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ON SYLOW NUMBERS OF SOME FINITE GROUPS

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Abstract. Let G be a finite group, let $\pi(G)$ be the set of primes p such that G contains an element of order p, and let $n_p(G)$ be the number of Sylow p-subgroups of G, that is, $n_p(G) = |\mathrm{Syl}_p(G)|$. Set $\mathrm{NS}(G) := \{n_p(G)|\ p \in \pi(G)\}$. In this paper, we will show that if $|G| = |S|$ and $\mathrm{NS}(G) = \mathrm{NS}(S)$, where S is one of the groups: the special projective linear groups $L_3(q)$, with $5 \nmid (q-1)$, the projective special unitary groups $U_3(q)$, the sporadic simple groups, the alternating simple groups, and the symmetric groups of degree prime r, then G is isomorphic to S. Furthermore, we will show that if G is a finite centerless group and $\mathrm{NS}(G) = \mathrm{NS}(L_2(17))$, then G is isomorphic to $L_2(17)$, and or G is isomorphic to $\mathrm{Aut}(L_2(17))$.

Keywords: finite group, simple group, Sylow subgroup.

1. Introduction

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. Let G be a finite group. Denote by $\pi(G)$ the set of primes p such that G contains an element of order p. A finite group G is called a simple K_n-group, if G is a simple group with $|\pi(G)| = n$. Also denote by (a, b) the greatest common divisor of positive integers a and b. Throughout this paper, we denote by $n_p(G)$ the number of Sylow $p-$ subgroup of G, that is, $n_p(G) = |\mathrm{Syl}_p(G)|$. We denote by $n_{2'}$ the largest positive odd divisor of the positive integer n. All other notations are standard, and we refer to [15], for example.

In 1992, Bi [10] showed that $L_2(p^k)$ can be characterized just by the orders of the order of normalizer of its Sylow subgroups. In other words, if G is a group and $|N_G(P)| = |N_{L_2(p^k)}(Q)|$, where $P \in \mathrm{Syl}_r(G)$ and $Q \in \mathrm{Syl}_r(L_2(p^k))$ for every prime

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r, then $G \cong L_2(p^k)$. This type of characterization is known for the following simple groups: $L_2(p^k)$ [10], $L_n(q)$ [9], $S_4(q)$ [13], the alternating simple groups [12], $U_n(q)$ [14], the sporadic simple groups [2] and ${}^2D_n(p^k)$ [1].

Set $\mathrm{NS}(G) := \{n_p(G) |\ p \in \pi(G)\}$. Let S be one of the above simple groups. Clearly if $n_p(G) = n_p(S)$ for every prime p and $|G| = |S|$, then $|N_G(P)| = |N_S(Q)|$, where $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_p(S)$. Thus by the above references, $G \cong S$. But when $\mathrm{NS}(G) = \mathrm{NS}(S)$ and $|G| = |S|$, then we don't know that $n_p(G) = n_p(S)$ for any prime p.

In [3], it is proved that if $\mathrm{NS}(G) = \mathrm{NS}(L_2(q))$ and $|G| = |L_2(q)|$, then $G \cong L_2(q)$. In this paper we have done this work for the groups: $L_3(q)$, where $5 \nmid (q-1)$, $U_3(q)$, the sporadic simple groups, the alternating simple groups and the symmetric groups S_r, where r is prime. In fact, we prove the following main theorem.

Main Theorem. Let G be a finite group. If $\mathrm{NS}(G) = \mathrm{NS}(S)$ and $|G| = |S|$, where $S \in \{L_3(q), U_3(q), A_n, S_r, M\}$ with r is prime, M is the sporadic simple group, and $5 \nmid (q-1)$ when $S = L_3(q)$, then $G \cong S$.

Furthermore, in this paper we pose another characterization by $\mathrm{NS}(G)$. In this new characterization, we want to know that if remove the condition order, then whether we can say $G \cong S$.

In the Theorem 6, we proved that if G is a centerless group and $\mathrm{NS}(G) = \mathrm{NS}(L_2(17))$, then $G \cong L_2(17)$ or $G \cong \mathrm{Aut}(L_2(17))$. Thus if the condition order replaced with $Z(G) = 1$, then it seems that this characterization works. This type of characterization is known for the groups: A_5 and A_6 [4] and $L_2(8)$ [3].

We denote by $k(\mathrm{NS}(G))$ the number of isomorphism classes of the finite groups H satisfying $\mathrm{NS}(G) = \mathrm{NS}(H)$. A finite group G is called n-recognizable by $\mathrm{NS}(G)$ if $k(\mathrm{NS}(G)) = n$. In the [4, 3], it is proved that $k(\mathrm{NS}(A_6)) = 5$ and $k(\mathrm{NS}L_2(8))) = 2$.

2. Preliminary Results

Lemma 1. [9] *Let G be a finite group. If $|N_G(R_1)| = |N_{L_n(q)}(R_2)|$ for every prime r, where $R_1 \in \mathrm{Syl}_r(G)$ and $R_2 \in \mathrm{Syl}_r(L_n(q))$, then $G \cong L_n(q)$.*

Lemma 2. [2] *Let S be sporadic simple group and p be the greatest element of $\pi(S)$. If G is a finite group such that $|G| = |S|$ and $|N_G(P_1)| = |N_S(P_2)|$, where $P_1 \in \mathrm{Syl}_p(G)$ and $P_2 \in \mathrm{Syl}_p(S)$, then $G \cong S$.*

Lemma 3. [12] *Let G be a finite group, $n \geq 4$ with $n \neq 8$, 10, and r the greatest prime not exceeding n. If $|G| = |A_n|$ and $|N_G(R)| = |N_{A_n}(S)|$, where $R \in \mathrm{Syl}_r(G)$ and $S \in \mathrm{Syl}_r(A_n)$, then $G \cong A_n$.*

Lemma 4. [12] *Let G be a finite group. If $|N_G(P_1)| = |N_{A_n}(P_2)|$ for every prime p, where $P_1 \in \mathrm{Syl}_p(G)$ and $P_2 \in \mathrm{Syl}_p(A_n)$, then $G \cong A_n$.*

Lemma 5. [6] *Let G be a finite group. If $|G| = |S_r|$, where r is prime and $|N_G(R)| = |N_{S_r}(S)|$, where $R \in \mathrm{Syl}_r(G)$ and $S \in \mathrm{Syl}_r(S_r)$, then $G \cong S_r$.*

Lemma 6. [14, 11] *Let G be a finite group. If $|N_G(P_1)| = |N_{U_n(q)}(P_2)|$ for every prime p, where $P_1 \in \mathrm{Syl}_p(G)$ and $P_2 \in \mathrm{Syl}_p(U_n(q))$, then $G \cong U_n(q)$.*

Lemma 7. [20] *Let $G = L_n(q)$, where $q = p^\alpha$. The orders of maximal tori of $L_n(q)$ are*

$$\Pi_{i=1}^{k}(q^{r_i} - 1)/(q-1)(n, q-1),\ (r_1, ..., r_k) \in Par(n).$$

Lemma 8. [18, Theorem 9.3.1] *Let G be a finite solvable group and $|G| = m \cdot n$, where $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $(m, n) = 1$. Let $\pi = \{p_1, ..., p_r\}$ and h_m be the number of $\pi-$Hall subgroups of G. Then $h_m = q_1^{\beta_1} \cdots q_s^{\beta_s}$ satisfies the following conditions for all $i \in \{1, 2, ..., s\}$:*

(1) $q_i^{\beta_i} \equiv 1$ *(mod p_j), for some p_j*
(2) *The order of some chief factor of G is divisible by $q_i^{\beta_i}$.*

Lemma 9. [19] *If G is a simple K_3- group, then G is isomorphic to one of the following groups: A_5, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$ or $U_4(2)$.*

Lemma 10. [16, Lemma 1] *Let G be a finite group and M be a normal subgroup of G. Then both $n_p(M)$ and $n_p(G/M)$ divide $n_p(G)$ and moreover $n_p(M)\ n_p(G/M) \mid n_p(G)$ for every prime p.*

Lemma 11. [5, Corollary 2.5] *Let r be a prime number, and let G be a simple such that $n_r(G) = r + 1$. Then G is isomorphic to $L_2(r)$.*

By the Sylow's theorem implies that if p is prime $n_p = 1 + pk$. If $p = 2$, then n_2 is odd. If $p \in \pi(G)$, then

$$\begin{cases} p \mid (n_p - 1) \\ \\ (p, n_p) = 1. \end{cases} \qquad (*)$$

In the proof of the main results, we often apply $(*)$ and the above comments.

3. Main Results

Theorem 1. *Let G be a finite group* $\mathrm{NS}(G) = \mathrm{NS}(L_3(q))$ *and $|G| = |L_3(q)|$ where $5 \nmid (q - 1)$, then $G \cong L_3(q)$.*

Proof. First, we get $\mathrm{NS}(L_3(q))$, where q is prime power. Let $q = p^n$ and $p \neq 2$. To find the number of Sylow p-subgroups in $L_3(q)$ first, look at $\mathrm{SL}_3(q)$. The normalizer of a Sylow p-subgroup is the set of upper triangular matrices with determinant 1, so the order of the normalizer is $q^3(q - 1)^2$. The order of the whole group $\mathrm{SL}_3(q)$ is $q^3(q^3 - 1)(q^2 - 1)$. Therefore, the number of Sylow p-subgroups is $(q + 1)(q^2 + q + 1)$. This will be the same as the number of Sylow p-subgroups of $L_3(q)$ because the canonical homomorphism from $\mathrm{SL}_3(q)$ to $L_3(q)$ yields a bijection on Sylow p-subgroups.

To find the number of Sylow 2-subgroups in $L_3(q)$, we consider the following cases.

(a) If $p \equiv 1 \pmod 4$, a Sylow 2-subgroups lies inside a group $C_{(q-1)} \times C_{(q-1)} : S_3$. Its normalizer has size $2(q-1)|q-1|_2$, where $|q-1|_2$ is the largest power of 2 dividing $q - 1$. Thus the number of Sylow 2-subgroups are $q^3|(q + 1)(q^3 - 1)|_{2'}$.

(b) if $p \equiv 3 \pmod 4$, a Sylow 2-subgroups lies inside a group $C_{(q^2-1)} : S_2$. Its normalizer has size $2(q - 1)|q + 1|_2$. Thus the number of Sylow 2-subgroups are $q^3|(q + 1)(q^3 - 1)|_{2'}$.

To find the number of Sylow 3-subgroups in $L_3(q)$, we consider the following cases.

(a) if $p \equiv 2 \pmod 3$, then a Sylow is a characteristic subgroup of a torus $C_{(q^2-1)}$. It is normalized by the torus normalizer of size $2(q^2 - 1)$. Thus the number of Sylow 3-subgroups are $q^3(q^3 - 1)/2$.

(b) if $p \equiv 1 \pmod 3$, then a Sylow lies inside a group $C_{(q-1)} \times C_{(q-1)} : S_3$. It is equal to its own normalizer. Thus the number of Sylow 3-subgroups are $q^3(q+1)(q^2+q+1)/6$.

Now let that r is not p, the field characteristic, and $r > 3$. Then r divides exactly one of $q+1$, $q-1$, q^2+q+1 this in turn implies that a Sylow S lies inside a maximal torus. By Lemma 7, the maximal tori are one of the following subgroups.

1. $C_{(q^2-1)}$,
2. $C_{(q-1)} \times C_{(q-1)}$,
3. C_{q^2+q+1}

So S is normalized by $N_G(T)$ where T is the torus, now we consider the following cases.

1. $N_G(T) = C_{(q^2-1)}.C_2$. This lies inside a copy of $\mathrm{GL}_2(q)$ which does not normalize the torus, with a little checking one can be sure that $N_G(T)$ really is the Sylow normalizer. Thus if $r \in \pi(q+1)$, then the number of Sylow r-subgroups are $q^3(q^3-1)/2$.

2. $N_G(T) = C_{(q-1)} \times C_{(q-1)} : S_3$. By [8], this is a maximal subgroup so we have our Sylow normalizer. Thus if $r \in \pi(q-1)$, then the number of Sylow r-subgroups are $q^3(q+1)(q^2+q+1)/6$.

3. $N_G(T) = C_{q^2+q+1} : C_3$. By [8], this is a maximal subgroup so we have our Sylow normalizer. Thus if $r \in \pi(q^2+q+1)$, then the number of Sylow r-subgroups are $q^3(q-1)(q^2-1)/3$.

Therefore if $p \neq 2$, then $\mathrm{NS}(L_3(q)) = \{(q+1)(q^2+q+1),\ q^3|(q+1)(q^3-1)|_{2'},\ q^3(q^3-1)/2,\ q^3(q+1)(q^2+q+1)/6,\ q^3(q-1)(q^2-1)/3\}$.

Now assume that $p = 2$. Arguing as above $n_2 = (q+1)(q^2+q+1)$. If $r \in \pi(q+1)$, then the number of Sylow r-subgroups are $q^3(q^3-1)/2$. If $r \in \pi(q-1)$, then the number of Sylow r-subgroups are $q^3(q+1)(q^2+q+1)/6$. If $r \in \pi(q^2+q+1)$, then the number of Sylow r-subgroups are $q^3(q-1)(q^2-1)/3$. Thus we need to get only the number of Sylow 3-subgroups.

If $q \equiv 2 \pmod 3$, then by [22] a Sylow 3-subgroup S is cyclic and $N_G(S) = C_{(q-1)} \times D_{2(q+1)}$. Thus the number of Sylow 3-subgroups are $q^3(q^3-1)/2$. If $q \equiv 4$ or $7 \pmod 9$, then a Sylow 3-subgroup S is 3^2 and $N_G(S) = 3^2.Q_8$. So the number of Sylow 3-subgroups are $q^3(q^2-1)(q^3-1)/216$.

If $q \equiv 1 \pmod 9$, then $N_G(S)$ is a quotient of $C_{3^2} \times C_{3^2} : S_3$ by a subgroup of order 3. Then the number of Sylow 3-subgroups are $q^3(q^2-1)(q^2+q+1)/162$. Therefore, $\mathrm{NS}(L_3(q)) = \{(q+1)(q^2+q+1),\ q^3(q^3-1)/2,\ q^3(q+1)(q^2+q+1)/6,\ q^3(q-1)(q^2-1)/3\}$ or $\{(q+1)(q^2+q+1),\ q^3(q^2-1)(q^3-1)/216,\ q^3(q^3-1)/2,\ q^3(q+1)(q^2+q+1)/6,\ q^3(q-1)(q^2-1)/3\}$, and or $\{(q+1)(q^2+q+1),\ q^3(q^2-1)(q^2+q+1)/162,\ q^3(q^3-1)/2,\ q^3(q+1)(q^2+q+1)/6,\ q^3(q-1)(q^2-1)/3\}$.

Now we will show that $G \cong L_3(q)$. Let $q = p^n$ and $p \neq 2$. By $\mathrm{NS}(G)$, $(q+1)(q^2+q+1)$ is the only member of $\mathrm{NS}(G)$ such that $p \nmid (q+1)(q^2+q+1)$. Thus by the Sylow's theorem, $n_p = (q+1)(q^2+q+1)$. Also $q^3 \mid (q+1)(q^3-1)|_{2'}$ is the only odd member of $\mathrm{NS}(G)$, so $n_2 = q^3|(q+1)(q^3-1)|_{2'}$.

If $r \in \pi(q+1)$, then by the Sylow's theorem and $\mathrm{NS}(G)$, $n_r = q^3(q^3-1)/2$.

If $r \in \pi(q-1)$, then $n_r = (q+1)(q^2+q+1)$ or $q^3(q+1)(q^2+q+1)/6$. Assume that $n_r = (q+1)(q^2+q+1)$. By the Sylow's theorem, $n_r = 1 + rk$, where k is a positive number. So $rk = q^3+2q^2+2q = q(q^2+2q+2)$. If $r = p$, then we get a contradiction. Thus $r \mid (q^2+2q+2)$. Since $r \in \pi(q-1)$ and $r \mid (q+3)(q-1) = q^2+2q-3$, $r \mid 5$. Thus $r = 5$. By the assumption $5 \nmid (q-1)$, which is a contradiction. Then

$n_r = q^3(q+1)(q^2+q+1)/6$. Therefore, $n_r(G) = n_r(L_3(q))$ for every r. Similar to the above discussion if $p = 2$, $n_r(G) = n_r(L_3(q))$ for every r where $5 \nmid (q-1)$. By the assumption $|G| = |L_3(q)|$, so $|N_G(R_1)| = |N_{L_3(q)}(R_2)|$ for every prime r, where $R_1 \in \mathrm{Syl}_r(G)$ and $R_2 \in \mathrm{Syl}_r(L_3(q))$. By Lemma 1, $G \cong L_3(q)$. □

Theorem 2. *Let G be a finite group such that* $\mathrm{NS}(G) = \mathrm{NS}(U_3(q))$ *and* $|G| = |U_3(q)|$, *then* $G \cong U_3(q)$.

Proof. First we find the number of Sylow r-subgroups in $U_3(q)$ for every prime r, where $q = 2^k$. By [22], the maximal subgroups of $U_3(q)$ are:

1. $q^3 : (q^2-1)/(3, q+1)$,
2. $(q+1)/(3, q+1) \times U_2(q)$,
3. $((q+1)(q+1)/(3, q+1)) \cdot S_3$,
4. $((q^2-q+1)/(3, q+1)) : 3$,
5. $U_3(2^m)$ if $k/m > 3$ is odd prime,
6. $U_3(2^m).3$ if m is odd and $k = 3m$.

Put Q_r a Sylow r-subgroup of $U_3(q)$ for $r \in \pi(U_3(q))$. By the maximal subgroups of $U_3(q)$ we have $N_{U_3(q)}(Q_2) = q^3 : (q^2-1)/(3, q+1)$, $N_{U_3(q)}(Q_r) = (q+1) \times D_{2(q-1)}/(3, q+1)$ for $r \in \pi(q+1)$, $N_{U_3(q)}(Q_r) = (q+1)^2 : S_3/(3, q+1)$ for $r \in \pi(q-1)$ and $N_{U_3(q)}(Q_r) = (q^2-q+1) : 3/(3, q+1)$ for $r \in \pi(q^2-q+1)$. Thus $n_2 = q^3+1$, $n_r = q^3(q-1)(q^2-q+1)/6$ for $r \in \pi(q+1)$, $n_r = q^3(q+1)(q^2-q+1)/2$ for $r \in \pi(q-1)$ and $n_r = q^3(q+1)^2(q-1)/3$ for $r \in \pi(q^2-q+1)$.

Therefore, if $p = 2$, then $\mathrm{NS}(U_3(q)) = \{q^3+1,\ q^3(q-1)(q^2-q+1)/6,\ q^3(q+1)(q^2-q+1)/2,\ q^3(q+1)^2(q-1)/3\}$.

Let $q = p^n$ where $p \neq 2$. By [8], we have the maximal subgroups of $U_3(q)$. Similar to the above discussion, $n_p = q^3+1$, $n_r = q^3(q-1)(q^2-q+1)/6$ for $r \in \pi(q+1)$, $n_r = q^3(q+1)(q^2-q+1)/2$ for $r \in \pi(q-1)$ and $n_r = q^3(q+1)^2(q-1)/3$ for $r \in \pi(q^2-q+1)$. For n_2 we note that if $q \equiv 1 \pmod 4$, then $n_2 = q^3(q^2-q+1)|(q^2-1)|_{2'}$ and if $q \equiv 3 \pmod 4$, then $n_2 = q^3(q^2-q+1)|(q-1)|_{2'}$.

Therefore, if $q \equiv 1 \pmod 4$, then $\mathrm{NS}(U_3(q)) = \{q^3+1,\ q^3(q^2-q+1)|(q^2-1)|_{2'},\ q^3(q-1)(q^2-q+1)/6,\ q^3(q+1)(q^2-q+1)/2,\ q^3(q+1)^2(q-1)/3\}$ and if $q \equiv 3 \pmod 4$, then $\mathrm{NS}(U_3(q)) = \{q^3+1,\ q^3(q^2-q+1)|(q-1)|_{2'},\ q^3(q-1)(q^2-q+1)/6,\ q^3(q+1)(q^2-q+1)/2,\ q^3(q+1)^2(q-1)/3\}$.

Now we will show that if $q = 2^n$, then $G \cong U_3(q)$. By $\mathrm{NS}(G)$, q^3+1 is the only odd member of $\mathrm{NS}(G)$, so $n_2 = q^3+1$. If $r \in \pi(q+1)$, then by the Sylow's theorem and $\mathrm{NS}(G)$, $n_r = q^3(q-1)(q^2-q+1)/6$.

If $r \in \pi(q-1)$, then $n_r = q^3+1$ or $q^3(q+1)(q^2-q+1)/2$. Assume that $n_r = q^3+1$. By the Sylow's theorem $n_r = 1 + rk$, where k is a positive number. So $rk = q^3$, then $r = p$, which is a contradiction. Thus $n_r = q^3(q+1)(q^2-q+1)/2$.

If $r \in \pi(q^2-q+1)$, then by $\mathrm{NS}(G)$, $n_r = q^3(q+1)^2(q-1)/3$. Thus $n_r(G) = n_r(U_3(q))$ for every r. Similar to the above discussion if $q = p^n$ where $p \neq 2$, $n_r(G) = n_r(U_3(q))$ for every r. Now by the assumption $|G| = |U_3(q)|$, so $|N_G(R_1)| = |N_{U_3(q)}(R_2)|$ for every prime r, where $R_1 \in \mathrm{Syl}_r(G)$ and $R_2 \in \mathrm{Syl}_r(U_3(q))$. By Lemma 6, $G \cong U_3(q)$. □

Theorem 3. *Let G be a finite group such that* $\mathrm{NS}(G) = \mathrm{NS}(M)$ *and* $|G| = |M|$ *where M is the sporadic simple group, then* $G \cong M$.

Proof. By Table 1 and 2 in [2] we can compute $\mathrm{NS}(M)$ for every sporadic simple group M. We claim that if p is the greatest element of $\pi(M)$, then $n_p(G) = n_p(M)$.

By NS(M) we have $p \mid n_q$ for every prime $q \neq p$. By $(*)$ we know $(p, n_p) = 1$. Thus the only member of the NS(G) that satisfy in the $(p, n_p) = 1$ is $n_p(M)$. Therefore, $n_p(G) = n_p(M)$. By the assumption $|G| = |M|$, so $|N_G(Q_1)| = |N_M(Q_2)|$, where $Q_1 \in \mathrm{Syl}_p(G)$ and $Q_2 \in \mathrm{Syl}_p(M)$. Now by Lemma 2, $G \cong M$. □

Theorem 4. *Let G be a finite group such that* $\mathrm{NS}(G) = \mathrm{NS}(A_n)$ *and* $|G| = |A_n|$ *where A_n is the alternating simple group, then $G \cong A_n$.*

Proof. Let $p \neq 2$ be prime and n be written in the scale of p in the form

$$n = a_0 + a_1 p + a_2 p^2 + \cdots + a_k p^k.$$

By [17] the number of Sylow subgroups of order p^m in the alternating group of degree n is $n!/a_0!a_1! \cdots a_k!\ p^m(p-1)^m$. If $p = 2$, then the the number of Sylow subgroups of order p^m is $n!/2p^m$. We claim that if r is the greatest element of $\pi(A_n)$, then $n_r(G) = n_r(A_n)$. Clearly $r \mid n!/a_0!a_1! \cdots a_k!\ p^m(p-1)^m = n_p$ for every prime $p \neq r$. Also $r \mid n!/2p^m = n_2$. By $(*)$ we know $(r, n_r) = 1$, so $n_r(G) = n_r(A_n)$. Let $n \neq 8$, 10. By the assumption $|G| = |A_n|$, so $|N_G(Q_1)| = |N_{A_n}(Q_2)|$, where $Q_1 \in \mathrm{Syl}_r(G)$ and $Q_2 \in \mathrm{Syl}_r(A_n)$. By Lemma 3, $G \cong A_n$.

Now let $n = 8$ or 10. We prove, for example if $n = 10$, then $G \cong A_{10}$ and afterwards if $n = 8$, then $G \cong A_8$.

If $n = 10$, then $\mathrm{NS}(G) = \mathrm{NS}(A_{10}) = \{4536, 11200, 14175, 14400\}$. Clearly $n_7(G) = 14400$. Since 14175 is odd by the Sylow theorem $n_2(G) = 14175$. Also because 3 divide 4536 and 14400, so $n_3(G) = 11200$.

Therefore, by arguing as above $n_5(G) = 4536$. Thus $n_p(G) = n_p(A_{10})$ for every prime p. Therefore, $|N_G(P_1)| = |N_{A_{10}}(P_2)|$ for every prime p, where $P_1 \in \mathrm{Syl}_p(G)$ and $P_2 \in \mathrm{Syl}_p(A_{10})$. By Lemma 4, $G \cong A_{10}$. □

Theorem 5. *Let G be a finite group such that* $\mathrm{NS}(G) = \mathrm{NS}(S_r)$ *and* $|G| = |S_r|$ *where S_r is the symmetric group of degree prime r. Then $G \cong S_r$.*

Proof. By [17] and similar to the proof of Theorem 4, $|N_G(Q_1)| = |N_{S_r}(Q_2)|$ where $Q_1 \in \mathrm{Syl}_r(G)$ and $Q_2 \in \mathrm{Syl}_r(S_r)$. Now it concludes from Lemma 5. □

Theorem 6. *Let G be a finite centerless group and* $\mathrm{NS}(G) = \mathrm{NS}(L_2(17)) = \{18, 136, 153\}$. *Then $G \cong L_2(17)$, and or $G \cong \mathrm{Aut}(L_2(17))$.*

Proof. First, we prove that $\pi(G) = \{2, 3, 17\}$. By the Sylow's theorem, $n_p \mid |G|$ for every p, hence by NS(G), $\{2, 3, 17\} \subseteq \pi(G)$. On the other hand by $(*)$, if $p \in \pi(G)$, then $p \mid (n_p - 1)$and $(p, n_p) = 1$, which implies that $p \in \{2, 3, 5, 17, 19\}$. By the Sylow's theorem, $n_2(G) = 153$, $n_3(G) = 136$ and $n_{17}(G) = 18$. If $5 \in \pi(G)$, then $n_5(G) = 136$ and if $19 \in \pi(G)$, then $n_{19}(G) = 153$. We prove that G is a non-solvable group. If G is a solvable group, then since $n_{17}(G) = 18$, by Lemma 8, $9 \equiv 1 (\mathrm{mod}\ 17)$, a contradiction. In follow, we will show that G is not a K_4 or K_5-group. Thus G is a K_3-group. We consider the following cases.

Case a. Let G be a K_4-group. Because $\{2, 3, 17\} \subseteq \pi(G)$, $\pi(G) = \{2, 3, 5, 17\}$ or $\{2, 3, 17, 19\}$. Let $\pi(G) = \{2, 3, 5, 17\}$. Since G is a finite group, it has a chief series. Let $1 = N_0 \trianglelefteq N_1 \trianglelefteq \ldots \triangleleft N_{r-1} \trianglelefteq N_r = G$ be a chief series of G. Since G is a non-solvable group, there exists a maximal number of non-negative integer i such that N_i/N_{i-1} is a simple group or the direct product of isomorphic simple groups and N_{i-1} is a maximal solvable normal subgroup of G. Now set $N_i := H$ and $N_{i-1} := N$. Hence G has the following normal series

$$1 \trianglelefteq N \lhd H \trianglelefteq G$$

such that H/N is a non-Abelian simple group, or H/N is a direct product of isomorphic non-Abelian simple groups. Since G is a K_4-group, $|\pi(H/N)| = 3$ or $|\pi(H/N)| = 4$. If $|\pi(H/N)| = 3$, then by Lemma 9, $\pi(H/N) = \{2, 3, 5\}$ or $\{2, 3, 17\}$.

If $\pi(H/N) = \{2, 3, 5\}$, then H/N is isomorphic to A_5, A_6 or $U_4(2)$, and or H/N is a direct product of isomorphic to A_5, A_6 or $U_4(2)$, by Lemma 9. By $n_p(H/N) \mid n_p(G)$ for every prime $p \in \pi(G)$, it is easy to check that this is impossible.

Let $\pi(H/N) = \{2, 3, 17\}$. We will show H/N is isomorphic to $L_2(17)$. Since $17 \mid |H/N|$ by Lemma 8, $n_{17}(H/N) \mid n_r(G) = 17 + 1$. Thus by Sylow's theorem $n_r(H/N) = 18$. We prove that H/N is a simple group.

Let $H/N \cong S_1 \times \cdots \times S_1$ where S_1 is a non-Abelian simple group. Since S_1 is a non-Abelian simple group, $n_{17}(S_1) \geq 18$. On the other hand, by Lemma 8, $n_{17}(S_1) \mid n_{17}(G) = 18$. Thus $n_{17}(S_1) = 18$. We have $18 = n_{17}(H/N) = n_{17}(S_1) \times \cdots \times n_{17}(S_1) = (18)^k$, where $k \geq 2$, a contradiction. Therefore, H/N is a simple group.

Since H/N is simple and $n_{17}(H/N) = 1 + 17$, by Lemma 11, $H/N \cong L_2(17)$.

Now set $\overline{H} := H/N \cong L_2(17)$ and $\overline{G} := G/N$. We have

$$L_2(17) \cong \overline{H} \cong \overline{H}C_{\overline{G}}(\overline{H})/C_{\overline{G}}(\overline{H}) \leq \overline{G}/C_{\overline{G}}(\overline{H}) = N_{\overline{G}}(\overline{H})/C_{\overline{G}}(\overline{H}) \leq \mathrm{Aut}(\overline{H}).$$

Let $K = \{x \in G | xN \in C_{\overline{G}}(\overline{H})\}$, then $G/K \cong \overline{G}/C_{\overline{G}}(\overline{H})$. Hence $L_2(17) \leq G/K \leq \mathrm{Aut}(L_2(17))$. Hence $G/K \cong L_2(17)$ or $G/K \cong \mathrm{Aut}(L_2(17))$. Let G/K isomorphic to $L_2(17)$, by Lemma 10, we have $n_2(K) = 1$, $n_3(K) = 1$, $n_{17}(K) = 1$ and $n_5(K) \mid 136$. We prove that $K = N$. Suppose that $K \neq N$. Since $N < K$ and N is a maximal solvable normal subgroup G, K is a non-solvable normal subgroup of G. Therefore K has the following normal series

$$1 \trianglelefteq N_1 \lhd H_1 \trianglelefteq K$$

such that H_1/N_1 is a non-Abelian simple group. But because $n_2(H_1/N_1) \mid n_2(K) = 1$, we get a contradiction. Thus $N = K$. Now we have $G/N \cong L_2(17)$. So $5 \in \pi(N)$ and the order of a Sylow 5-subgroup in G and N are equal. As N is normal in G thus the number of Sylow 5-subgroups of G and N are equal. Therefore, the number of Sylow 5-subgroups of N is 136. We know that N is a solvable group, thus by Lemma 10, $17 \equiv 1 \pmod 5$, a contradiction. Similarly if $G/K \cong \mathrm{Aut}(L_2(17))$ we can get a contradiction. Hence $|\pi(H/N)| \neq 3$.

Let $|\pi(H/N)| = 4$. Since $n_{17}(G) = 18$ and $n_{17}(H/N) \mid n_{17}(G)$, we conclude that $n_{17}(H/N) = 18$ and H/N is a simple K_4-group. Because $n_{17}(H/N) = 18$, then arguing as above $H/N \cong L_2(17)$. On the other hand, H/N was a K_4-group; a contradiction. If $\pi(G) = \{2, 3, 17, 19\}$, then similarly we can get a contradiction.

Case b. Let G be a K_5group, then $\pi(G) = \{2, 3, 5, 17, 19\}$. Similar to the proof of the Case a, G has the following normal series

$$1 \trianglelefteq N \lhd H \trianglelefteq G$$

such that H/N is a non-Abelian simple group or H/N is a direct product of isomorphic non-Abelian simple groups. Since G is a K_5-group, so $|\pi(H/N)| = 3$ or $|\pi(H/N)| = 4$ and or $|\pi(H/N)| = 5$. Similar to the proof of the Case a, if $|\pi(H/N)| = 3$, then $\pi(H/N) = \{2, 3, 5\}$ or $\{2, 3, 17\}$ and we get a contradiction. If $|\pi(H/N)| = 4$ or 5, then $17 \in \pi(H/N)$ and similar to the Case a, we can get a contradiction.

Therefore, $\pi(G) = \{2, 3, 17\}$. The group G has the following normal series

$$1 \unlhd N \lhd H \unlhd G$$

uch that H/N is a simple K_3-group or H/N is a direct product of simple K_3-groups. Similar to the Case a, it is easy to prove that $H/N \cong L_2(17)$. Set $\overline{H} := H/N \cong L_2(17)$ and $\overline{G} := G/N$. Thus we have

$$L_2(17) \cong \overline{H} \cong \overline{H}C_{\overline{G}}(\overline{H})/C_{\overline{G}}(\overline{H}) \leq \overline{G}/C_{\overline{G}}(\overline{H}) = N_{\overline{G}}(\overline{H})/C_{\overline{G}}(\overline{H}) \leq \mathrm{Aut}(\overline{H}).$$

Let $K = \{x \in G \mid xK \in C_{\overline{G}}(\overline{H})\}$, then $G/K \cong \overline{G}/C_{\overline{G}}(\overline{H})$. Hence $L_2(17) \leq G/K \leq \mathrm{Aut}(L_2(17))$. So G/K isomorphic to $L_2(17)$ or $\mathrm{Aut}(L_2(17))$. Let G/K isomorphic to $L_2(17)$. By Lemma 10, $n_p(K) = 1$ for every prime $p \in \pi(G)$. Then K is a nilpotent subgroup of G.

We claim that $K = 1$. Let Q be a Sylow $q-$subgroup of K, since K is nilpotent, then Q is normal in G. We have that if $P \in \mathrm{Syl}_p(G)$, then Q normalizes P and so if $p \neq q$, then $P \leq N_G(Q) = G$. Also we note that KP/K is a Sylow p-subgroup of G/K. On the other hand, if $R/K = N_{G/K}(\ KP/K)$, then $R = N_G(P)K$. We know that $n_p(G) = n_p(G/K)$, so $|G : R| = |G : N_G(P)|$. Thus $R = N_G(P)$ and therefore, $K \leq N_G(P)$. Because K is nilpotent, so Q normalizes P and $Q \leq N_G(P)$. Since $P \leq N_G(Q)$ and $Q \leq N_G(P)$, that implies that $[P, Q] \leq P$ and $[P, Q] \leq Q$, then $[P, Q] \leq P \cap Q = 1$. So $P \leq C_G(Q)$ and $Q \leq C_G(P)$, in other words, P and Q centralize each other. Let $C = C_G(Q)$, then C contains a full Sylow $p-$subgroup of G for all primes p different from q, and thus $|G : C|$ is a power of q. Now let S be a Sylow $q-$subgroup of G. Then $G = CS$. Furthermore, if $Q > 1$, then $C_Q(S)$ is nontrivial, and we see that $C_Q(S) \leq Z(G)$. Since by the assumption $Z(G) = 1$, it follows that $Q = 1$. Since q is arbitrary, $K = 1$. Therefore, G is isomorphic to $L_2(17)$. Arguing as above if G/K isomorphic to $\mathrm{Aut}(L_2(17))$, G is isomorphic to $\mathrm{Aut}(L_2(17))$. □

Conjecture. Let G be a finite group such that $\mathrm{NS}(G) = \mathrm{NS}(M)$ and $|G| = |M|$ where M is an arbitrary simple group. Then $G \cong M$.

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