type dominates countably many types, the ordered quotient $\mathrm{RKT}'(T)/\sim_{\mathrm{RK}}$ can be linearly ordered only for small T. Moreover, as the height of $\mathrm{RKT}'(T)$ is countable for $T\in\mathcal{T}_c$, this quotient has continuum many incomparable elements, i. e., the width equals to continuum:

Proposition 3.2. *The width of any premodel set* $\langle X;\leq\rangle$ *equals to continuum.*

Proof. Assume that the width of a preordered set $\langle X;\leq\rangle$ is less than continuum. Consider a maximal antichain Y. By the assumption, we have $|Y|=\lambda<2^\omega$. We associate to each element $y\in Y$ a maximal chain C_y. Each chain C_y is countable since the height is countable and each $\sim$-class is countable too. Now we note that $X=\bigcup\{\triangle(c)\mid c\in C_y, y\in Y\}$ since $\langle X;\leq\rangle$ is upward directed. Then, as each lower cone $\triangle(c)$ is countable, we obtain $|X| \leq \lambda \cdot \omega \cdot \omega < 2^\omega$ that contradicts the condition $|X| = 2^\omega$. □

4. Distributions for countable models of a theory by $\leq_{\rm RK}$-sequences

Recall that by Tarski–Vaught criterion, a set A in a structure $\mathcal{M}$ of language L forms an elementary substructure if and only if for any formula $\varphi(x_0, x_1, \dots, x_n)$ of the language L and for any elements $a_1, \dots, a_n \in A$ if $\mathcal{M} \models \exists x_0\, \varphi(x_0, a_1, \dots, a_n)$ then there is an element $a_0 \in A$ such that $\mathcal{M} \models \varphi(a_0, a_1, \dots, a_n)$. It means that each formula $\varphi(\bar{x})$ over a finite set $A_0 \subseteq A$ and belonging to a type over A_0 has a realization $\bar{a} \in A$.

Let $\mathcal{M}$ be a model of a countable theory T and $\mathbf{q} = (q_n)_{n\in\omega}$ be a $\leq_{\rm RK}$-*sequence* of types of T, i. e., a sequence of non-principal types q_n with $q_n \leq_{\rm RK} q_{n+1}$, $n \in \omega$. We denote by $U(\mathcal{M}, \mathbf{q})$ the set of all realizations in $\mathcal{M}$ of types of T dominated by some types in $\mathbf{q}$. The $\leq_{\rm RK}$-sequence $\mathbf{q}$ is called *elementary submodel* if for any consistent formula $\varphi(\bar{y})$ of T, some type in $\mathbf{q}$ dominates a type $p(\bar{y}) \in S(T)$ containing the formula $\varphi(\bar{y})$, and if the formula $\varphi(\bar{y})$ is equal to $\exists x\, \psi(x, \bar{y})$ then the type $p(\bar{y})$ extends to a type $p'(x, \bar{y}) \in S(T)$ dominated by a type in $\mathbf{q}$ and such that $\psi(x, \bar{y}) \in p'$.

Theorem 4.1. *For any ω-homogeneous model $\mathcal{M}$ of a countable theory T and for any $\leq_{\rm RK}$-sequence $\mathbf{q}$ of types in $S(T)$, realized in $\mathcal{M}$, the following conditions are equivalent:*

(1) *some (countable) subset of $U(\mathcal{M}, \mathbf{q})$ is a universe of elementary submodel of $\mathcal{M}$;*

(2) $\mathbf{q}$ *is an elementary submodel $\leq_{\rm RK}$-sequence.*

Proof. (1) ⇒ (2) is implied by Tarski–Vaught criterion.

(2) ⇒ (1). Let $\mathbf{q}$ be an elementary submodel $\leq_{\rm RK}$-sequence. Using elements of $U(\mathcal{M}, \mathbf{q})$ we construct, by induction, a countable elementary submodel of $\mathcal{M}$. On the initial step we enumerate, by natural numbers, all consistent with T formulas $\varphi(x, \bar{y})$ such that the enumeration ν starts with some formula $\varphi_0(x)$ and each formula appears infinitely times. We choose a realization $a_0 \in U(\mathcal{M}, \mathbf{q})$ of the formula $\varphi_0(x)$ and put $A_0 = \{a_0\}$. Assume that, on step n, a finite set $A_n \subset U(\mathcal{M}, \mathbf{q})$ is defined, the type of this set is dominated by some type in $\mathbf{q}$, and all possible tuples of elements in A_n are substituted in initially enumerated formulas $\varphi(x, \bar{y})$ instead of tuples $\bar{y}$ such that there are infinitely many numbers for each formula, where tuples of elements in A_n are not substituted. We assume also that the results $(\varphi(x, \bar{y}))^{\bar{y}}_{\bar{a}}$ of substitutions have the same numbers as before, a substitution is carried out for the formula with the number $n+1$, and this formula has the form $\varphi(x, \bar{a})$. If $\mathcal{M} \models \neg\exists x \varphi(x, \bar{a})$, we put $A_{n+1} = A_n$. If $\mathcal{M} \models \exists x \varphi(x, \bar{a})$, we add fictitiously to the tuple $\bar{a}$ all missing elements of A_n and choose an existing, by conjecture, type $p'(x, \bar{y})$ extending the type $p(\bar{y}) = {\rm tp}(A_n)$, where $\varphi(x, \bar{y}) \in p'$ and the types p, p' are dominated by some types in $\mathbf{q}$. We take for a_{n+1} a realization in $U(\mathcal{M}, \mathbf{q})$ of the type $p'(x, A_n)$ (that exists since the model $\mathcal{M}$ is ω-homogeneous) and put $A_{n+1} = A_n \cup \{a_{n+1}\}$.

It is easy to see, using a mechanism of consistency [8], that $\bigcup\limits_{n\in\omega} A_n$ is a universe of required elementary submodel of $\mathcal{M}$. □

Since every ω-saturated structure is ω-homogeneous, Theorem 4.1 implies

Corollary 4.2. *For any ω-saturated model $\mathcal{M}$ of a countable theory T and for any $\leq_{\rm RK}$-sequence $\mathbf{q}$ of types in $S(T)$, the following conditions are equivalent:*

(1) *some (countable) subset of $U(\mathcal{M},\mathbf{q})$ is a universe of elementary submodel of $\mathcal{M}$;*

(2) $\mathbf{q}$ *is an elementary submodel $\leq_{\rm RK}$-sequence.*

Note that in the proof of Theorem 4.1 we essentially use that the model $\mathcal{M}$ is ω-homogeneous and all types of the sequence $\mathbf{q}$ are realized in $\mathcal{M}$. Possibly the types of a $\leq_{\rm RK}$-sequence $\mathbf{q}$ are not realized in an ω-homogeneous model $\mathcal{M}$ but are realized in some other ω-homogeneous model $\mathcal{M}'$, where Theorem 4.1 can be applied.

Example 4.3. Consider the theory $T_{\rm iup}$. By Theorem 4.1, each countable model of $T_{\rm iup}$ realizes a dense set X of 1-types (where $\bigcup X$ contains all formulas

$$P_{i_1}(x)\wedge\ldots\wedge P_{i_m}(x)\wedge\neg P_{j_1}(x)\wedge\ldots\wedge\neg P_{j_n}(x),$$

$\{i_1,\ldots,i_m\}\cap\{j_1,\ldots,j_n\}=\varnothing$) and vice versa, for each countable dense set X of types, there is an (ω-homogeneous) model of $T_{\rm iup}$ such that the set of types of its elements is equal to X.

Take two countable disjoint dense sets Y_0 and Y_1 of 1-types, and ω-homogeneous models $\mathcal{M}_0$ and $\mathcal{M}_1$ containing exactly one realization of each type in Y_0 and Y_1 respectively. Then there are $\leq_{\rm RK}$-sequences $\mathbf{q}_i$ of types with realizations from given sets of realizations of types in Y_i, $i=0,1$. Here, all types in $\mathbf{q}_i$ are realized in $\mathcal{M}_i$ and are omitted in $\mathcal{M}_{1-i}$, $i=0,1$.

By Theorem 4.1, each elementary submodel $\leq_{\rm RK}$-sequence $\mathbf{q}$ corresponds to some set of isomorphism types of countable models of a theory T, which can vary from 1 to 2^ω. We denote this set by $I^m_{\mathbf{q}}(T)$. Here m denotes the word “maximum” and we take all isomorphism types related to $\mathbf{q}$.

The sets $I^m_{\mathbf{q}}(T)$ can have nonempty intersections (for instance, having a prime model $\mathcal{M}_0$ its isomorphism type belongs to each set $I^m_{\mathbf{q}}(T)$) and can be disjoint (as in Example 4.3).

Distributing isomorphism types of countable model to pairwise disjoint sets, related to $\leq_{\rm RK}$-sequences $\mathbf{q}$ (and not related to the other $\leq_{\rm RK}$-sequences) and denoting the cardinalities of these sets by $I_{\mathbf{q}}$, we have the equality

$$I(T,\omega)=\sum_{\mathbf{q}} I_{\mathbf{q}}=2^\omega.$$

5. Three classes of countable models

Recall [25] that a model $\mathcal{M}$ of a theory T is called *limit* if $\mathcal{M}$ is not prime over tuples and $\mathcal{M}=\bigcup\limits_{n\in\omega}\mathcal{M}_n$ for some elementary chain $(\mathcal{M}_n)_{n\in\omega}$ of prime models of T over tuples. In this case, the model $\mathcal{M}$ is said to be *limit over a sequence* $\mathbf{q}$ *of types*, where $\mathbf{q}=(q_n)_{n\in\omega}$, $\mathcal{M}_n=\mathcal{M}_{q_n}$, $n\in\omega$. If some type p is co-finite in $\mathbf{q}$ then the limit model over $\mathbf{q}$ is said to be *limit over the type p.*

Consider a countable complete theory T. Denote by $\mathbf{P}$ and $\mathbf{P}(T)$, $\mathbf{L}$ and $\mathbf{L}(T)$, and $\mathbf{NPL}$ and $\mathbf{NPL}(T)$, respectively, the set of isomorphism types of prime over tuples, limit, and other countable models of T, and by $P(T)$, $L(T)$, and ${\rm NPL}(T)$ the cardinalities of these sets.

By the definition, each value $P(T)$, $L(T)$, and $\mathrm{NPL}(T)$ may vary from 0 to 2^ω and the following equality holds:

$$I(T,\omega) = P(T) + L(T) + \mathrm{NPL}(T).$$

Since $I(T,\omega) = 2^\omega$ for theories T in $\mathcal{T}_c$, some value $P(T)$, $L(T)$, or $\mathrm{NPL}(T)$ is equal to 2^ω.

The tuple $(P(T), L(T), \mathrm{NPL}(T))$ is called a *triple of distribution of countable models of* T, or a *spectrum triple of countable models of* T, and is denoted by $\mathrm{cm}_3(T)$.

Definition 5.1. A theory T is called *p-zero* (respectively *l-zero*, npl-*zero*) if $P(T) = 0$ ($L(T) = 0$, $\mathrm{NPL}(T) = 0$).

A theory T is called *p-categorical* (respectively *l-categorical*, npl-*categorical*) if $P(T) = 1$ ($L(T) = 1$, $\mathrm{NPL}(T) = 1$).

A theory T is called *p-Ehrenfeucht* (respectively *l-Ehrenfeucht*, npl-*Ehrenfeucht*) if $1 < P(T) < \omega$ ($1 < L(T) < \omega$, $1 < \mathrm{NPL}(T) < \omega$).

A theory T is called *p-countable* (respectively *l-countable*, npl-*countable*) if $P(T) = \omega$ ($L(T) = \omega$, $\mathrm{NPL}(T) = \omega$).

A theory T is called *p-continuum* (respectively *l-continuum*, npl-*continuum*) if $P(T) = 2^\omega$ ($L(T) = 2^\omega$, $\mathrm{NPL}(T) = 2^\omega$).

By the definition, each p-zero theory is l-zero.

Recall [25] that the p-categoricity of a small theory T is equivalent to its countable categoricity as well as to the absence of limit models. The p-Ehrenfeuchtness of T means that the structure $\mathrm{RK}(T)$ is finite and has at least two elements. The theory T is Ehrenfeucht if and only if T is p-Ehrenfeucht and $L(T) < \omega$. Besides, every small theory is npl-zero, i. e., each its countable model is prime over a tuple or is limit. Since by Vaught's and Morley's theorems ([28], [17]), $I(T,\omega) \in (\omega \setminus \{0,2\}) \cup \{\omega, \omega_1, 2^\omega\}$ and for small theories T, the inequalities $1 \le P(T) \le \omega$ hold, we have the following

Theorem 5.2. *For any small theory T, the triple* $\mathrm{cm}_3(T)$ *has one of the following values:*

(1) $(1,0,0)$ (*any p-categorical theory, being ω-categorical, is l-zero and* npl-*zero*)*;*

(2) $(\lambda_1, \lambda_2, 0)$*, where* $2 \le \lambda_1 \le \omega$*,* $\lambda_2 \in (\omega \setminus \{0\}) \cup \{\omega, \omega_1, 2^\omega\}$ (*for non-ω-categorical small theories*).

As shown in [23], [24], [25], all values, pointed out in Theorem 5.2 (for $\lambda_2 \ne \omega_1$), have realizations in the class of small theories.

Similarly to Theorem 5.2, for the classification of theories in the class $\mathcal{T}_c$, the problem arises for the description of all possible triples $(\lambda_1, \lambda_2, \lambda_3)$ realized by $\mathrm{cm}_3(T)$ for theories $T \in \mathcal{T}_c$.

Examples in Section 1 confirm the realizability of triples $(0, 0, 2^\omega)$ and $(2^\omega, 2^\omega, 0)$ in the class $\mathcal{T}_c$ (by the p-zero, npl-continuum theory T_{iup} and the p-continuum, npl-zero theory T_{sdup} respectively). Some fusion of theories T_{iup} and T_{sdup} substantiates the realizability of triple $(2^\omega, 2^\omega, 2^\omega)$. E. A. Palyutin noted that the theory T_{sipe} realizes the triple $(1, 0, 2^\omega)$. This triple is also realized by the theory T_{sier}.

The following theorem produces a characterization for the class of npl-zero theories.

Theorem 5.3. *A countable model $\mathcal{M}$ of a theory $T \in \mathcal{T}_c$ is prime over a finite set or limit if and only if each tuple $\bar{a} \in M$ can be extended to a tuple $\bar{b} \in M$ such that every consistent formula $\varphi(\bar{x}, \bar{b})$ belongs to an isolated type over $\bar{b}$.*

Proof. If for a tuple $\bar{b} \in M$, every consistent formula $\varphi(\bar{x}, \bar{b})$ belongs to an isolated type over $\bar{b}$, then there is a model $\mathcal{M}(\bar{b}) \preccurlyeq \mathcal{M}$. If any tuple $\bar{a}$ can be extended to a tuple $\bar{b}$ of described form then repeating the proof of Proposition 4.1 in [25], we obtain a representation of $\mathcal{M}$ as a union of elementary chain of prime models over finite sets. Thus, $\mathcal{M}$ is prime over a finite set or limit.

If a tuple $\bar{a} \in M$ can not be extended to a tuple $\bar{b} \in M$ such that each consistent formula $\varphi(\bar{x}, \bar{b})$ belongs to an isolated type over $\bar{b}$, then $\bar{a}$ is not contained in prime models over tuples, being elementary submodels of $\mathcal{M}$, whence the model $\mathcal{M}$ is neither prime over a tuple nor limit. □

Theorem 5.3 implies

Corollary 5.4. *A theory $T \in \mathcal{T}_c$ is* npl*-zero if and only if for any* (*countable*) *model $\mathcal{M}$ of T, each tuple $\bar{a} \in M$ can be extended to a tuple $\bar{b} \in M$ such that every consistent formula $\varphi(\bar{x}, \bar{b})$ belongs to an isolated type over $\bar{b}$.*

Below we describe some families of triples $(\lambda_1, \lambda_2, \lambda_3)$ that cannot be realized by $\mathrm{cm}_3(T)$, where $T \in \mathcal{T}_c$.

Proposition 5.5. *There is no theory $T \in \mathcal{T}_c$ such that* $\mathrm{cm}_3(T)$ *has any of the following:*

(1) $(\lambda_1, 2^\omega, \lambda_3)$, *where* $\lambda_1, \lambda_3 < 2^\omega$;

(2) $(2^\omega, \lambda_2, \lambda_3)$, *where* $\lambda_2, \lambda_3 < 2^\omega$.

Proof. (1) If $P(T) < 2^\omega$ and $\mathrm{NPL}(T) < 2^\omega$ then there are less than continuum many types realized in models representing isomorphism types in the classes $\mathbf{P}(T)$ and $\mathbf{NPL}(T)$. Since each type, realized in a limit model, is also realized in a prime model over a tuple, there are continuum many types not realized in countable models of T, that is impossible.

(2) Assume that $\mathrm{NPL}(T) < 2^\omega$. Then there are $< 2^\omega$ types in $S(T)$, over which prime models do not exist. Therefore, for any type $p \in S(T)$ there are continuum many types $q \in S(T)$ extending p and having models $\mathcal{M}_q$. Since there are continuum many types q and the model $\mathcal{M}_p$ is countable, then there are continuum many these non-domination-equivalent types q dominating p and not dominated by p. Hence, for any model $\mathcal{M}_p$ there are continuum many possibilities for elementary extensions by pairwise non-isomorphic models $\mathcal{M}_q$ non-isomorphic to $\mathcal{M}_p$. Since the process of extension of models $\mathcal{M}_p$ by continuum many models $\mathcal{M}_q$ can be continued unboundedly many times, there are continuum many pairwise non-isomorphic limit models, i. e., $L(T) = 2^\omega$. □

The following proposition gives a sufficient condition for the existence of continuum many prime models over finite sets under the assumption of existence of uncountably many models.

Proposition 5.6. *Assume there are uncountably many types $p(\bar{x})$ of a theory $T \in \mathcal{T}_c$ such that for each consistent formula $\varphi(\bar{a}, \bar{y})$, $\models p(\bar{a})$, there is a principal formula $\psi(\bar{a}, \bar{y})$ with $\psi(\bar{a}, \bar{y}) \vdash \varphi(\bar{a}, \bar{y})$ and this formula can be chosen independently of the types p. Then $P(T) = 2^\omega$.*

Proof. Since there are uncountably many types $p(\bar{x})$, we have neighbourhoods $\chi_\delta(\bar{x})$ of these types, $\delta \in 2^{<\omega}$, each belonging to uncountably many given types $p(\bar{x})$ and satisfies the following conditions:

- $\chi_\delta(\bar{x}) \equiv (\chi_{\delta\hat{\ }0}(\bar{x}) \vee \chi_{\delta\hat{\ }1}(\bar{x}))$;
- $\models \neg\exists\bar{x}(\chi_{\delta\hat{\ }0}(\bar{x}) \wedge \chi_{\delta\hat{\ }1}(\bar{x}))$.

For each sequence $\delta \in 2^\omega$, the local consistency implies the consistency of the set $\Phi_\delta(\bar{x})$ of formulas $\chi_{\delta\restriction n}(\bar{x})$, $n \in \omega$. Hence there are continuum many types in $S^{l(\bar{x})}(\varnothing)$. Moreover, since the formulas ψ can be chosen independently of realizations of types p, by compactness each set $\Phi_\delta(\bar{x})$ has a completion $q(\bar{x}) \in S(\varnothing)$ such that for any consistent formula $\varphi(\bar{a}, \bar{y})$, $\models q(\bar{a})$, there is $\psi(\bar{a}, \bar{y})$ with $\psi(\bar{a}, \bar{y}) \vdash \varphi(\bar{a}, \bar{y})$ and this formula does not depend on q. Thus, there is a model $\mathcal{M}_q$ and $P(T) = 2^\omega$. □

By Proposition 5.6, we have a partial solution of a variant of the Vaught's problem proposed by E. A. Palyutin as the implication $P(T) > \omega \Rightarrow P(T) = 2^\omega$. Namely, this implication is true for prime models over realizations of types p having the specified, as in the proposition, *uniform choice property* of formulas ψ by formulas φ.

6. Operators acting on a class of structures

Consider a non-principal 1-type $p_\infty(x)$ and formulas $\varphi_n(x) \in p_\infty(x)$, $n \in \omega$, such that $\varphi_0(x) = (x \approx x)$, $\vdash \varphi_{n+1}(x) \to \varphi_n(x)$, $\{\varphi_n(x) \mid n \in \omega\} \vdash p_\infty(x)$. The formula $\mathrm{Col}_n(x) = \varphi_n(x) \wedge \neg\varphi_{n+1}(x)$ is the *n-th approximation* of $p_\infty(x)$, or the *n-th color*. Then the type $p_\infty(x)$ is isolated by the set $\{\neg\mathrm{Col}_n \mid n \in \omega\}$ of formulas.

The *operator of continuum partition* $\mathfrak{c}\text{-Partition}(\mathcal{A}, \mathcal{A}_0, Y, \{R_i^{(2)}\}_{i\in\omega})$ (a partition of a set into continuum many disjoint sets providing the absence of prime model over a type $p_\infty(x)$) takes for its input:

(1) a predicate structure $\mathcal{A}$;

(2) a substructure $\mathcal{A}_0 \subset \mathcal{A}$, where its universe is equal to an infinite set for solutions of a formula $\psi(x)$ in $\mathcal{A}$, the substructure generates unique non-principal 1-type $p_\infty(x) \in S(\varnothing)$ and $p_\infty(x)$ is realized in $\mathcal{A}_0$;

(3) an infinite set Y with $Y \cap A = \varnothing$;

(4) a sequence $(R_i^{(2)})_{i\in\omega}$ of binary predicate symbols.

We suppose that A_0 is the domain of predicates R_i, Y is their range,

$$\vdash R_i(x, y) \to R_0(x, y), \quad i > 0.$$

The action of the operator is defined by the following schemes of formulas:

(1) $\forall x\, \exists^\infty y(\mathrm{Col}_0(x) \to R_0(x, y))$;

(2) $\forall x, x'\, (\neg(x \approx x') \to \neg\exists y(R_0(x, y) \wedge R_0(x', y)))$, i. e., R_0-images of distinct element satisfying $\psi(x)$ are disjoint and an equivalence relation on Y with infinitely many infinite classes is refined by the formula $R_0(x, y)$;

(3) $\forall x(\mathrm{Col}_n(x) \to \exists^\infty y(R_0(x, y) \wedge \bigwedge\limits_{i=1}^{n} R_i^{\delta_i}(x, y)) \wedge \neg\exists z \bigvee\limits_{i>n} R_i(x, z))$ for all possible binary tuples $(\delta_1, \ldots, \delta_n)$, i. e., for any element $a \in A_0$ of color n, the set of solutions for the formula $R_0(a, y)$ is divided, by $R_n(x, y)$, into 2^n disjoint sets, each of which is infinite.

Thus, the set of solutions for the formula $R_0(a, y)$, where $a \models p_\infty(x)$, is divided by $R_n(x, y)$ into continuum many disjoint sets similar to the example with unary independent predicates. For output of the operator, we obtain a structure $\mathcal{B}$ with continuum many non-principal types $\{R_i^{\delta_i}(a,y) \mid i \in \omega \setminus \{0\}\}$, and there are no prime models over the type $p_\infty(x)$.

The *operator of allocation for a countable subset* ω-$\mathsf{Alloc}(\mathcal{A}, \mathbf{q}_\omega, \mathcal{A}_0, \{R_j^{(2)}\}_{j\in\omega})$ (which among all types selects a countable family of types, connected in a natural way, providing the existence of a prime model, over a finite set, which realizes the chosen family of types) takes for its input:

(1) a predicate structure $\mathcal{A}$ with a continuum-set $\mathbf{q}$ of non-principal 1-types;

(2) a countable subset $\mathbf{q}_\omega \subset \mathbf{q}$;

(3) a substructure $\mathcal{A}_0 \subset \mathcal{A}$ with unique non-principal 1-type $p_\infty(x) \in S(\varnothing)$ and such that $p_\infty(x)$ is realized in $\mathcal{A}_0$;

(4) a sequence $(R_j^{(2)})_{j\in\omega}$ of binary predicate symbols.

Denote by $\mathrm{Col}_{ij}(x)$ approximations of types $q_j(x) \in \mathbf{q}_\omega$, $j \in \omega$. Then the type q_j is isolated by the set of formulas $\{\neg\mathrm{Col}_{ij}(x) \mid i \in \omega\}$. At the operator's action, we assume that A_0 is the domain of predicates R_{ij} and their range contains the set of realizations for types in $\mathbf{q}_\omega$. The action of the operator is defined by the following schemes of formulas:

(1) $\forall x(\mathrm{Col}_i(x) \to \bigwedge\limits_{k\geq i} \exists^\infty y(R_j(x,y) \wedge \mathrm{Col}_{kj}(y)) \wedge \bigwedge\limits_{k<i} \neg\exists y(R_j(x,y) \wedge \mathrm{Col}_{kj}(y)))$,
i. e., for any element $a \in A_0$ of the i-th color, there are infinitely many images of each color k, $k \geq i$, and there are no images of colors k, $k < i$;

(2) $\forall x, x'(\neg(x \approx x') \to \neg\exists y(R_j(x,y) \wedge R_j(x',y)))$, i. e., images of distinct elements belonging to A_0 are disjoint.

If the continuum-set $\mathbf{q}$ of non-principal types is obtained by the operator $\mathfrak{c}$-$\mathsf{Partition}$ (and there are no prime models over each type in $\mathbf{q}$) then after passing all colors Col by all predicates R_j, the countable subset $\mathbf{q}_\omega$ is selected and, using a generic construction for a structure with required properties, there exists a prime model $\mathcal{M}_{p_\infty}$ over a realization of p_∞ and realizing exactly all types in $\mathbf{q}_\omega$. If $\mathbf{q}_\omega$ is dense in $\mathbf{q}$ with respect to natural topology then, assuming that types in $\mathbf{q}_\omega$ are free (are not connected with $a \in A_0$), we can remove elements in $\mathbf{q}_\omega$ and obtain new prime model $\mathcal{M}_{p_\infty,\tilde{\mathbf{q}}_\omega}$, $\tilde{\mathbf{q}}_\omega \subset \mathbf{q}_\omega$, an elementary submodel of $\mathcal{M}_{p_\infty,\mathbf{q}_\omega}$. But having links of the dense set $\mathbf{q}_\omega$ with the type p_∞ by predicates, the removing of a type in $\mathbf{q}_\omega$ leads to the removing of p_∞. Hence applying the operator ω-Alloc with input parameters, satisfying the conditions above, there are no other (non-isomorphic) prime models being an elementary submodel of $\mathcal{M}_{p_\infty,\mathbf{q}_\omega}$. Thus if we focus on this property, the given operator is called the *operator of ban for downward movement* and it is denoted by $\mathsf{banDown}$ with the same input parameters.

The *operator of ban for upward movement* $\mathsf{banUp}(\mathcal{A}, \mathcal{A}_1, \mathcal{A}_2, Z, \{R_n^{(3)}\}_{n\in\omega})$ (which produces the absence of prime model over a type of a pair) takes for its input:

(1) a predicate structure $\mathcal{A}$;

(2) two disjoint substructures $\mathcal{A}_1$ and $\mathcal{A}_2$ of $\mathcal{A}$ with unique non-principal 1-types p_1 and p_2, being realized in $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively;

(3) an infinite set Z such that $A_1 \cap Z = \varnothing$ and $A_2 \cap Z = \varnothing$;

(4) a sequence $(R_n^{(3)})_{n\in\omega}$ of ternary predicate symbols.

We denote approximations of p_1 and p_2 by Col_{i1} and Col_{i2}, $i \in \omega$, respectively. The set $A_1 \times A_2$ is the domain of predicates R_n, and Z is their range, $\vdash R_n(x,y,z) \to$

$R_0(x,y,z)$, $i>0$. The action of the operator is defined by the following schemes of formulas:

(1) $\forall x,y(\mathrm{Col}_{01}(x)\wedge\mathrm{Col}_{02}(y)\to\exists^\infty zR_0(x,y,z))$;

(2) $\forall x,y,x',y'(\neg(x\approx x')\wedge\neg(y\approx y')\to\neg\exists z(R_n(x,y,z)\wedge R_n(x',y',z)))$, i. e., R_n-images of distinct pairs $(a_1,a_2)\in A_1\times A_2$ are disjoint and the set Z is divided into infinitely many infinite equivalence classes;

(3) $\forall x,y(\mathrm{Col}_{k1}(x)\wedge\mathrm{Col}_{n2}(y)\to$

$$\to\exists^\infty z(R_0(x,y,z)\wedge\bigwedge_{1\le i\le\min(k,n)}R_i^{\delta_i}(x,y,z))\wedge\neg\exists z\bigvee_{i>\min(k,n)}R_i(x,y,z))$$

for all possible binary tuples $(\delta_1,\dots,\delta_{\min(k,n)})$.

Hence, if a pair (a_1,a_2) has the (∞,∞)-color, the set of solutions for the formula $R_0(a_1,a_2,z)$ is divided on continuum many parts. Thus, there is a prime model over each realization of $p_1(x)$ and of $p_2(y)$, but there are no prime models over types $q(x,y)\supset p_1(x)\cup p_2(y)$.

The *operator for construction of limit models over a type*, $\mathsf{typeLim}(p,\lambda,\{R_i^{(2)}\}_{i\in\omega})$ (which produces a limit model over given type) takes for its input:

(1) a non-principal 1-type $p(x)$;

(2) a number $\lambda\in\omega+1$ of limit models over $p(x)$;

(3) a sequence $(R_i^{(2)})_{i\in\omega}$ of binary predicate symbols.

We suppose that predicates R_i act on a set of realizations of $p(x)$ so that $R_i(a,y)\vdash p(y)$ and $\models\exists yR_i(a,y)$ and realizations of $R_i(a,y)$ do not semi-isolate a, where $a\models p(x)$. We construct a tree of R_i-extensions over a realization a_0 of p. Consider sequences $i_0,\dots,i_n,\dots\in 2^\omega$ corresponding to paths defined by formulas $R_{i_0}(a_1,a_0)\wedge\dots\wedge R_{i_n}(a_{n+1},a_n)\wedge\dots$. There are 2^ω extensions. As shown in [23], [24], given finite or countable number λ of limit models can be obtained by some family of identities (see [23], [24] for details).

If $\lambda=n\in\omega\setminus\{0\}$ we use the following identities:

(1) $n-1\approx m$, $m\ge n$;

(2) $mm\approx m$, $m<n$;

(3) $n_1n_2\dots n_s\approx n_s$, $\min\{n_1,n_2,\dots,n_{s-1}\}>n_s$.

If $\lambda=\omega$ we introduce identities:

(1) $nn\approx n$, $n\in\omega$;

(2) $n_1n_2\dots n_s\approx n_s$, $\min\{n_1,n_2,\dots,n_{s-1}\}>n_s$;

(3) $n_1n_2\approx n_1(n_1+1)(n_2+2)\dots(n_2-1)n_2$, $n_1<n_2$.

The *operator for construction of limit models over a $\le_{RK}$-sequence*

$$\mathsf{seqLim}((q_n)_{n\in\omega},\lambda,\{R_i^{(2)}\}_{i\in\omega})$$

(which produces a limit model over a RK-sequence of types) takes for its input:

(1) a $\le_{RK}$-sequence $(q_n)_{n\in\omega}$;

(2) a number $\lambda\in\omega+1$ of limit models over the sequence $(q_n)_{n\in\omega}$;

(3) a sequence $(R_i^{(2)})_{i\in\omega}$ of binary predicate symbols.

Consider types q_n and q_{n+1}. Since they belong to the $\le_{RK}$-sequence, there is a formula $\varphi(x,y)$ such that $q_{n+1}(y)\cup\{\varphi(x,y)\}$ is consistent and $q_{n+1}(y)\cup\{\varphi(x,y)\}\vdash q_n(x)$. We assume that predicates R_i act so that $R_i(x,y)\vdash\varphi(x,y)$ and for every $a\models q_{n+1}(y)$, $R_i(x,a)\vdash q_n(x)$. Below we consider numbers i instead of predicates

R_i. Then for the $\leq_{RK}$-sequence $(q_n)_{n\in\omega}$, there are ω^ω sequences $s = i_1, \ldots, i_n, \ldots$ corresponding to $R_{i_1}(a_1^s, a_0^s) \wedge \ldots \wedge R_{i_n}(a_n^s, a_{n-1}^s) \wedge \ldots$, where $a_n^s \models q_n(x)$, $n \in \omega$.

By the sequence $(q_n)_{n\in\omega}$, we construct sequences of prime models $\mathcal{M}_{q_n}$ over realizations of q_n, where $(n+1)$-th model is an elementary extension of n-th one. Any limit model is a union of countable chain of a sequence of prime models over tuples. Predicates R_i, $i \in \omega$, connect realizations of types in (q_n) and produce required number of limit models. As shown in [25], the problem of extension of a theory producing a given number of limit models over (q_n) is reduced to a quotient of the set ω^ω by an identification of some words such that the result of this factorization contains as many classes as there are limit models.

For $n \in \omega \setminus \{0\}$ limit models, we use the following identities (as in [25]):

(1) $n - 1 \approx m$, $m \geq n$;

(2) $n_0 n_1 \ldots n_s \approx \underbrace{n_s \ldots n_s}_{s+1 \text{ times}}$, $\max\{n_0, n_1, \ldots, n_{s-1}\} < n_s$.

For countably many limit models, we take identities:

(1) $n_0 n_1 \ldots n_s \approx \underbrace{n_s \ldots n_s}_{s+1 \text{ times}}$, $\max\{n_0, n_1, \ldots, n_{s-1}\} < n_s$;

(2) $n_0 n_1 \ldots n_s \approx n_0(n_0 + 1) \ldots (n_0 + s)$, $n_0 + s \leq n_s$;

(3) $n_0 n_1 \ldots n_s \approx n_0(n_0 + 1) \ldots (n_0 + t) \underbrace{(n_0 + t) \ldots (n_0 + t)}_{s-t \text{ times}}$, $n_0 + s$, $n_0 + t = n_s$, $t > 0$, $s > t$.

7. Distributions of prime and limit models for finite Rudin–Keisler preorders

If $\widetilde{\mathbf{M}}$ is a $\sim_{RK}$-class containing an isomorphism type $\mathbf{M}$ of a prime model over a tuple, then as usual we denote by $\mathrm{IL}(\widetilde{\mathbf{M}})$ the number of limit models, being unions of elementary chains of models, whose isomorphism types belong to the class $\widetilde{\mathbf{M}}$.

Clearly, for theories T with finite structures $\mathrm{RK}(T)$, any limit model is limit over a type.

The following two theorems show that for p-Ehrenfeucht small theories, the number of countable models is defined by the number of prime models over tuples and by the distribution function IL of numbers of limit models over types. Assuming the Continuum Hypothesis, all possible basic characteristics are realized.

Theorem 7.1 ([22], [24]). *Any small theory T with a finite Rudin–Keisler preorder satisfies the following conditions:*

(a) $\mathrm{RK}(T)$ *contains a least element* $\mathbf{M}_0$ (*the isomorphism type of a prime model*), *and* $\mathrm{IL}(\widetilde{\mathbf{M}_0}) = 0$;

(b) $\mathrm{RK}(T)$ *contains the greatest* $\sim_{\mathrm{RK}}$-*class* $\widetilde{\mathbf{M}_1}$ (*the class of isomorphism types of all prime models over realizations of powerful types*), *and* $|\mathrm{RK}(T)| > 1$ *implies* $\mathrm{IL}(\widetilde{\mathbf{M}_1}) \geq 1$;

(c) *if* $|\widetilde{\mathbf{M}}| > 1$, *then* $\mathrm{IL}(\widetilde{\mathbf{M}}) \geq 1$.

Moreover, we have the following decomposition formula:

$$I(T, \omega) = |\mathrm{RK}(T)| + \sum_{i=0}^{|\mathrm{RK}(T)/\sim_{\mathrm{RK}}|-1} \mathrm{IL}(\widetilde{\mathbf{M}_i}),$$

where $\widetilde{\mathbf{M}_0},\ldots,\widetilde{\mathbf{M}_{|\mathrm{RK}(T)/\sim_{\mathrm{RK}}|-1}}$ *are all elements of the partially ordered set* $\mathrm{RK}(T)/\sim_{\mathrm{RK}}$ *and* $\mathrm{IL}(\widetilde{\mathbf{M}_i})\in\omega\cup\{\omega,\omega_1,2^\omega\}$ *for each* i*.*

Theorem 7.2 ([23], [24]). *For any finite preordered set* $\langle X;\leq\rangle$ *with a least element* x_0 *and the greatest class* $\widetilde{x_1}$ *in the ordered factor set* $\langle X;\leq\rangle/\sim$ *with respect to* $\sim$ *(where* $x\sim y\Leftrightarrow x\leq y$ *and* $y\leq x$*), and for any function* $f\colon X/\sim\,\to\omega\cup\{\omega,2^\omega\}$*, satisfying the conditions* $f(\widetilde{x_0})=0$*,* $f(\widetilde{x_1})>0$ *for* $|X|>1$*, and* $f(\widetilde{y})>0$ *for* $|\widetilde{y}|>1$*, there exist a small theory* T *and an isomorphism* $g\colon\langle X;\leq\rangle\,\widetilde{\rightarrow}\,\mathrm{RK}(T)$ *such that* $\mathrm{IL}(g(\widetilde{y}))=f(\widetilde{y})$ *for any* $\widetilde{y}\in X/\sim$*.*

Note that by criterion of existence of a prime model, a theory T with continuum many types is p-categorical if and only if there is a unique $\equiv_{\mathrm{RK}}$-class $S\subset S(T)$ such that for any realization $\bar a$ of some (any) type in S every consistent formula $\varphi(\bar x,\bar a)$ is implied by an isolated formula with parameters $\bar a$.

Similarly, a theory T with continuum many types is p-Ehrenfeucht if and only if there are finitely many pairwise non-$\equiv_{\mathrm{RK}}$-equivalent types p_j, $j<n$, $1<n<\omega$, such that for any j and for some (any) realization $\bar a_j$ of p_j every consistent formula $\varphi(\bar x,\bar a_j)$ is implied by an isolated formula with parameters $\bar a$.

The proofs of the following assertions are identical to corresponding proofs for the class of small theories ([1], [22].

Proposition 7.3. *If* $\mathcal{M}_p$ *and* $\mathcal{M}_q$ *are domination-equivalent non-isomorphic models then there exists a model that is limit over the type* p *and a model that is limit over the type* q*.*

Proposition 7.4. *If types* p *and* q *are domination-equivalent, and there exist a limit model over* p *and a prime model over* q*, then there exists a model that is limit both over* p *and over* q*.*

Theorem 7.5. *Let* $p(\bar x)$ *be a complete type of a countable theory* T*. The following conditions are equivalent:*

(1) *there exists a limit model over* p*;*

(2) *there exists a model* $\mathcal{M}_p$ *and the relation* I_p *of isolation on a set of realizations of* p *in a* (*any*) *model* $\mathcal{M}\models T$ *realizing* p *is non-symmetric;*

(3) *there exists a model* $\mathcal{M}_p$ *and, in some* (*any*) *model* $\mathcal{M}\models T$ *realizing* p*, there exist realizations* $\bar a$ *and* $\bar b$ *of* p *such that the type* $\mathrm{tp}(\bar b/\bar a)$ *is principal and* $\bar b$ *does not semi-isolate* $\bar a$ *and, in particular,* SI_p *is non-symmetric on the set of realizations of* p *in* $\mathcal{M}$*.*

By Proposition 7.3, we have the following analogue of Theorem 7.1 for the class $\mathcal{T}_c$.

Proposition 7.6. *Every theory* $T\in\mathcal{T}_c$ *with a finite Rudin–Keisler preorder satisfies the following: if* $|\widetilde{\mathbf{M}}|>1$ *then* $\mathrm{IL}(\widetilde{\mathbf{M}})\geq 1$*. Moreover, we have the following decomposition formula:*

$$I(T,\omega)=|\mathrm{RK}(T)|+\sum_{i=0}^{|\mathrm{RK}(T)/\sim_{\mathrm{RK}}|-1}\mathrm{IL}(\widetilde{\mathbf{M}_i})+\mathrm{NPL}(T),$$

where $\widetilde{\mathbf{M}_0},\ldots,\widetilde{\mathbf{M}_{|\mathrm{RK}(T)/\sim_{\mathrm{RK}}|-1}}$ *are all elements of the partially ordered set* $\mathrm{RK}(T)/\sim_{\mathrm{RK}}$ *and* $\mathrm{IL}(\widetilde{\mathbf{M}_i})\in\omega\cup\{\omega,\omega_1,2^\omega\}$ *for each* i*,* $0\leq\mathrm{NPL}(T)\leq 2^\omega$*.*

The following theorem is an analogue of Theorem 7.2 for the class $\mathcal{T}_c$.

Theorem 7.7. *For any finite preordered set* $\langle X; \leq\rangle$ *and for any function*

$$f\colon X/{\sim} \to \omega \cup \{\omega, 2^\omega\}$$

such that $f(\widetilde{x}) > 0$ *for* $|\widetilde{x}| > 1$ (*where* $x \sim y \Leftrightarrow x \leq y$ *and* $y \leq x$), *there exists a theory* $T \in \mathcal{T}_c$ (*without prime models*) *and an isomorphism* $g\colon \langle X; \leq\rangle \mathrel{\tilde{\to}} \mathrm{RK}(T)$ *such that* $\mathrm{IL}(g(\widetilde{x})) = f(\widetilde{x})$ *for any* $\widetilde{x} \in X/{\sim}$.

Proof. Denote the cardinality of X by m and consider the theory T_0 of unary predicates P_i, $i < m$, forming a partition of a set A into m disjoint infinite sets with a coloring Col: $A \to \omega \cup \{\infty\}$ such that for any $i < m$, $j \in \omega$, there are infinitely many realizations for each type $\{\mathrm{Col}_j(x) \wedge P_i(x)\}$, $\{\neg\mathrm{Col}_j(x) \mid j \in \omega\} \cup \{P_i(x)\} = p_i(x)$. In this case, each set of formulas isolates a complete type.

Let $X_1, \ldots, X_n$ be connected components of the preordered set $\langle X; \leq\rangle$, consisting of $m_1, \ldots, m_n$ elements respectively, $m_1 + \ldots + m_n = m$. Now we assume that each element in X corresponds to a predicate P_i, $i < m$.

We expand the theory T_0 to a theory T_1 by binary predicates Q_{kl}, whose domain coincides with the set of solutions for the formula $P_k(x)$ and the range is the set of solutions for the formula $P_l(x)$; we connect types p_k and p_l if corresponding elements x_k and x_l in X belong to a common connected component and x_l covers x_k. Moreover, the coloring Col will be 1-inessential and Q_{kl}-ordered [22]:

(1) for any $i \geq j$, there are elements $x, y \in M$ such that

$$\models \mathrm{Col}_i(x) \wedge \mathrm{Col}_j(y) \wedge Q_{kl}(x, y) \wedge P_k(x) \wedge P_l(y);$$

(2) if $i < j$ then there are no elements $u, v \in M$ such that

$$\models \mathrm{Col}_i(u) \wedge \mathrm{Col}_j(v) \wedge Q_{kl}(u, v) \wedge P_k(u) \wedge P_l(v).$$

Applying a generic construction we get that if $a \models p_l(y)$ then the formula $Q_{kl}(x, a)$ is isolating and $p_l(y) \cup Q_{kl}(x, y) \vdash p_k(x)$, moreover, realizations of p_k do not semi-isolate realizations of p_l. Thus the set of non-principal 1-types $p_i(x)$ has a preorder corresponding to the preorder $\leq$.

We construct, by induction, an expansion of T_1 to a required theory T.

On initial step, we expand the theory T_1 by binary predicates $\{R_i^{(2)}\}_{i\in\omega}$ and apply the operator of continuum partition $\mathfrak{c}\text{-}\mathsf{Partition}(\mathcal{A}, \mathcal{A} \restriction P_0, Y, \{R_i^{(2)}\}_{i\in\omega}) = \mathcal{B}$, where $\mathcal{A}$ is a model of T_1. We consider an arbitrary connected component X_i and enumerate its elements so that if $x_k > x_l$ then $k > l$. On further m_i steps, we apply the operator of allocation for a countable subset $\omega\text{-}\mathsf{Alloc}(\mathcal{B}, \mathbf{q}_\omega, \mathcal{A} \restriction P_{l_i}, \{R_j^{(2)}\}_{j\in\omega})$, where $l_1, \ldots, l_i$ are numbers of elements forming the connected component X_i, $\mathbf{q}_\omega$ is a countable dense subset of set $\mathbf{q}$ of 1-types for the structure $\mathcal{B}$. We organize a similar process for all connected components in X. Now for all types corresponding to elements in distinct connected components and to maximal elements in a common component, we apply the operator of ban for upward movement $\mathsf{banUp}(\mathcal{A}, \mathcal{A} \restriction P_i, \mathcal{A} \restriction P_j, \{R_\Delta^{(3)}\})$, expanding the theory by disjoint families ternary predicates $R_n^{(3)}$, $n \in \omega$.

The required number of limit models can be done applying for each $g(\widetilde{x})$ the operator $\mathsf{typeLim}(g(\widetilde{x}), f(\widetilde{x}), \{R_i^{g(\widetilde{x})}\}_{i\in\omega})$ expanding the theory by predicates $R_i^{g(\widetilde{x})}$. □

By the proof of Theorem 7.7, positive values $P(T)$ for the class $\mathcal{T}_c$ can be defined by prime models not prime over $\varnothing$. Modifying the proof, one can realize an arbitrary finite preordered set $\langle X;\leq\rangle$ with a least element by $\mathrm{RK}(T)$ for a theory $T\in\mathcal{T}_c$ with a prime model over $\varnothing$.

By the construction for the proof of Theorem 7.7, we get

Corollary 7.8. *For any cardinalities $\lambda_1\in\omega\setminus\{0\}$ and $\lambda_2\in\omega\cup\{\omega,2^\omega\}$, there is a theory $T\in\mathcal{T}_c$ such that $\mathrm{cm}_3(T)=(\lambda_1,\lambda_2,2^\omega)$.*

Proof. Constructing the required theory $T\in\mathcal{T}_c$ we take a set X in Theorem 7.7 with $|X|=\lambda_1$ and use the operator typeLim with input parameters such that the sum of $f(\widetilde{x})$ is equal to λ_2. □

8. Distributions of prime and limit models for countable Rudin–Keisler preorders

We say (as in [25]) that a family $\mathbf{Q}$ of $\leq_{\mathrm{RK}}$-sequences $\mathbf{q}$ of types *represents* a $\leq_{\mathrm{RK}}$-sequence $\mathbf{q}'$ of types if any limit model over $\mathbf{q}'$ is limit over some $\mathbf{q}\in\mathbf{Q}$.

Theorem 8.1 ([25]). *Any small theory T satisfies the following conditions:*

(a) *the structure $\mathrm{RK}(T)$ is upward directed and has a least element $\mathbf{M}_0$ (the isomorphism type of prime model of T), $\mathrm{IL}(\widetilde{\mathbf{M}_0})=0$;*

(b) *if $\mathbf{q}$ is a $\leq_{\mathrm{RK}}$-sequence of non-principal types q_n, $n\in\omega$, such that each type q of T is related by $q\leq_{\mathrm{RK}}q_n$ for some n, then there exists a limit model over $\mathbf{q}$; in particular, $I_l(T)\geq 1$ and the countable saturated model is limit over $\mathbf{q}$, if $\mathbf{q}$ exists;*

(c) *if $\mathbf{q}$ is a $\leq_{\mathrm{RK}}$-sequence of types q_n, $n\in\omega$, and $(\mathcal{M}_{q_n})_{n\in\omega}$ is an elementary chain such that any co-finite subchain does not consist of pairwise isomorphic models, then there exists a limit model over $\mathbf{q}$;*

(d) *if $\mathbf{q}'=(q'_n)_{n\in\omega}$ is a subsequence of $\leq_{\mathrm{RK}}$-sequence $\mathbf{q}$, then any limit model over $\mathbf{q}$ is limit over $\mathbf{q}'$;*

(e) *if $\mathbf{q}=(q_n)_{n\in\omega}$ and $\mathbf{q}'=(q'_n)_{n\in\omega}$ are $\leq_{\mathrm{RK}}$-sequences of types such that for some $k,m\in\omega$, since some n, any types q_{k+n} and q'_{m+n} are related by $\mathcal{M}_{q_{k+n}}\simeq\mathcal{M}_{q'_{m+n}}$, then any model $\mathcal{M}$ is limit over $\mathbf{q}$ if and only if $\mathcal{M}$ is limit over $\mathbf{q}'$.*

Moreover, the following decomposition formula holds:

$$I(T,\omega)=|\mathrm{RK}(T)|+\sum_{\mathbf{q}\in\mathbf{Q}}\mathrm{IL}_{\mathbf{q}},$$

where $\mathrm{IL}_{\mathbf{q}}\in\omega\cup\{\omega,\omega_1,2^\omega\}$ is the number of limit models related to the $\leq_{\mathrm{RK}}$-sequence $\mathbf{q}$ and not related to extensions and to restrictions of $\mathbf{q}$ that used for the counting of all limit models of T, and the family $\mathbf{Q}$ of $\leq_{\mathrm{RK}}$-sequences of types represents all $\leq_{\mathrm{RK}}$-sequences, over which limit models exist.

Theorem 8.2 ([25]). *Let $\langle X,\leq\rangle$ be at most countable upward directed preordered set with a least element x_0, $f\colon Y\to\omega\cup\{\omega,2^\omega\}$ be a function with at most countable set Y of $\leq_0$-sequences, i. e., of sequences in $X\setminus\{x_0\}$ forming $\leq$-chains, and satisfying the following conditions:*

(a) *$f(y)\geq 1$ if for any $x\in X$ there exists some x' in the sequence y such that $x\leq x'$;*

(b) *$f(y)\geq 1$ if any co-finite subsequence of y does not contain pairwise equal elements;*

(c) *$f(y)\leq f(y')$ if y' is a subsequence of y;*

(d) *$f(y)=f(y')$ if $y=(y_n)_{n\in\omega}$ and $y'=(y'_n)_{n\in\omega}$ are sequences such that there exist some $k,m\in\omega$ for which $y_{k+n}=y'_{m+n}$ since some n.*

Then there exists a small theory T and an isomorphism g: $\langle X, \leq\rangle \overset{\sim}{\rightarrow} \mathrm{RK}(T)$ such that any value $f(y)$ is equal to the number of limit models over $\leq_{\mathrm{RK}}$-sequence $(q_n)_{n\in\omega}$, corresponding to the $\leq_0$-sequence $y = (y_n)_{n\in\omega}$, where $g(y_n)$ is the isomorphism type of the model $\mathcal{M}_{q_n}$, $n \in \omega$.

Repeating the proof of Theorem 8.1 we obtain

Theorem 8.3. *Any theory $T \in \mathcal{T}_c$ satisfies the following conditions:*

(a) *if $\mathbf{q}$ is a $\leq_{\mathrm{RK}}$-sequence of types q_n, $n \in \omega$, and $(\mathcal{M}_{q_n})_{n\in\omega}$ is an elementary chain such that any co-finite subchain does not consist of pairwise isomorphic models, then there exists a limit model over $\mathbf{q}$;*

(b) *if $\mathbf{q}' = (q'_n)_{n\in\omega}$ is a subsequence of $\leq_{\mathrm{RK}}$-sequence $\mathbf{q}$, then any limit model over $\mathbf{q}$ is limit over $\mathbf{q}'$;*

(c) *if $\mathbf{q} = (q_n)_{n\in\omega}$ and $\mathbf{q}' = (q'_n)_{n\in\omega}$ are $\leq_{\mathrm{RK}}$-sequences of types such that for some $k, m \in \omega$, since some n, any types q_{k+n} and q'_{m+n} are related by $\mathcal{M}_{q_{k+n}} \simeq \mathcal{M}_{q'_{m+n}}$, then any model $\mathcal{M}$ is limit over $\mathbf{q}$ if and only if $\mathcal{M}$ is limit over $\mathbf{q}'$.*

Moreover, the following decomposition formula holds:

$$I(T, \omega) = |\mathrm{RK}(T)| + \sum_{\mathbf{q}\in\mathbf{Q}} \mathrm{IL}_{\mathbf{q}} + \mathrm{NPL}(T),$$

where $\mathrm{IL}_{\mathbf{q}} \in \omega \cup \{\omega, \omega_1, 2^\omega\}$ is the number of limit models related to the $\leq_{\mathrm{RK}}$-sequence $\mathbf{q}$ and not related to extensions and to restrictions of $\mathbf{q}$ that used for the counting of all limit models of T, and the family $\mathbf{Q}$ of $\leq_{\mathrm{RK}}$-sequences of types represents all $\leq_{\mathrm{RK}}$-sequences, over which limit models exist.

Similarly Theorem 7.2, Theorem 8.2 has a generalization for the class $\mathcal{T}_c$:

Theorem 8.4. *Let $\langle X, \leq\rangle$ be at most countable preordered set, f: $Y \to \omega \cup \{\omega, 2^\omega\}$ be a function with at most countable set Y of $\leq$-sequences, i. e., of sequences in X forming $\leq$-chains, and satisfying the following conditions:*

(a) *$f(y) \geq 1$ if any co-finite subsequence of y does not contain pairwise equal elements;*

(b) *$f(y) \leq f(y')$ if y' is a subsequence of y;*

(c) *$f(y) = f(y')$ if $y = (y_n)_{n\in\omega}$ and $y' = (y'_n)_{n\in\omega}$ are sequences such that there exist some $k, m \in \omega$ for which $y_{k+n} = y'_{m+n}$ since some n.*

Then there exists a theory $T \in \mathcal{T}_c$ and an isomorphism g: $\langle X, \leq\rangle \overset{\sim}{\rightarrow} \mathrm{RK}(T)$ such that any value $f(y)$ is equal to the number of limit models over $\leq_{\mathrm{RK}}$-sequence $(q_n)_{n\in\omega}$, corresponding to the $\leq$-sequence $y = (y_n)_{n\in\omega}$, where $g(y_n)$ is the isomorphism type of the model $\mathcal{M}_{q_n}$, $n \in \omega$.

Proof. We assume that X is countable since for a finite X the proof repeats the construction for the proof of Theorem 7.7. Now we consider the theory T_0 of unary predicates P_i, $i \in \omega$, forming, with the type $p_\infty(x) = \{\neg P_i(x) \mid i \in \omega\}$, a partition of a set A into disjoint infinite classes with a coloring Col: $A \to \omega \cup \{\infty\}$ such that for any $i, j \in \omega$, there are infinitely many realizations of types $\{\mathrm{Col}_j(x) \wedge P_i(x)\}$, $\{\neg\mathrm{Col}_j(x) \mid i \in \omega\} \cup \{P_i(x)\} = p_i(x)$, $\{\mathrm{Col}_j(x)\} \cup p_\infty(x)$, $\{\neg\mathrm{Col}_j(x) \mid j \in \omega\} \cup p_\infty(x)$. Here, each set of formulas isolates a complete type. We connect the type $\{\neg\mathrm{Col}_j(x) \mid j \in \omega\} \cup p_\infty(x)$ with the type $p_0(x)$ by an extension of T_0 to a theory T_1 with a binary predicate Q_0 such that for all $j \in \omega$, we have:

(1) $\forall x, y\, (\mathrm{Col}_j(x) \wedge P_0(x) \wedge Q_0(x, y) \to \mathrm{Col}_j(y) \wedge P_j(y))$;

(2) $\forall x, y\, (\mathrm{Col}_j(y) \wedge P_j(y) \wedge Q_0(x, y) \to \mathrm{Col}_j(x) \wedge P_0(x))$;

(3) Q_0 is a bijection between sets of solutions for the formulas $\mathrm{Col}_j(x) \wedge P_0(x)$ and $\mathrm{Col}_j(y) \wedge P_j(y)$.

These conditions allow not to care about the type $p_\infty(x)$ with respect to existence of a prime model over it, because $p_0(x)$ and $p_\infty(x)$ are strongly RK-equivalent.

Let $X_1, \ldots, X_n, \ldots$ be connected components in the preordered set $\langle X, \leq\rangle$. We consider a one-to-one correspondence between X and the set of predicates $P_i(x)$, $i \in \omega$.

Similar to the proof of Theorem 7.7, we expand the theory T_1 to a theory T_2 by binary predicates Q_{kl} with domains $P_k(x)$ and ranges P_l, and connect types p_k and p_n if corresponding elements in X lay in common connected component and an element x_l corresponding to p_l covers an element x_k corresponding to p_k. Moreover, using a generic construction, the coloring Col should be 1-inessential and Q_{kl}-ordered.

The rest of the proof repeats arguments for the proof of Theorem 7.7, where the operator ω-Alloc of allocation for a countable set is applied countably many times, for non-principal types corresponding to elements in X. In this case, if non-principal types are not exhausted, we apply the operator $\mathfrak{c}$-$\mathsf{Partition}$ of continuum partition for remaining types.

For the required number of limit models with respect to a sequence $(q_n)_{n\in\omega}$, we expand the theory by predicates $R_i^{(q_n)}$, $i \in \omega$, and apply the operator

$$\mathsf{seqLim}((q_n)_{n\in\omega}, f(y), \{R_i^{(q_n)}\}_{i\in\omega}),$$

where y is a sequence in Y corresponding to the sequence $(q_n)_{n\in\omega}$. □

By the construction for the proof of Theorem 8.4, we obtain

Corollary 8.5. *For any cardinality $\lambda \in \omega \cup \{\omega, 2^\omega\}$, there is a theory $T \in \mathcal{T}_c$ such that $\mathrm{cm}_3(T) = (\omega, \lambda, 2^\omega)$.*

Proof. Constructing the required theory $T \in \mathcal{T}_c$ we take a set X in Theorem 8.4 with $|X| = \omega$ and the operator seqLim such that the sum of $f(y)$ is equal to λ. □

9. Interrelations of classes **P**, **L**, and **NPL** in theories with continuum many types. Distributions of triples $\mathrm{cm}_3(T)$ in the class $\mathcal{T}_c$

Theorem 9.1. *Let $\langle X, \leq\rangle$ be at most countable preordered set, where X is a disjoint union of some sets P and* NPL*, $f\colon Y \to \omega \cup \{\omega, 2^\omega\}$ be a function with at most countable set Y of $(P, \leq)$-sequences, i. e., of sequences in P forming $\leq$-chains, and satisfying the following conditions:*

(a) *$f(y) \geq 1$ if any co-finite subsequence of y does not contain pairwise equal elements;*

(b) *$f(y) \leq f(y')$ if y' is a subsequence of y;*

(c) *$f(y) = f(y')$ if $y = (y_n)_{n\in\omega}$ and $y' = (y'_n)_{n\in\omega}$ are sequences such that there exist some $k, m \in \omega$ for which $y_{k+n} = y'_{m+n}$ since some n.*

Then there is a theory $T \in \mathcal{T}_c$ and an isomorphism $g\colon \langle X, \leq\rangle \xrightarrow{\sim} \mathrm{CM}_0(T)$ to a substructure $\mathrm{CM}_0(T) = \langle \mathbf{CM}_0(T); \leq_{\mathrm{RK}}\rangle$ of $\mathrm{CM}(T)$, with $\mathbf{CM}_0(T) \subset \mathbf{P}(T) \cup \mathbf{NPL}(T)$ and satisfying the following:

(1) *$g(P) = \mathbf{P}(T)$, $g(\mathrm{NPL}) = \mathbf{CM}_0(T) \cap \mathbf{NPL}(T)$;*

(2) *each value* $f(y)$ *is equal to the number of limit models over a* $\leq_{\rm RK}$*-sequence* $(q_n)_{n\in\omega}$ *corresponding to the* $\leq$*-sequence* $y = (y_n)_{n\in\omega}$*, where* $g(y_n)$ *is the isomorphism type of the model* $\mathcal{M}_{q_n}$*,* $n \in \omega$*.*

Proof. The construction of a preordered set of types, isomorphic to the structure $\langle X, \leq\rangle$ and without prime models over the type p_0, is similar to the proof of Theorem 8.4. Then for each non-principal type p_i, corresponding to an element in P, we apply the operator of allocation for a countable subset ${\rm dss}(\mathcal{A}, \mathbf{q}_\omega, \mathcal{A} \upharpoonright P_i, \{R_n\}_{n\in\omega})$. If there are types p_i, corresponding to elements in NPL, we apply, for these types, the operator of continuum partition $\mathfrak{c}$-$\mathsf{Partition}(\mathcal{A}, \mathcal{A} \upharpoonright P_i, Z, \{R_n\}_{n\in\omega})$. For all types, corresponding to elements in distinct connected components in $\langle X, \leq\rangle$ as well as to maximal elements in a common component, we apply the operator of ban for upward movement. For removing of prime models over remaining continuum many types, we apply, for n-tuples of elements, the operator of continuum partition, using $(n + 1)$-ary predicates. The required number of limit models is obtained by the operator for construction of limit models over a sequence of types. □

Theorem 9.2. *In the conditions of Theorem* 9.1*, there is a theory* $T \in \mathcal{T}_c$ *and an isomorphism* g: $\langle X, \leq\rangle \,\widetilde{\rightarrow}\, {\rm CM}_0(T)$ *to a substructure* ${\rm CM}_0(T) = \langle \mathbf{CM}_0(T); \leq_{\rm RK}\rangle$ *of* ${\rm CM}(T)$*, with* $\mathbf{CM}_0(T) \subset \mathbf{P}(T) \cup \mathbf{NPL}(T)$ *and satisfying the following:*

(1) $g(P) = \mathbf{CM}_0(T) \cap \mathbf{P}(T)$*,* $g({\rm NPL}) = \mathbf{NPL}(T)$*;*

(2) *each value* $f(y)$ *is equal to the number of limit models over a* $\leq_{\rm RK}$*-sequence* $(q_n)_{n\in\omega}$ *corresponding to the* $\leq$*-sequence* $y = (y_n)_{n\in\omega}$*, where* $g(y_n)$ *is the isomorphism type of the model* $\mathcal{M}_{q_n}$*,* $n \in \omega$*.*

Proof. It is similar to the proof of Theorem 9.1 with the only difference that before we use the operator of continuum partition and then, if non-principal types p_i are not exhausted, we apply the operator of allocation for a countable set. For getting prime models over remaining continuum many types, we apply, for n-tuples of elements, the operator of allocation for a countable set, using $(n+1)$-ary predicates. The required number of limit models is obtained by the operator for construction of limit models over a sequence of types. □

By the construction for the proof of Theorem 9.2, we obtain

Corollary 9.3. *For any cardinalities* $\lambda \in \omega \cup \{\omega, 2^\omega\}$ *there is a theory* $T \in \mathcal{T}_c$ *such that* ${\rm cm}_3(T) = (2^\omega, 2^\omega, \lambda)$*.*

Proof. It suffices to apply the operator ω-Alloc for n-tuples of elements in the proof of Theorem 9.2. For the operator seqLim we put the values 2^ω for the second input parameter. □

Proposition 5.5 and Corollaries 7.8, 8.5, 9.3 imply the following analogue of Theorem 5.2 for the class $\mathcal{T}_c$.

Theorem 9.4. *Under the assumption of Continuum Hypothesis, for any theory* T *in the class* $\mathcal{T}_c$*, the triple* ${\rm cm}_3(T)$ *has one of the following values:*

(1) $(2^\omega, 2^\omega, \lambda)$*, where* $\lambda \in \omega \cup \{\omega, 2^\omega\}$*;*

(2) $(0, 0, 2^\omega)$*;*

(3) $(\lambda_1, \lambda_2, 2^\omega)$*, where* $\lambda_1 \geq 1$*,* $\lambda_1, \lambda_2 \in \omega \cup \{\omega, 2^\omega\}$*.*

All these values have realizations in the class $\mathcal{T}_c$*.*

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Roman Andreevich Popkov
Novosibirsk State Technical University,
K. Marx avenue, 20
630073, Novosibirsk, Russia.
E-mail address: r-popkov@yandex.ru

Sergey Vladimirovich Sudoplatov
Sobolev Institute of Mathematics,
Academician Koptyug avenue, 4
630090, Novosibirsk, Russia.
Novosibirsk State Technical University,
K. Marx avenue, 20
630073, Novosibirsk, Russia;
Novosibirsk State University,
ul. Pirogova, 2,
630090, Novosibirsk, Russia
E-mail address: sudoplat@math.nsc.ru