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# CASEY'S THEOREM IN HYPERBOLIC GEOMETRY

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Abstract. We obtain a hyperbolic version of Casey's theorem. A similar result is obtained in spherical geometry as well.

**Keywords:** Casey's theorem, Ptolemy's theorem, hyperbolic plane, spherical plane.

## 1. Introduction

Let $Q$ be a convex quadrilateral on the Euclidean plane $\mathbb{E}^2$, the hyperbolic plane $\mathbb{H}^2$ or the 2-sphere $\mathbb{S}^2$. Let us denote the side lengths of $Q$ by $a, b, c, d$ and its diagonals by $e, f$ as shown on Fig 1. We say $Q$ is *cyclic* if it is circumscribed to a circle in $\mathbb{E}^2$, or $\mathbb{S}^2$, or a curve of constant geodesic curvature in $\mathbb{H}^2$ (i.e. circle, horocycle or one branch of equidistant line).

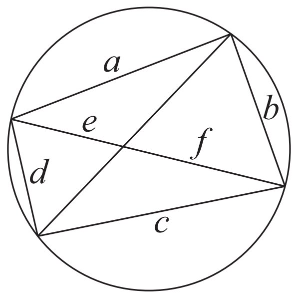


Fig. 1

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Ptolemy's classical theorem (see, for example, [4], pp. 42–43) states that if a quadrilateral is cyclic then the product of its diagonals is equal to the sum of the products of the pairs of opposite sides (Fig. 1)

$$a\,c + b\,d = e\,f.$$

In 1881 Irish mathematician John Casey generalized Ptolemy's theorem in the following way (see [5], p. 103).

**Casey's theorem.** *Let circles $O_1, O_2, O_3, O_4$ on the Euclidean plane $\mathbb{E}^2$ are internally tangent to a circle $O$ in prescribed order. Denote by $t_{ij}$ the length of the common external tangent of the circles $O_i, O_j$. Then*

$$t_{12}\,t_{34} + t_{23}\,t_{14} = t_{13}\,t_{24}\,.$$

We say that a common tangent of two circles is *external* if it does not intersect the segment joining the centers of the circles (see Fig. 2).

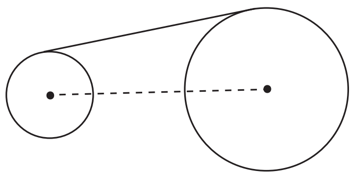

FIG. 2

A spherical version of Ptolemy's theorem was established by Darboux and Frobenius. It can be also found in [8] (p. 180, proposition 4). In hyperbolic geometry the Ptolemy identity was proved by T. Kubota [6] and J.E. Valentine [11].

Many classical theorems of Euclidean geometry have natural analogs in non-Euclidean ones. Here are just some of them directly connected with quadrilaterals. The Brahmagupta formula expresses the area of cyclic quadrilateral in terms of its side lengths. Spherical and hyperbolic variants of this formula can be found in papers by W.J. M'Clelland, T. Preston [8] and A.D. Mednykh [9] respectively. Bretschneider's formula relates the area of a convex quadrilateral and lengths of its sides and diagonals. Non-Euclidean analogs of Bretschneider's formula are given in [2], [3] and [12]. A formula for the area of hyperbolic trapezoid in terms of side lengths was presented in [10]. Some relations for hyperbolic tangential quadrilateral in terms of distances between the vertices and the points of tangency were found in [7].

In the present paper, we produce hyperbolic and spherical versions of Casey's theorem.

## 2. Casey's theorem in hyperbolic geometry

**Theorem 1.** *Let circles $O_1, O_2, O_3, O_4$ on the hyperbolic plane $\mathbb{H}^2$ be internally tangent to a circle $O$ in the prescribed order. Denote by $t_{ij}$ the length of the common external tangent of the circles $O_i, O_j$. Then*

$$\operatorname{sh}\frac{t_{12}}{2}\,\operatorname{sh}\frac{t_{34}}{2} + \operatorname{sh}\frac{t_{23}}{2}\,\operatorname{sh}\frac{t_{14}}{2} = \operatorname{sh}\frac{t_{13}}{2}\,\operatorname{sh}\frac{t_{24}}{2}.$$

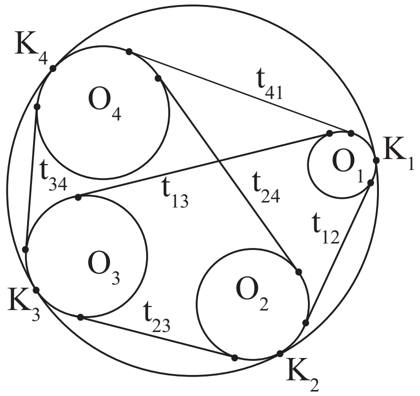


FIG. 3

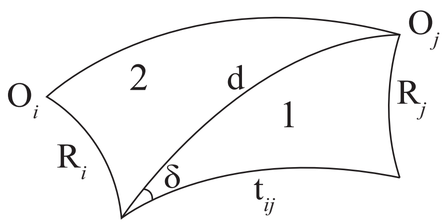


FIG. 4

*Proof.* Denote by $O_i$ the centers of the circles respectively (Fig. 3). Let $R_i$ be the radii of the circles $O_i$ and $K_i$ be the points of tangency with circle $O$.

Consider the quadrilateral with sides $O_iO_j$, $R_i$, $R_j$ and $t_{ij}$ (Fig. 4). Denote by $d$ the diagonal which divides the quadrilateral into triangles 1 and 2. Since radii $R_i$ and $R_j$ are orthogonal to common tangent $t_{ij}$, we can apply the hyperbolic Pythagorean theorem for triangle 1 (see, e.g., [1])

$$\operatorname{ch} d = \operatorname{ch} R_i \, \operatorname{ch} t_{ij}. \tag{1}$$

Denote the angle between diagonal $d$ and tangent $t_{ij}$ by $\delta$. Then by the Sine rule for hyperbolic triangle 1 we have

$$\frac{\operatorname{sh} d}{\sin \pi/2} = \frac{\operatorname{sh} R_i}{\sin \delta}$$

or

$$\sin \delta = \frac{\operatorname{sh} R_i}{\operatorname{sh} d}.$$

We use the second Cosine rule for hyperbolic triangle 2 (see, e.g., [1])

$$\operatorname{ch} O_iO_j = \operatorname{ch} R_j \, \operatorname{ch} d - \operatorname{sh} R_j \, \operatorname{sh} R_i$$

and get

$$\operatorname{ch} d = \frac{\operatorname{ch} O_iO_j + \operatorname{sh} R_j \, \operatorname{sh} R_i}{\operatorname{ch} R_j}.$$

Substituting $\operatorname{ch} d$ in (1) we obtain

$$\operatorname{ch} t_{ij} = \frac{\operatorname{ch} O_iO_j + \operatorname{sh} R_j \ \operatorname{sh} R_i}{\operatorname{ch} R_j \operatorname{ch} R_i}$$

or

$$\operatorname{ch} O_iO_j = \operatorname{ch} t_{ij} \ \operatorname{ch} R_j \operatorname{ch} R_i - \operatorname{sh} R_j \ \operatorname{sh} R_i. \tag{2}$$

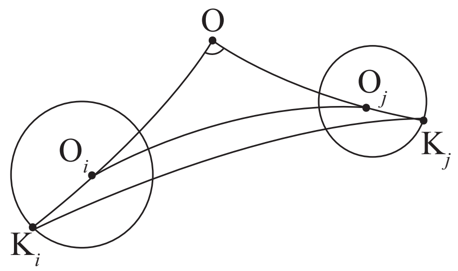


FIG. 5

We apply the first Cosine rule for triangle $OK_iK_j$ where $OK_i = OK_j = R$ (see Fig. 5)

$$\operatorname{ch} K_iK_j = \operatorname{ch}^2 R - \operatorname{sh}^2 R \ \cos \angle K_iOK_j$$

or

$$\cos \angle K_iOK_j = \frac{\operatorname{ch}^2 R - \operatorname{ch} K_iK_j}{\operatorname{sh}^2 R}. \tag{3}$$

Using the first Cosine rule for hyperbolic triangle $OO_iO_j$, where $OO_i = R - R_i$, $OO_j = R - R_j$, we have

$$\operatorname{ch} O_iO_j = \operatorname{ch}(R - R_i) \ \operatorname{ch}(R - R_j) - \operatorname{sh}(R - R_i) \ \operatorname{sh}(R - R_j) \ cos\angle O_iOO_j \, .$$

Since $\angle O_iOO_j = \angle K_iOK_j$ then by (3) the latter expression takes the following form

$$\operatorname{ch} O_iO_j = \operatorname{ch}(R - R_i) \ \operatorname{ch}(R - R_j) - \operatorname{sh}(R - R_i) \ \operatorname{sh}(R - R_j) \ \frac{\operatorname{ch}^2 R - \operatorname{ch} K_iK_j}{\operatorname{sh}^2 R}.$$

Using (2) for the left-hand side, we get

$$\operatorname{ch} t_{ij} \ \operatorname{ch} R_j \operatorname{ch} R_i - \operatorname{sh} R_j \ \operatorname{sh} R_i$$
$$= \operatorname{ch}(R - R_i) \ \operatorname{ch}(R - R_j) - \operatorname{sh}(R - R_i) \ \operatorname{sh}(R - R_j) \ \frac{\operatorname{ch}^2 R - \operatorname{ch} K_iK_j}{\operatorname{sh}^2 R}.$$

Hence,

$$\operatorname{ch} K_iK_j$$
$$= \frac{\operatorname{sh}^2 R \,(\operatorname{ch} t_{ij} \ \operatorname{ch} R_j \operatorname{ch} R_i - \operatorname{sh} R_j \ \operatorname{sh} R_i - \operatorname{ch}(R - R_i) \ \operatorname{ch}(R - R_j))}{\operatorname{sh}(R - R_i) \ \operatorname{sh}(R - R_j)} + \operatorname{ch}^2 R.$$

So, as $\operatorname{sh}^2 \frac{x}{2} = \frac{\operatorname{ch} x - 1}{2}$, we have

$$\operatorname{sh}^2 \frac{K_iK_j}{2}$$
$$= \frac{\operatorname{sh}^2 R \,(\operatorname{ch} t_{ij}\ \operatorname{ch} R_i \operatorname{ch} R_j - \operatorname{sh} R_i\ \operatorname{sh} R_j - \operatorname{ch}(R-R_i)\ \operatorname{ch}(R-R_j))}{2\ \operatorname{sh}(R-R_i)\ \operatorname{sh}(R-R_j)} + \frac{\operatorname{ch}^2 R - 1}{2}.$$

We reduce the right-hand side to common denominator and use the difference formula for hyperbolic cosine; then the latter expression is simplified

$$\operatorname{sh}^2 \frac{K_iK_j}{2} = \frac{\operatorname{sh}^2 R}{2\ \operatorname{sh}(R-R_i)\ \operatorname{sh}(R-R_j)} \cdot \tag{4}$$
$$\Big[\operatorname{ch} t_{ij} \operatorname{ch} R_i \operatorname{ch} R_j - \operatorname{sh} R_i\ \operatorname{sh} R_j - \big(\operatorname{ch}(R-R_i)\ \operatorname{ch}(R-R_j) - \operatorname{sh}(R-R_i)\ \operatorname{sh}(R-R_j)\big)\Big]$$
$$= \frac{\operatorname{sh}^2 R}{2\ \operatorname{sh}(R-R_i)\ \operatorname{sh}(R-R_j)} \Big[\operatorname{ch} t_{ij} \operatorname{ch} R_i \operatorname{ch} R_j - \operatorname{sh} R_i\ \operatorname{sh} R_j - \operatorname{ch}(R-R_i-R+R_j)\Big]$$
$$= \frac{\operatorname{sh}^2 R}{2 \operatorname{sh}(R-R_i) \operatorname{sh}(R-R_j)} (\operatorname{ch} t_{ij} \operatorname{ch} R_i \operatorname{ch} R_j - \operatorname{sh} R_i \operatorname{sh} R_j - \operatorname{ch} R_i \operatorname{ch} R_j + \operatorname{sh} R_i \operatorname{sh} R_j)$$
$$= \frac{\operatorname{sh}^2 R\ \operatorname{ch} R_i\ \operatorname{ch} R_j}{2\ \operatorname{sh}(R-R_i)\ \operatorname{sh}(R-R_j)} (\operatorname{ch} t_{ij} - 1).$$

Points of tangency of circles $O_1, O_2, O_3, O_4$ by circle $O$ form a cyclic quadrilateral $K_1K_2K_3K_4$ in $\mathbb{H}^2$. According to hyperbolic Ptolemy's theorem (see [11]), segments $K_iK_j$ satisfy the equation

$$\operatorname{sh} \frac{K_1K_3}{2}\ \operatorname{sh} \frac{K_2K_4}{2} = \operatorname{sh} \frac{K_1K_2}{2}\ \operatorname{sh} \frac{K_3K_4}{2} + \operatorname{sh} \frac{K_2K_3}{2}\ \operatorname{sh} \frac{K_1K_4}{2}. \tag{5}$$

We take the square root in (4) and substitute expressions for $\operatorname{sh} \frac{K_iK_j}{2}$ in (5). Then we divide the resulting equation by

$$\operatorname{sh}^2 R \sqrt{\frac{\operatorname{ch} R_1\ \operatorname{ch} R_2\ \operatorname{ch} R_3\ \operatorname{ch} R_4}{2\ \operatorname{sh}(R-R_1)\ \operatorname{sh}(R-R_2)\ \operatorname{sh}(R-R_3)\ \operatorname{sh}(R-R_4)}}$$

to obtain

$$\sqrt{(\operatorname{ch} t_{13} - 1)\,(\operatorname{ch} t_{24} - 1)} = \sqrt{(\operatorname{ch} t_{12} - 1)\,(\operatorname{ch} t_{34} - 1)} + \sqrt{(\operatorname{ch} t_{23} - 1)\,(\operatorname{ch} t_{14} - 1)}.$$

Using the power reduction formula for hyperbolic sine we find

$$\operatorname{sh} \frac{t_{13}}{2}\ \operatorname{sh} \frac{t_{24}}{2} = \operatorname{sh} \frac{t_{12}}{2}\ \operatorname{sh} \frac{t_{34}}{2} + \operatorname{sh} \frac{t_{23}}{2}\ \operatorname{sh} \frac{t_{14}}{2}.$$

□

## 3. Casey's theorem in spherical geometry

The analog of Casey's theorem holds in spherical geometry as well.

**Theorem 2.** *Let circles $O_1, O_2, O_3, O_4$ on the sphere $\mathbb{S}^2$ be internally tangent to a circle $O$ in the prescribed order. Denote by $t_{ij}$ the length of the common external tangent of the circles $O_i, O_j$. Then*

$$\sin\frac{t_{12}}{2}\sin\frac{t_{34}}{2}+\sin\frac{t_{23}}{2}\sin\frac{t_{14}}{2}=\sin\frac{t_{13}}{2}\sin\frac{t_{24}}{2}.$$

One can prove the spherical version of the theorem using the same arguments as in the hyperbolic case. The only difference is to replace hyperbolic trigonometry relations by corresponding spherical ones.

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