ON SYLOW NUMBERS OF SOME FINITE GROUPS

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Abstract. Let $G$ be a finite group, let $\pi(G)$ be the set of primes $p$ such that $G$ contains an element of order $p$, and let $n_p(G)$ be the number of Sylow $p$-subgroups of $G$, that is, $n_p(G) = |\text{Syl}_p(G)|$. Set $\text{NS}(G) := \{n_p(G) | p \in \pi(G)\}$. In this paper, we will show that if $|G| = |S|$ and $\text{NS}(G) = \text{NS}(S)$, where $S$ is one of the groups: the special projective linear groups $L_3(q)$, with $5 \mid (q-1)$, the projective special unitary groups $U_3(q)$, the sporadic simple groups, the alternating simple groups, and the symmetric groups of degree prime $r$, then $G$ is isomorphic to $S$. Furthermore, we will show that if $G$ is a finite centerless group and $\text{NS}(G) = \text{NS}(L_2(17))$, then $G$ is isomorphic to $L_2(17)$, and or $G$ is isomorphic to $\text{Aut}(L_2(17))$.

Keywords: finite group, simple group, Sylow subgroup.

1. Introduction

If $n$ is an integer, then we denote by $\pi(n)$ the set of all prime divisors of $n$. Let $G$ be a finite group. Denote by $\pi(G)$ the set of primes $p$ such that $G$ contains an element of order $p$. A finite group $G$ is called a simple $K_n$-group, if $G$ is a simple group with $|\pi(G)| = n$. Also denote by $(a,b)$ the greatest common divisor of positive integers $a$ and $b$. Throughout this paper, we denote by $n_p(G)$ the number of Sylow $p$-subgroup of $G$, that is, $n_p(G) = |\text{Syl}_p(G)|$. We denote by $n_{2^t}$ the largest positive odd divisor of the positive integer $n$. All other notations are standard, and we refer to [15], for example.

In 1992, Bi [10] showed that $L_2(p^k)$ can be characterized just by the orders of the order of normalizer of its Sylow subgroups. In other words, if $G$ is a group and $|N_G(P)| = |N_{L_2(p^k)}(Q)|$, where $P \in \text{Syl}_r(G)$ and $Q \in \text{Syl}_r(L_2(p^k))$ for every prime
when

where

A

replaced with

p

any prime

Lemma 4. Let $S$ be a prime not exceeding $p$. In this paper we have done this work for the groups: $L_3(q), U_3(q), A_5, S_5, P$, the sporadic simple groups [2] and $^2D_n(p^k)$ [1].

Set $NS(G) := \{ n_p(G) \mid p \in \pi(G) \}$, where $G$ is a centerless group. Clearly if $n_p(G) = n_p(S)$ for every prime $p$ and $|G| = |S|$, then $|NS(G)| = |NS(S)|$, where $P \in Syl_p(G)$ and $Q \in Syl_p(S)$. Thus by the above references, $G \cong S$. But when $NS(G) = NS(S)$ and $|G| = |S|$, then we don’t know that $n_p(G) = n_p(S)$ for any prime $p$.

In [3], it is proved that if $NS(G) = NS(L_2(q))$ and $|G| = |L_2(q)|$, then $G \cong L_2(q)$. In this paper we have done this work for the groups: $L_3(q)$, where $5 \nmid (q-1), U_3(q)$, the sporadic simple groups, the alternating simple groups and the symmetric groups $S_r$, where $r$ is prime. In fact, we prove the following main theorem.

**Main Theorem.** Let $G$ be a finite group. If $NS(G) = NS(S) = |G| = |S|$, where $S \in \{ L_3(q), U_3(q), A_5, S_5, P \}$ with $r$ is prime, $M$ is the sporadic simple group, and $5 \nmid (q-1)$ when $S = L_3(q)$, then $G \cong S$.

Furthermore, in this paper we pose another characterization by $NS(G)$. In this new characterization, we want to know that if remove the condition order, then whether we can say $G \cong S$.

In the Theorem 6, we proved that if $G$ is a centerless group and $NS(G) = NS(L_2(17))$, then $G \cong L_2(17)$ or $G \cong Aut(L_2(17))$. Thus if the condition order replaced with $Z(G) = 1$, then it seems that this characterization works. This type of characterization is known for the groups: $A_5$ and $A_6$ [4] and $L_2(8)$ [3].

We denote by $k(\text{NS}(G))$ the number of isomorphism classes of the finite groups $H$ satisfying $NS(G) = NS(H)$. A finite group $G$ is called $n$-recognizable by $NS(G)$ if $k(\text{NS}(G)) = n$. In the [4, 3], it is proved that $k(\text{NS}(A_6)) = 5$ and $k(\text{NS}(L_2(8))) = 2$.

2. Preliminary Results

Lemma 1. [9] Let $G$ be a finite group. If $|G| = |L_n(q)(R_2)|$ for every prime $r$, where $R_1 \in Syl_1(G)$ and $R_2 \in Syl_1(L_n(q))$, then $G \cong L_n(q)$.

Lemma 2. [2] Let $S$ be sporadic simple group and $p$ be the greatest element of $\pi(S)$. If $G$ is a finite group such that $|G| = |S|$ and $|G| = |N_{S}(P_2)|$, where $P_1 \in Syl_p(G)$ and $P_2 \in Syl_p(S)$, then $G \cong S$.

Lemma 3. [12] Let $G$ be a finite group, $n \geq 4$ with $n \neq 8, 10$, and $r$ the greatest prime not exceeding $n$. If $|G| = |A_n|$ and $|G| = |N_{A_n}(S)|$, where $R \in Syl_1(G)$ and $S \in Syl_1(A_n)$, then $G \cong A_n$.

Lemma 4. [12] Let $G$ be a finite group. If $|N_{G}(P_1)| = |N_{A_n}(P_2)|$ for every prime $p$, where $P_1 \in Syl_p(G)$ and $P_2 \in Syl_p(A_n)$, then $G \cong A_n$.

Lemma 5. [6] Let $G$ be a finite group. If $|G| = |S_r|$, where $r$ is prime and $|G| = |N_{S_r}(S_r)|$, where $R \in Syl_1(G)$ and $S \in Syl_1(S_r)$, then $G \cong S_r$.

Lemma 6. [14, 11] Let $G$ be a finite group. If $|N_{G}(P_1)| = |U_{n,q}(P_2)|$ for every prime $p$, where $P_1 \in Syl_p(G)$ and $P_2 \in Syl_p(U_{n,q}(q))$, then $G \cong U_{n,q}(q)$.

Lemma 7. [20] Let $G = L_n(q)$, where $q = p^\alpha$. The orders of maximal tori of $L_n(q)$ are

$$\Pi_{i=1}^k (q^{r_i} - 1)/(q-1)(n, q - 1), (r_1, ..., r_k) \in Par(n).$$
Lemma 8. [18, Theorem 9.3.1] Let \( G \) be a finite solvable group and \(|G| = m \cdot n\), where \( m = p_1^{a_1} \cdots p_r^{a_r} \), \( (m, n) = 1 \). Let \( \pi = \{p_1, \ldots, p_r\} \) and \( h_m \) be the number of \( \pi \)-Hall subgroups of \( G \). Then \( h_m = q_1^{3^1} \cdots q_s^{3^s} \) satisfies the following conditions for all \( i \in \{1, 2, \ldots, s\}\):

1. \( q_i^{3^i} \equiv 1 \pmod{p_j} \), for some \( p_j \)
2. The order of some chief factor of \( G \) is divisible by \( q_i^{3^i} \).

Lemma 9. [19] If \( G \) is a simple \( K_3 \)-group, then \( G \) is isomorphic to one of the following groups: \( A_5 \), \( A_6 \), \( L_2(7) \), \( L_2(8) \), \( L_2(17) \), \( L_3(3) \), \( U_3(3) \) or \( U_4(2) \).

Lemma 10. [16, Lemma 1] Let \( G \) be a finite group and \( M \) be a normal subgroup of \( G \). Then both \( n_p(M) \) and \( n_p(G/M) \) divide \( n_p(G) \) and moreover \( n_p(M) \cdot n_p(G/M) \) divides \( n_p(G) \) for every prime \( p \).

Lemma 11. [5, Corollary 2.5] Let \( r \) be a prime number, and let \( G \) be a simple such that \( n_r(G) = r + 1 \). Then \( G \) is isomorphic to \( L_2(r) \).

By the Sylow’s theorem implies that if \( p \) is prime \( n_p = 1 + pk \). If \( p = 2 \), then \( n_2 \) is odd. If \( p \in \pi(G) \), then

\[
\begin{align*}
   p & \mid (n_p - 1) \\
   (p, n_p) & = 1.
\end{align*}
\]

In the proof of the main results, we often apply \((*)\) and the above comments.

3. Main Results

Theorem 1. Let \( G \) be a finite group \( NS(G) = NS(L_3(q)) \) and \(|G| = |L_3(q)| \) where \( 5 \nmid (q - 1) \), then \( G \cong L_3(q) \).

**Proof.** First, we get \( NS(L_3(q)) \), where \( q \) is prime power. Let \( q = p^n \) and \( p \neq 2 \). To find the number of Sylow \( p \)-subgroups in \( L_3(q) \) first, look at \( SL_3(q) \). The normalizer of a Sylow \( p \)-subgroup is the set of upper triangular matrices with determinant 1, so the order of the normalizer is \( q^3(q^3 - 1)^2 \). The order of the whole group \( SL_3(q) \) is \( q^3(q^3 - 1)(q^3 - 1) \). Therefore, the number of Sylow \( p \)-subgroups is \( (q + 1)(q^2 + q + 1) \). This will be the same as the number of Sylow \( p \)-subgroups of \( L_3(q) \) because the canonical homomorphism from \( SL_3(q) \) to \( L_3(q) \) yields a bijection on Sylow \( p \)-subgroups.

To find the number of Sylow 2-subgroups in \( L_3(q) \), we consider the following cases.

(a) If \( p \equiv 1 \pmod{4} \), a Sylow 2-subgroups lies inside a group \( C_{(q-1)} \times C_{(q-1)} : S_3 \). Its normalizer has size \( 2(q - 1)|q - 1|_2 \), where \(|q - 1|_2 \) is the largest power of 2 dividing \( q - 1 \). Thus the number of Sylow 2-subgroups are \( q^3[(q + 1)(q^3 - 1)]_2 \).

(b) if \( p \equiv 3 \pmod{4} \), a Sylow 2-subgroups lies inside a group \( C_{(q^2 - 1)} : S_2 \). Its normalizer has size \( 2(q - 1)|q + 1|_2 \). Thus the number of Sylow 2-subgroups are \( q^3[(q + 1)(q^3 - 1)]_2 \).

To find the number of Sylow 3-subgroups in \( L_3(q) \), we consider the following cases.

(a) if \( p \equiv 2 \pmod{3} \), then a Sylow is a characteristic subgroup of a torus \( C_{(q^2 - 1)} \). It is normalized by the torus normalizer of size \( 2(q^2 - 1) \). Thus the number of Sylow 3-subgroups are \( q^3(q^3 - 1)/2 \).
(b) if $p \equiv 1 \pmod{3}$, then a Sylow lies inside a group $C(q^{r-1}) \times C(q^{r-1}) : S_3$. It is equal to its own normalizer. Thus the number of Sylow $3$-subgroups are $q^{3}(q + 1)(q^2 + q + 1)/6$.

Now let that $r$ is not $p$, the field characteristic, and $r > 3$. Then $r$ divides exactly one of $q + 1$, $q - 1$, $q^2 + q + 1$ this in turn implies that a Sylow $3$ lies inside a maximal torus. By Lemma 7, the maximal tori are one of the following subgroups.

1. $C(q^{2r-1})$
2. $C(q^{-1}) \times C(q^{-1})$
3. $C_{q^2 + q + 1}$

So $S$ is normalized by $N_G(T)$ where $T$ is the torus, now we consider the following cases.

1. $N_G(T) = C(q^{2r-1}).C_2$. This lies inside a copy of $GL_2(q)$ which does not normalize the torus, with a little checking one can be sure that $N_G(T)$ really is the Sylow normalizer. Thus if $r \in \pi(q + 1)$, then the number of Sylow $r$-subgroups are $q^{3}(q^2 - 1)/2$.

2. $N_G(T) = C(q^{-1}) \times C(q^{-1}) : S_3$. By [8], this is a maximal subgroup so we have our Sylow normalizer. Thus if $r \in \pi(q - 1)$, then the number of Sylow $r$-subgroups are $q^{3}(q + 1)(q^2 + q + 1)/6$.

3. $N_G(T) = C_{q^2 + q + 1} : C_2$. By [8], this is a maximal subgroup so we have our Sylow normalizer. Thus if $r \in \pi(q^2 + q + 1)$, then the number of Sylow $r$-subgroups are $q^{3}(q - 1)(q^2 - 1)/3$.

Now assume that $p = 2$. Arguing as above $n_2 = (q + 1)(q^2 + q + 1)$. If $r \in \pi(q + 1)$, then the number of Sylow $r$-subgroups are $q^{3}(q^2 - 1)/2$. If $r \in \pi(q - 1)$, then the number of Sylow $r$-subgroups are $q^{3}(q + 1)(q^2 + q + 1)/6$. If $r \in \pi(q^2 + q + 1)$, then the number of Sylow $r$-subgroups are $q^{3}(q - 1)(q^2 - 1)/3$. Thus we need to get only the number of Sylow $3$-subgroups.

If $q \equiv 2 \pmod{3}$, then by [22] a Sylow $3$-subgroup $S$ is cyclic and $N_G(S) = C(q^{-1}) \times D_{2(q^2+1)}$. Thus the number of Sylow $3$-subgroups are $q^{3}(q - 1)/2$. If $q \equiv 4$ or $7 \pmod{9}$, then a Sylow $3$-subgroup $S$ is $3^2.Q_8$. So the number of Sylow $3$-subgroups are $q^{3}(q^2 - 1)(q^2 - 1)/216$.

If $q \equiv 1 \pmod{3}$, then $N_G(S)$ is a quotient of $C_{3^2} \times C_{3^2} : S_3$ by a subgroup of order $3$. Then the number of Sylow $3$-subgroups are $q^{3}(q^2 - 1)(q^2 + q - 1)/162$.

Therefore, $NS(L_3(q)) = \{(q + 1)(q^2 + q - 1)/2, q^3(q + 1)(q^2 + q + 1)/6, q^3(q - 1)(q^2 - 1)/3\}$. or $\{(q + 1)(q^2 + q + 1), q^3(q^2 - 1)(q^2 - 1)/2, q^3(q + 1)(q^2 + q + 1)/6, q^3(q - 1)(q^2 - 1)/3\}$, and or $\{(q + 1)(q^2 + q + 1), q^3(q^2 - 1)(q^2 + q - 1)/162, q^3(q^2 - 1)/2, q^3(q + 1)(q^2 + q + 1)/6, q^3(q - 1)(q^2 - 1)/3\}$.

Now we will show that $G \cong L_3(q)$. Let $q = p^a$ and $p \neq 2$. By $NS(G)$, $(q^2 + q + 1)$ is the only member of $NS(G)$ such that $p \nmid (q + 1)(q^2 + q + 1)$. Thus by the Sylow’s theorem, $n_p = (q + 1)(q^2 + q + 1)$. Also $q^3 \nmid (q + 1)(q^2 - 1)$ is the only odd member of $NS(G)$, so $n_2 = q^3(q + 1)(q^2 - 1)/2$.

If $r \in \pi(q + 1)$, then by the Sylow’s theorem and $NS(G)$, $n_r = q^3(q^2 - 1)/2$.

If $r \in \pi(q - 1)$, then $n_r = (q + 1)(q^2 + q + 1)$ or $q^3(q + 1)(q^2 + q + 1)/6$. Assume that $n_r = (q + 1)(q^2 + q + 1)$. By the Sylow’s theorem, $n_r = 1 + rk$, where $k$ is a positive number. So $rk = q^2 + 2q^2 + 2q = q(q^2 + 2q + 2)$. If $r = p$, then we get a contradiction. Thus $r \nmid (q^2 + 2q + 2)$. Since $r \in \pi(q - 1)$ and $r \nmid (q + 3)(q - 1) = q^2 + 2q - 3$, $r \nmid 5$. By the assumption $5 \nmid (q - 1)$, which is a contradiction.
Let the assumption be made. By the Sylow's theorem, \( n_r = n_r(L_2(q)) \) for every \( r \). Similar to the above discussion if \( p = 2 \), \( n_r(G) = n_r(L_2(q)) \) for every \( r \) where \( 5 \nmid (q - 1) \). By the assumption \( |G| = |L_3(q)| \), so \( |N_G(R_1)| = |N_{L_3(q)}(R_2)| \) for every prime \( r \), where \( R_1 \in \text{Syl}_r(G) \) and \( R_2 \in \text{Syl}_r(L_3(q)) \). By Lemma 1, \( G \cong L_3(q) \).

**Theorem 2.** Let \( G \) be a finite group such that \( \text{NS}(G) = \text{NS}(U_3(q)) \) and \( |G| = |U_3(q)| \), then \( G \cong U_3(q) \).

**Proof.** First we find the number of Sylow \( r \)-subgroups in \( U_3(q) \) for every prime \( r \), where \( q = 2^k \). By [22], the maximal subgroups of \( U_3(q) \) are:

1. \( q^3 : (q^2 - 1)/(3, q + 1) \),
2. \( (q + 1)/(3, q + 1) \times U_2(q) \),
3. \( (q + 1)/(3, q + 1) : S_3 \),
4. \( (q^2 - q + 1)/(3, q + 1) : 3 \),
5. \( U_3(2^m) \) if \( k/m > 3 \) is odd prime,
6. \( U_3(2^m).3 \) if \( m \) is odd and \( k = 3m \).

Put \( Q_r \) a Sylow \( r \)-subgroup of \( U_3(q) \) for \( r \in \pi(U_3(q)) \). By the maximal subgroups of \( U_3(q) \) we have \( N_{U_3(q)}(Q_2) = q^3 : (q^2 - 1)/(3, q + 1) \), \( N_{U_3(q)}(Q_r) = (q + 1) \times D_{2(q-1)}(3, q + 1) \) for \( r \in \pi(q + 1) \), \( N_{U_3(q)}(Q_r) = (q + 1)^2 : S_3 \) for every \( r \) in \( \pi(q - 1) \) and \( N_{U_3(q)}(Q_r) = (q^2 - q + 1) : 3 \) for \( r \in \pi(q^2 - q + 1) \). Thus \( n_2 = q^3 + 1, n_r = q^3(q - 1)(q^2 - q + 1)/6 \) for \( r \in \pi(q + 1) \), \( n_r = q^3(q^2 - q + 1)/2 \) for \( r \in \pi(q - 1) \) and \( n_r = q^3(q^2 - q + 1)(q - 1)/3 \) for \( r \in \pi(q^2 - q + 1) \).

Therefore, if \( p = 3 \), then \( \text{NS}(U_3(q)) = \{ q^3 + 1, q^3(q - 1)(q^2 - q + 1)/6, q^3(q^2 - q + 1)(q - 1)/3 \} \).

Let \( q = p^k \) where \( p \neq 2 \). By [8], we have the maximal subgroups of \( U_3(q) \). Similar to the above discussion, \( n_p = q^3 + 1 \), \( n_r = q^3(q^2 - q + 1)/6 \) for \( r \in \pi(q + 1) \), \( n_r = q^3(q - 1)(q^2 - q + 1)/2 \) for \( r \in \pi(q - 1) \) and \( n_r = q^3(q^2 - q + 1)(q - 1)/3 \) for \( r \in \pi(q^2 - q + 1) \).

Therefore, if \( r \equiv 1 \pmod{4} \), then \( n_2 = q^3(q^2 - q + 1)(q - 1)/2 \). If \( r \equiv 3 \pmod{4} \), then \( n_2 = q^3(q^2 - q + 1)(q - 1)/3 \).

Now we will show that if \( q = 2^p \), then \( G \cong U_3(q) \). By \( \text{NS}(G) \), \( q + 1 \) is the only odd member of \( \text{NS}(G) \), so \( n_2 = q^3 + 1 \). If \( r \in \pi(q + 1) \), then by the Sylow's theorem and \( \text{NS}(G) \), \( n_r = q^3(q^2 - q + 1)/6 \).

If \( r \in \pi(q - 1) \), then \( n_r = q^3 + 1 \) or \( q^3(q^2 - q + 1)/2 \). Assume that \( n_r = q^3 + 1 \). By the Sylow's theorem \( n_r = 1 + rk \), where \( k \) is a positive number. So \( rk = q^3 \), then \( r = p \), which is a contradiction. Thus \( n_r = q^3(q^2 - q + 1)/6 \).

If \( r \in \pi(q - 1) \), then by \( \text{NS}(G) \), \( n_r = q^3(q^2 - q + 1)/3 \). Thus \( n_r(G) = n_r(U_3(q)) \) for every \( r \). Similar to the above discussion if \( q = p^k \) where \( p \neq 2 \), \( n_r(G) = n_r(U_3(q)) \) for every \( r \). Now by the assumption \( |G| = |U_3(q)| \), so \( |N_G(R_1)| = |N_{U_3(q)}(R_2)| \) for every prime \( r \), where \( R_1 \in \text{Syl}_r(G) \) and \( R_2 \in \text{Syl}_r(U_3(q)) \). By Lemma 6, \( G \cong U_3(q) \).

**Theorem 3.** Let \( G \) be a finite group such that \( \text{NS}(G) = \text{NS}(M) \) and \( |G| = |M| \) where \( M \) is the sporadic simple group, then \( G \cong M \).

**Proof.** By Table 1 and 2 in [2] we can compute \( \text{NS}(M) \) for every sporadic simple group \( M \). We claim that if \( p \) is the greatest element of \( \pi(M) \), then \( n_p(G) = n_p(M) \).
By \(\text{NS}(M)\) we have \(p \mid n_p\) for every prime \(q \neq p\). By \((*)\) we know \((p, n_p) = 1\). Thus the only member of the \(\text{NS}(G)\) that satisfy in the \((p, n_p) = 1\) is \(n_p(M)\). Therefore, \(n_p(G) = n_p(M)\). By the assumption \(|G| = |M|\), so \(|N_{G_1}(Q_1)| = |N_{M_1}(Q_2)|\), where \(Q_1 \in \text{Syl}_p(G)\) and \(Q_2 \in \text{Syl}_p(M)\). Now by Lemma 2, \(G \cong M\).

\[\quad\]

**Theorem 4.** Let \(G\) be a finite group such that \(\text{NS}(G) = \text{NS}(A_6)\) and \(|G| = |A_6|\)
where \(A_6\) is the alternating simple group, then \(G \cong A_6\).

**Proof.** Let \(p \neq 2\) be prime and \(n\) be written in the scale of \(p\) in the form

\[
n = a_0 + a_1 p + a_2 p^2 + \cdots + a_k p^k.
\]

By [17] the number of Sylow subgroups of order \(p^m\) in the alternating group of degree \(n\) is \(n!/[a_1! a_2! \cdots a_k! p^m (p-1)^m]\). If \(p = 2\), then the number of Sylow subgroups of order \(p^m\) is \(n!/[2p^m]\). We claim that if \(r\) is the greatest element of \(\pi(A_2)\), then \(n_r(G) = n_r(A_2)\). Clearly \(r \mid n!/[a_1! a_2! \cdots a_k! p^m (p-1)^m]\), since \(p \neq 2\). Also \(r \mid n!/[2p^m]\) if \(\pi = 1\), so \(n_r(G) = n_r(A_2)\).

Let \(n \neq 8, 10\). By the assumption \(|G| = |A_6|\), so \(|N_{G_1}(Q_1)| = |N_{A_6}(Q_2)|\), where \(Q_1 \in \text{Syl}_p(G)\) and \(Q_2 \in \text{Syl}_p(A_6)\). By Lemma 3, \(G \cong A_6\).

Now let \(n = 8\) or 10. We prove, for example if \(n = 10\), then \(G \cong A_{10}\) and afterwards if \(n = 8\), then \(G \cong A_8\).

If \(n = 10\), then \(\text{NS}(G) = \text{NS}(A_{10}) = \{4536, 11200, 14175, 14400\}\). Clearly \(n_2(G) = 14400\). Since 14175 is odd by the Sylow theorem \(n_2(G) = 14175\). Also because 3 divides 4536 and 14400, so \(n_3(G) = 11200\).

Therefore, by arguing as above \(n_3(G) = 4536\). Thus \(n_p(G) = n_p(A_{10})\) for every prime \(p\). Therefore, \(|N_{G_1}(P_1)| = |N_{A_{10}}(P_2)|\) for every prime \(p\), where \(P_1 \in \text{Syl}_p(G)\) and \(P_2 \in \text{Syl}_p(A_{10})\). By Lemma 4, \(G \cong A_{10}\). \(\square\)

**Theorem 5.** Let \(G\) be a finite group such that \(\text{NS}(G) = \text{NS}(S_r)\) and \(|G| = |S_r|\)
where \(S_r\) is the symmetric group of degree prime \(r\). Then \(G \cong S_r\).

**Proof.** By [17] and similar to the proof of Theorem 4, \(|N_{G_1}(Q_1)| = |N_{S_r}(Q_2)|\) where \(Q_1 \in \text{Syl}_p(G)\) and \(Q_2 \in \text{Syl}_p(S_r)\). Now it concludes from Lemma 5. \(\square\)

**Theorem 6.** Let \(G\) be a finite centerless group and \(\text{NS}(G) = \text{NS}(L_2(17))\) = \{18, 136, 153\}. Then \(G \cong L_2(17)\), and or \(G \cong \text{Aut}(L_2(17))\).

**Proof.** First, we prove that \(\pi(G) = \{2, 3, 17\}\). By the Sylow’s theorem, \(n_p \mid |G|\) for every \(p\), hence by \(\text{NS}(G)\), \(\{2, 3, 17\} \subseteq \pi(G)\). On the other hand by \((*)\), if \(p \in \pi(G)\), then \(p \mid (n_p - 1)\) and \((p, n_p) = 1\), which implies that \(p \in \{2, 3, 5, 17, 19\}\). By the Sylow’s theorem, \(n_2(G) = 153\), \(n_3(G) = 136\) and \(n_7(G) = 18\). If \(5 \in \pi(G)\), then \(n_5(G) = 136\) and if \(19 \in \pi(G)\), then \(n_{19}(G) = 153\). We prove that \(G\) is a non-solvable group. If \(G\) is a solvable group, then since \(n_7(G) = 18\), by Lemma 8, \(9 \equiv 1(\mod 17)\), a contradiction. In follow, we will show that \(G\) is not a \(K_4\) or \(K_5\)-group. Thus \(G\) is a \(K_3\)-group. We consider the following cases.

**Case a.** Let \(G\) be a \(K_1\)-group. Because \(\{2, 3, 17\} \subseteq \pi(G)\), \(\pi(G) = \{2, 3, 5, 17\}\) or \(\{2, 3, 17, 19\}\). Let \(\pi(G) = \{2, 3, 5, 17\}\). Since \(G\) is a finite group, it has a chief series. Let \(1 = N_0 \leq N_1 \leq \cdots \leq N_{i-1} \leq N_i = G\) be a chief series of \(G\). Since \(G\) is a non-solvable group, there exists a maximal number of non-negative integer \(i\) such that \(N_i/N_{i-1}\) is a simple group or the direct product of isomorphic simple groups and \(N_{i-1}\) is a maximal soluble normal subgroup of \(G\). Now set \(N_i = H\) and \(N_{i-1} := N\). Hence \(G\) has the following normal series

\[
n = a_0 + a_1 p + a_2 p^2 + \cdots + a_k p^k.
\]
such that $H/N$ is a non-Abelian simple group, or $H/N$ is a direct product of isomorphic non-Abelian simple groups. Since $G$ is a $K_4$-group, $|\pi(H/N)| = 3$ or $|\pi(H/N)| = 4$. If $|\pi(H/N)| = 3$, then by Lemma 9, $\pi(H/N) = \{2, 3, 5\}$ or $\{2, 3, 17\}$.

If $\pi(H/N) = \{2, 3, 5\}$, then $H/N$ is isomorphic to $A_5$, $A_6$ or $U_4(2)$, and or $H/N$ is a direct product of isomorphic to $A_5$, $A_6$ or $U_4(2)$, by Lemma 9. By $n_p(H/N) | n_p(G)$ for every prime $p \in \pi(G)$, it is easy to check that this is impossible.

Let $\pi(H/N) = \{2, 3, 17\}$. We will show $H/N$ is isomorphic to $L_2(17)$. Since $17 \mid |H/N|$ by Lemma 8, $n_{17}(H/N) | n_5(G) = 17 + 1$. Thus by Sylow’s theorem $n_5(H/N) = 18$. We prove that $H/N$ is a simple group.

Let $H/N \cong S_1 \times \cdots S_1$ where $S_1$ is a non-Abelian simple group. Since $S_1$ is a non-Abelian simple group, $n_{17}(S_1) \geq 18$. On the other hand, by Lemma 8, $n_{17}(S_1) \mid n_{17}(G) = 18$. Thus $n_{17}(S_1) = 18$. We have $18 = n_{17}(H/N) = n_{17}(S_1) \times \cdots \times n_{17}(S_1) = (18)^k$, where $k \geq 2$, a contradiction. Therefore, $H/N$ is a simple group.

Since $H/N$ is simple and $n_{17}(H/N) = 1 + 17$, by Lemma 11, $H/N \cong L_2(17)$. Now set $\overline{H} := H/N \cong L_2(17)$ and $\overline{G} := G/N$. We have $L_2(17) \cong \overline{H} \cong \overline{G}/C_{\overline{G}}(\overline{H}) \leq \overline{G}/C_{\overline{G}}(\overline{H}) = N_{\overline{G}}(\overline{H})/C_{\overline{G}}(\overline{H}) \leq \text{Aut}^*(\overline{H})$.

Let $K = \{x \in G | xN \subset C^*(\overline{H})\}$, then $G/K \cong \overline{G}/C^*(\overline{H})$. Hence $L_2(17) \leq G/K \leq \text{Aut}(L_2(17))$. Hence $\overline{G}/K \cong L_2(17)$ or $G/K \cong \text{Aut}(L_2(17))$. Let $G/K$ isomorphic to $L_2(17)$, by Lemma 10, we have $n_2(K) = 1$, $n_3(K) = 1$, $n_{17}(K) = 1$ and $n_5(K) = 136$. We prove that $K = N$. Suppose that $K \neq N$. Since $N < K$ and $N$ is a maximal solvable normal subgroup $G$, $K$ is a non-solvable normal subgroup of $G$. Therefore $K$ has the following normal series

$$1 \leq N_1 \triangleleft H_1 \leq K$$

such that $H_1/N_1$ is a non-Abelian simple group. But because $n_2(H_1/N_1) | n_2(K) = 1$, we get a contradiction. Thus $N = K$. Now we have $G/N \cong L_2(17)$. So $5 \in \pi(N)$ and the order of a Sylow 5-subgroup in $G$ and $N$ are equal. As $N$ is normal in $G$ thus the number of Sylow 5-subgroups of $G$ and $N$ are equal. Therefore, the number of Sylow 5-subgroups of $N$ is 136. We know that $N$ is a solvable group, thus by Lemma 10, $17 \equiv 1 \pmod{5}$, a contradiction. Similarly if $G/K \cong \text{Aut}(L_2(17))$ we can get a contradiction. Hence $|\pi(H/N)| \neq 3$.

Let $|\pi(H/N)| = 4$. Since $n_{17}(G) = 18$ and $n_{17}(H/N) | n_{17}(G)$, we conclude that $n_{17}(H/N) = 18$ and $H/N$ is a simple $K_4$-group. Because $n_{17}(H/N) = 18$, then arguing as above $H/N \cong L_2(17)$. On the other hand, $H/N$ was a $K_4$-group; a contradiction. If $\pi(G) = \{2, 3, 17, 19\}$, then similarly we can get a contradiction.

**Case b.** Let $G$ be a $K_4$-group, then $\pi(G) = \{2, 3, 5, 17, 19\}$. Similar to the proof of the Case a, $G$ has the following normal series

$$1 \leq N_1 \triangleleft H_1 \leq K$$

such that $H_1/N_1$ is a non-Abelian simple group or $H/N$ is a direct product of isomorphic non-Abelian simple groups. Since $G$ is a $K_4$-group, so $|\pi(H/N)| = 3$ or $|\pi(H/N)| = 4$ and or $|\pi(H/N)| = 5$. Similar to the proof of the Case a, if $|\pi(H/N)| = 3$, then $\pi(H/N) = \{2, 3, 5\}$ or $\{2, 3, 17\}$ and we get a contradiction. If $|\pi(H/N)| = 4$ or $5$, then $17 \in \pi(H/N)$ and similar to the Case a, we can get a contradiction.
Therefore, $\pi(G) = \{2, 3, 17\}$. The group $G$ has the following normal series

$$1 \leq N \trianglelefteq H \leq G$$

such that $H/N$ is a simple $K_3$–group or $H/N$ is a direct product of simple $K_3$–groups. Similar to the Case a, it is easy to prove that $H/N \cong L_2(17)$. Set $\overline{H} := H/N \cong L_2(17)$ and $\overline{G} := G/N$. Thus we have

$$L_2(17) \cong \overline{H} \cong \overline{H}/C_{\overline{G}}(\overline{H})/C_{\overline{G}}(\overline{H}) \leq \overline{G}/C_{\overline{G}}(\overline{H}) = N_{\overline{G}}(\overline{H})/C_{\overline{G}}(\overline{H}) \leq \text{Aut}(\overline{H}).$$

Let $K = \{x \in G \mid xK \in C_{\overline{G}}(\overline{H})\}$, then $G/K \cong \overline{G}/C_{\overline{G}}(\overline{H})$. Hence $L_2(17) \leq G/K$ isomorphic to $L_2(17)$ or $\text{Aut}(L_2(17))$. Let $G/K$ isomorphic to $L_2(17)$. By Lemma 10, $n_p(K) = 1$ for every prime $p \in \pi(G)$. Then $K$ is a nilpotent subgroup of $G$.

We claim that $K = 1$. Let $Q$ be a Sylow $q$–subgroup of $K$, since $K$ is nilpotent, then $Q$ is normal in $G$. We have that if $P \in \text{Syl}_q(G)$, then $Q$ normalizes $P$ and so if $p \neq q$, then $P \leq N_G(Q) = G$. Also we note that $K P/K$ is a Sylow $p$–subgroup of $G/K$. On the other hand, if $R/K = N_{G/K}(K P/K)$, then $R = N_G(P)K$. We know that $n_p(G) = n_p(G/K)$, so $|G : R| = |G : N_G(P)|$. Thus $R = N_G(P)$ and therefore, $K \leq N_G(P)$. Because $K$ is nilpotent, so $Q$ normalizes $P$ and $Q \leq N_G(P)$. Since $P \leq N_G(Q)$ and $Q \leq N_G(P)$, that implies that $|P, Q| \leq P$ and $|P, Q| \leq Q$, then $|P, Q| \leq P \cap Q = 1$. So $P \leq C_G(Q)$ and $Q \leq C_G(P)$, in other words, $P$ and $Q$ centralize each other. Let $C = C_G(Q)$, then $C$ contains a full Sylow $p$–subgroup of $G$ for all primes $p$ different from $q$, and thus $|G : C|$ is a power of $q$. Now let $S$ be a Sylow $q$–subgroup of $G$. Then $G = CS$. Furthermore, if $Q > 1$, then $C_G(S)$ is nontrivial, and we see that $C_G(S) \leq Z(G)$. Since by the assumption $Z(G) = 1$, it follows that $Q = 1$. Since $q$ is arbitrary, $K = 1$. Therefore, $G$ is isomorphic to $L_2(17)$. Arguing as above if $G/K$ isomorphic to $\text{Aut}(L_2(17))$, $G$ is isomorphic to $\text{Aut}(L_2(17))$.

\begin{conjecture}
Let $G$ be a finite group such that $\text{NS}(G) = \text{NS}(M)$ and $|G| = |M|$ where $M$ is an arbitrary simple group. Then $G \cong M$.
\end{conjecture}

4. ACKNOWLEDGMENT

The authors are thankful to the referee for carefully reading the paper and for his suggestions and remarks.

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