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DE RHAM REGULARIZATION OPERATORS IN ORLICZ SPACES OF DIFFERENTIAL FORMS ON RIEMANNIAN MANIFOLDS

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Abstract. In his classical monograph *Variétés Différentiables* (Paris: Hermann, 1955), G. de Rham introduced smoothing operators on currents on a differentiable manifold. We study some properties of the restrictions of these operators to Orlicz spaces of differential forms on a Riemannian manifold. In particular, we prove that if an N-function Φ is Δ_2-regular then the L_Φ-cohomology of a Riemannian manifold can be calculated with the use of smooth L^Φ-forms.

Keywords: Riemannian manifold, differential form, de Rham regularization operator, Orlicz space, operator of exterior derivation, L_Φ-cohomology.

Introduction

Let X be a smooth manifold and let $D'(X)$ be the space of de Rham currents of all degrees on X.

In his classical monograph [1], de Rham considered some operators R and A depending on a sequence of positive parameters $\varepsilon_1, \varepsilon_2, \dots$ with the following properties:

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(1) If T is a k-current on X then RT is a k-current and AT is a $(k-1)$-current and

$$RT - T = dAT + AdT.$$

(2) The supports of RT and AT are contained in an arbitrarily small neighborhood of $\operatorname{supp} T$ provided that the parameters ε_i are sufficiently small.

(3) $RT \in C^\infty$ and if $T \in C^\infty$ then $AT \in C^\infty$.

(4) If "all ε_i tend to zero" then $RT \to T$ and $AT \to 0$ in $D'(X)$.

(5) If T has compact support in X then $RT \to T$ in the space of smooth forms on X with compact support.

In [2], Gol′dshtein, Kuz′minov, and Shvedov studied the de Rham operators R and A in the spaces of differential forms $L^p(X, \Lambda^k)$ with modulus integrable to the power p on a Riemannian manifold X. It turned out, that, under an appropriate choice of a family of parameters $\varepsilon_{i,j}$, the de Rham operators $R_{(i)}$ and A_i corresponding to the sequence $\{\varepsilon_{i,k}\}_{k\in\mathbb{N}}$ take L^p-forms to L^p-forms and if $\omega \in L^p(X, \Lambda^k)$ (respectively, $\omega \in \Omega^k_{p,q}(X) := \{\theta \in L^p(X, \Lambda^k) \mid d\theta \in L^q(X, \Lambda^{k+1})\}$) then $R_{(i)}\omega \to \omega$ as $i \to \infty$ in L^p (respectively, in $\Omega^k_{p,q}(X)$).

In the present article, we generalize the results of [2] to Orlicz spaces of differential forms $L^\Phi(X, \Lambda^k)$. Section 1 contains some general information on N-functions and Orlicz spaces. In Section 2, we introduce Orlicz spaces of differential forms on a Riemannian manifold, the corresponding analogs of the spaces $\Omega_{p,q}$, and L_Φ-cohomology. In Section 3, we consider local averaging operators R_ε and A_ε, prove that they take L^p into L^p, and establish some estimates for their norms. Finally, in Section 4, the properties of the "local" averaging operators are used for constructing regularization operators $R_{(i)}$ and $A_{(i)}$ on a Riemannian manifold X. The main properties of $R_{(i)}$ and $A_{(i)}$ are collected in a theorem.

Observe that Orlicz spaces of differential forms first appeared in [4] in the context of domains in $\mathbb{R}^n$.

1. N-Functions and Orlicz Function Spaces

Definition 1.1. A function $\Phi : \mathbb{R} \to \mathbb{R}$ is called an *N-function* if

(i) Φ is even and convex;

(ii) $\Phi(x) = 0 \Longleftrightarrow x = 0$;

(iii) $\lim\limits_{x\to 0} \frac{\Phi(x)}{x} = 0$; $\quad \lim\limits_{x\to\infty} \frac{\Phi(x)}{x} = \infty$.

An N-function Φ has left and right derivatives (which can differ only on an at most countable set, see, for instance, [6, Theorem 1, p. 7]). The left derivative φ of Φ (we write $\varphi = \Phi'$ below) is left continuous, nondecreasing on $(0, \infty)$, and such that $0 < \varphi(t) < \infty$ for $t > 0$, $\varphi(0) = 0$, $\lim\limits_{t\to\infty} \varphi(t) = \infty$. The function

$$\psi(s) = \inf\{t > 0 : \varphi(t) > s\}, \quad s > 0,$$

is called the *left inverse* of φ.

The functions Φ, Ψ given by

$$\Phi(x) = \int_0^{|x|} \varphi(t)dt, \quad \Psi(x) = \int_0^{|x|} \psi(t)dt$$

are called *complementary N-functions*.

The N-function Ψ complementary to an N-function Φ can also be expressed as

$$\Psi(y) = \sup\{x|y| - \Phi(x) \,:\, x \geq 0\}, \quad y \in \mathbb{R}.$$

N-functions are classified in accordance with their growth rates as follows:

Definition 1.2. An N-function Φ is said to satisfy the Δ_2-condition for large x (for small x, for all x), which is written as $\Phi \in \Delta_2(\infty)$ ($\Phi \in \Delta_2(0)$, or $\Phi \in \Delta_2$), if there exist constants $x_0 > 0$, $K > 2$ such that $\Phi(2x) \leq K\varphi(x)$ for $x \geq x_0$ (for $0 \leq x \leq x_0$, or for all $x \geq 0$); and it satisfies the ∇_2-condition for large x (for small x, or for all x), which is denoted symbolically as $\Phi \in \nabla_2(\infty)$ ($\Phi \in \nabla_2(0)$, or $\Phi \in \nabla_2$) if there are constants $x_0 > 0$ and $c > 1$ such that $\Phi(x) \leq \frac{1}{2c}\Phi(cx)$ for $x \geq x_0$ (for $0 \leq x \leq x_0$, or for all $x \geq 0$).

Henceforth, let Φ be an N-function and let (Ω, Σ, μ) be a measure space.

Definition 1.3. The set $\tilde{L}^\Phi = \tilde{L}^\Phi(\Omega) = \tilde{L}^\Phi(\Omega, \Sigma, \mu)$ is defined to be the set of measurable functions $f : \Omega \to \mathbb{R}$ such that

$$\rho_\Phi(f) := \int_\Omega \Phi(f) d\mu < \infty.$$

Proposition 1.1. [7] *The set $\tilde{L}^\Phi$ is a vector space in the following cases:*
(i) $\mu(\Omega) < \infty, \quad \Phi \in \Delta_2(\infty)$*;*
(ii) $\mu(\Omega) = \infty, \quad \Phi \in \Delta_2$.
(iii) Ω *is countable,* μ *is the counting measure on* $\Omega, \quad \Phi \in \Delta_2(0)$.

Definition 1.4. The linear space

$$L^\Phi = L^\Phi(\Omega) = L^\Phi(\Omega, \Sigma, \mu) = \{f : \Omega \to \mathbb{R} \text{ measurable } : \rho_\Phi(af) < \infty \text{ for } \textit{some } a > 0\}$$

is called an *Orlicz space* on (Ω, Σ, μ).

For an Orlicz space $L^\Phi = L^\Phi(\Omega, \Sigma, \mu)$, the N-function Φ is called Δ_2-*regular* if $\Phi \in \Delta_2(\infty)$ when $\mu(\Omega) < \infty$ or $\Phi \in \Delta_2$ when $\mu(\Omega) = \infty$ or $\Phi \in \Delta_2(0)$ for μ a counting measure.

Let Ψ be the complementary N-function to Φ.

Below we as usual identify two functions equal outside a set of measure zero.

If $f \in L^\Phi$ then the functional $\|\cdot\|_\Phi$ (called *the Orlicz norm*) defined by

$$\|f\|_\Phi = \|f\|_{L^\Phi(\Omega)} = \sup\left\{\left|\int_\Omega fg\, d\mu\right| : \rho_\Psi(g) \leq 1\right\}$$

is a seminorm. It becomes a norm if μ satisfies the *finite subset property* (see [6, p. 59]): if $A \in \Sigma$ and $\mu(A) > 0$ then there exists $B \in \Sigma$, $B \subset A$, such that $0 < \mu(B) < \infty$.

The equivalent *gauge* (or *Luxemburg*) *norm* of a function $f \in L^\Phi$ is defined by the formula

$$\|f\|_{(\Phi)} = \|f\|_{L^{(\Phi)}(\Omega)} = \inf\left\{k > 0 \,:\, \rho_\Phi\left(\frac{f}{k}\right) \leq 1\right\}.$$

This is a norm without any constraint on the measure μ (see [6, p. 54, Theorem 3]).

2. Orlicz Spaces of Differential Forms

Let X be a Riemannian manifold. Given $x \in X$, denote by $(\omega(x), \theta(x))$ the inner product of exterior k-forms $\omega(x)$ and $\theta(x)$ on T_xX. This gives a function $x \mapsto (\omega(x), \theta(x))$ on X.

Let $\Phi : \mathbb{R} \to \mathbb{R}$ and $\Psi : \mathbb{R} \to \mathbb{R}$ be two complementary N-functions. Denote by $\tilde{L}^{\Phi}(X, \Lambda^k)$ the class of all measurable k-forms ω such that

$$\rho_{\Phi}(\omega) := \int_X \Phi(|\omega(x)|) d\mu_X < \infty.$$

Here $d\mu_X$ stands for the volume element of the Riemannian manifold X. We will identify k-forms differing on a set of measure zero.

Given a (not necessarily orientable) Riemannian manifold X, introduce the space $L^{\Phi}(X, \Lambda^k)$ as the class of all measurable k-forms ω satisfying the condition

$$\rho_{\Phi}(\alpha\omega) < \infty \text{ for some } \alpha > 0.$$

Obviously, $\tilde{L}^{\Phi}(X, \Lambda^k) \subset L^{\Phi}(X, \Lambda^k)$.

As in the case of Orlicz function spaces, the space $L^{\Phi}(X, \Lambda^k)$ is endowed with two equivalent norms: the *gauge norm*

$$\|\omega\|_{(\Phi)} = \|\omega\|_{L^{(\Phi)}(X)} = \inf\left\{k > 0 \,:\, \rho_{\Phi}\left(\frac{\omega}{k}\right) \le 1\right\}.$$

and the *Orlicz norm*

$$\|\omega\|_{\Phi} = \|\omega\|_{L^{\Phi}(X)} = \sup\left\{\left|\int_X (\omega(x), \theta(x))\, d\mu_X\right| : \theta \in \tilde{L}^{\Psi}(X, \Lambda^k),\ \rho_{\Psi}(\theta) \le 1\right\}$$

As in the case of function spaces, it can be proved that $L^{\Phi}(X, \Lambda^k)$, endowed with one of these norms, is a Banach space.

Obviously, the gauge norm of a k-form ω is nothing but the gauge norm of its modulus function $|\omega|$. That the same holds for the Orlicz norm is the contents of the following

Lemma 2.1. *The Orlicz norm of a k-form $\omega \in L^{\Phi}(X, \Lambda^k)$ coincides with the Orlicz norm of its modulus function $|\omega|$.*

Proof. For $\theta \in L^{\Psi}(X, \Lambda^k)$ with $\rho_{\Psi}(\theta) \le 1$ we have

$$\left|\int_X (\omega(x), \theta(x)) d\mu_X\right| \le \int_X |\omega(x)||\theta(x)| d\mu_X \le \sup_{\substack{g \in L^{\Psi}(X),\\ \rho_{\Psi}(g) \le 1}} \left|\int_X |\omega(x)| g(x) d\mu_X\right| = \|\,|\omega|\,\|_{\Phi}.$$

Thus,

$$\|\omega\|_{\Phi} = \sup_{\substack{\theta \in L^{\Psi}(X, \Lambda^k),\\ \rho_{\Psi}(\theta) \le 1}} \left|\int_X (\omega(x), \theta(x))\, d\mu_X\right| \le \|\,|\omega|\,\|_{\Phi}.$$

On the other hand, let $(g_m)_{m \in \mathbb{N}}$ be a sequence of functions in $L^{\Psi}(X)$ with $\rho_{\Psi}(g_m) \le 1$ such that

$$\left|\int_X |\omega(x)| g_m(x) d\mu_X\right| \to \|\,|\omega|\,\|_{\Phi} \text{ as } m \to \infty.$$

Since

$$\left|\int_X |\omega(x)| g_m(x) d\mu_X\right| \le \int_X |\omega(x)||g_m(x)| d\mu_X \le \|\,|\omega|\,\|_{\Phi},$$

we also have

$$\int_X |\omega(x)||g_m(x)|d\mu_X \to \| \, |\omega| \, \|_\Phi \text{ as } m \to \infty.$$

Consider the sequence $(\theta_m)_{m\in\mathbb{N}}$ of k-forms θ_m defined by

$$\theta_m(x) = \begin{cases} |g_m(x)| \frac{\omega(x)}{|\omega(x)|} & \text{if } \omega(x) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\rho_\Psi(\theta_m) = \rho_\Psi(g_m) \leq 1$ and

$$\left| \int_X (\omega(x), \theta_m(x)) d\mu_X \right| = \left| \int_X |\omega(x)||g_m(x)| d\mu_X \right| \to \| \, |\omega| \, \|_\Phi$$

as $m \to \infty$. Therefore,

$$\| \, |\omega| \, \|_\Phi \leq \sup_{\substack{\theta \in L^\Psi(X, \Lambda^k), \\ \rho_\Psi(\theta) \leq 1}} \left| \int_X (\omega(x), \theta(x)) d\mu_X \right| = \|\omega\|_\Phi.$$

□

Lemma 2.2. *Suppose that we have a family of differential forms $x \mapsto \omega(x,y)$ of the same degree k on a Riemannian manifold X depending on a parameter y belonging to a σ-finite measure space Y such that the function $(x,y) \mapsto |\omega(x,y)|$ is measurable on $X \times Y$. Assume that the form $\{x \mapsto \omega(x,y)\}$ lies in L^Φ for every y and the integral $I(x) = \int_Y \omega(x,y) dy$ exists for almost every $x \in X$. Then*

$$\|I\|_\Phi \leq \int_Y \|\omega(\cdot, y)\|_\Phi \, dy. \tag{1}$$

Proof. Let Ψ be the N-function dual to Φ. By Lemma 2.1, we can pass from the forms ω and I to the functions $|\omega|$ and $|I|$.

Take a function g on X with $\rho_\Psi(g) \leq 1$. Using Tonelli's theorem, we infer

$$\begin{aligned} \left| \int_X |I(x)| g(x) d\mu_X \right| &\leq \int_X |I(x)||g(x)| \, d\mu_X \leq \int_X \left(\int_Y |\omega(x,y)||g(x)| \, dy \right) d\mu_X \\ &= \int_Y \left(\int_X |\omega(x,y)||g(x)| \, d\mu_X \right) dy \leq \int_Y \|\omega(\cdot, y)\|_\Phi \, dy. \end{aligned}$$

Since this is fulfilled for every k-form θ with $\rho_\Psi(\theta) \leq 1$, we can take the supremum of the left-hand side to obtain (1). □

A differential $(k+1)$-form $\theta \in L^1_{\text{loc}}(X, \Lambda^{k+1})$ is called *the weak exterior derivative* (or *differential*) of a k-form $\omega \in L^1_{\text{loc}}(X, \Lambda^k)$ (which is written as $d\omega = \theta$) if, for every orientable domain $V \subset \text{Int} M$,

$$\int_V \theta \wedge u = (-1)^{k+1} \int_V \omega \wedge du$$

for any $u \in \mathcal{D}^{n-k}(V)$, where $\mathcal{D}^l(V)$ is the set of smooth l-forms on X with compact support included in $\text{Int}\, V$.

Suppose that X is a Riemannian manifold of dimension n.

Let Φ_1 and Φ_2 be N-functions. For $1 \leq k \leq n$, put

$$\Omega^{k-1}_{\Phi_1, \Phi_2}(X) = \left\{ \omega \in L^{\Phi_1}(X, \Lambda^{k-1}) \, : \, d\omega \in L^{\Phi_2}(X, \Lambda^k) \right\}.$$

This is a Banach space with the graph norm.

Consider the spaces

$$Z^k_{\Phi_2}(X) = \{\omega \in L^{\Phi_2}(X, \Lambda^k) \,:\, d\omega = 0\};$$
$$B^k_{\Phi_1,\Phi_2}(X) = \{\omega \in L^{\Phi_2}(X, \Lambda^k) \,:\, \omega = d\beta \text{ for some } \beta \in L^{\Phi_1}(X, \Lambda^{k-1})\}.$$

Denote by $\overline{B}^k_{\Phi_1,\Phi_2}(X)$ the closure of $B^k_{\Phi_1,\Phi_2}(X)$ in $L^{\Phi_2}(X, \Lambda^k)$. The quotient spaces

$$H^k_{\Phi_1,\Phi_2}(X) := Z^k_{\Phi_2}(X)/B^k_{\Phi_1,\Phi_2}(X)$$

and

$$\overline{H}^k_{\Phi_1,\Phi_2}(X) := Z^k_{\Phi_2}(X)/\overline{B}^k_{\Phi_1,\Phi_2}(X)$$

are called the kth L_{Φ_1,Φ_2}-*cohomology* and the kth *reduced* L_{Φ_1,Φ_2}-*cohomology* of the Riemannian manifold X, the latter cohomology being a Banach space.

If $\Phi_1 = \Phi_2 = \Phi$ then we use the notation $\Omega_\Phi(X)$, $H^k_\Phi(X)$, and $\overline{H}^k_\Phi(X)$ instead of $\Omega_{\Phi,\Phi}(X)$, $H^k_{\Phi,\Phi}(X)$, and $\overline{H}^k_{\Phi,\Phi}(X)$ respectively. Thus, the L_Φ-*cohomology* $H^k_\Phi(X)$ (respectively, the *reduced* L_Φ-*cohomology* $\overline{H}_\Phi(X)$) is the cohomology (respectively, reduced cohomology) of the cochain complex $\{\Omega^*_\Phi(X), d\}$.

3. De Rham Regularization Operators in $\mathbb{R}^n$

Pass to the study of the de Rham operators R and A on the Orlicz spaces $L^\Phi(X, \Lambda^k)$.

Consider first the case when $X = U$ is an open set in $\mathbb{R}^n$ endowed with a Riemannian metric that can be different from the standard metric of the Euclidean space. Assume for convenience that U contains the closed unit ball $B \subset \mathbb{R}^n$. In this case, every fiber of the tangent bundle TU can be identified with $\mathbb{R}^n$. This identification makes it possible to regard every form $\omega \in L^1_{\text{loc}}(U, \Lambda^k)$ as a locally integrable function on U taking values in te space $A^k(\mathbb{R}^n)$ of k-linear skew-symmetric functions on $\mathbb{R}^n$. The Riemannian metric on U and the standard metric of $\mathbb{R}^n$ generate in $A^k(\mathbb{R}^n)$ for every $x \in U$ the norms $|\cdot|_x$ and $|\cdot|$. These norms are connected with each other by the equality $|\omega|_x = H_x|\omega|$, where $H_x : A^k(\mathbb{R}^n) \to A^k(\mathbb{R}^n)$ is a linear operator depending continuously on $x \in U$.

Denote by $\rho(x)dx$ the volume element of U.

In [1], de Rham constructed a group of transformations $s : \mathbb{R}^n \times U \to U$ satisfying the following conditions:

(a) for every $v \in \mathbb{R}^n$, the transformation $s_v : U \to U$ defined by $s_v(x) = s(v, x)$ is smooth and coincides with the identity mapping outside B;

(b) for every $x \in B$, the mapping $\alpha_x : \mathbb{R}^n \to B$ defined by the equality $\alpha_x(v) = s(v, x)$ is a diffeomorphism of $\mathbb{R}^n$ onto the interior of B.

It was observed in [2] that the mapping $s : \mathbb{R}^n \times U \to U$ is smooth.

In view of the above, for a k-form ω, the induced k-form $s^*_v\omega(x)$ can be regarded as a function $S_\omega : U \times \mathbb{R}^n \to A^k(\mathbb{R}^n)$ of the arguments $x \in U$ and $v \in \mathbb{R}^n$.

Let $I = [0, 1] \subset \mathbb{R}$. The cylinder $I \times U$ will be regarded as embedded in $\mathbb{R}^{n+1}$ as follows: $I \times U \subset \mathbb{R}^1 \times \mathbb{R}^n = \mathbb{R}^{n+1}$. Define let $\mu_v : I \times U \to U$ by setting $\mu_v(t, x) = s_{tv}(x)$. Given a vector $v = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$, put $\tilde{v} = (0, v_1, v_2, \dots, v_n) \in \mathbb{R}^{n+1}$. Consider the operator $\gamma : A^{k+1}(\mathbb{R}^{n+1}) \to A^k(\mathbb{R}^n)$ defined by

$$\gamma\omega(v_1, \dots, v_k) = \omega(e_1, \tilde{v}_1, \dots, \tilde{v}_k), \quad e_1 = (1, 0, \dots, 0).$$

By what was observed at the beginning of the section, for a k-form ω, the form $\gamma(\mu_v^*\omega)(t,x)$ can be considered as a function $G_\omega : U \times I \times \mathbb{R}^n \to A^{k-1}(\mathbb{R}^n)$ of the arguments $x \in U$, $t \in [0,1]$, and $v \in \mathbb{R}^n$.

Let f be a smooth function with compact support on $\mathbb{R}^n$ satisfying the following conditions: $\operatorname{supp} f \subset B$, $\int_{\mathbb{R}^n} f(v)\,dv = 1$, $f(v) \geq 0$, $f(v) = f(-v)$ for all v. Put

$$\tau = f(v) dv_1 \wedge \cdots \wedge dv_n.$$

Following [1, 2], given $\varepsilon > 0$ and a k-form ω, consider the operators defined by the equalities

$$R_\varepsilon^*\omega(x) = \int_{\mathbb{R}^n} S_\omega(x, \varepsilon v)\tau(v); \quad x \in U. \tag{2}$$

$$A_\varepsilon^*\omega(x) = \int_{\mathbb{R}^n} \left(\int_0^1 G_\omega(x, t, \varepsilon v) dt \right) \tau(v); \quad x \in U. \tag{3}$$

We will need the following lemma, proved in [2].

Lemma 3.1. *For any form ω on U, we have*

$$|S_\omega(x,\, v)|_x \leq C(v) |\omega(s_v x)|_{s_v x}$$

$$|G_\omega(x,\, t,\, v)|_x \leq M(v) |\omega(s_{tv} x)|_{s_{tv} x}$$

where $C(v) \to 1$ and $M(v) \to 0$ as $v \to 0$.

Lemma 3.2. *For any $\varepsilon > 0$, R_ε^* maps $L^\Phi(U, \Lambda^k)$ into $L^\Phi(U, \Lambda^k)$; moreover,*

$$\|R_\varepsilon^*\|_\Phi \leq C_1(\varepsilon), \tag{4}$$

where $C_1(\varepsilon) \to 1$ as $\varepsilon \to 0$.

Proof. By Lemma 2.2, it suffices to estimate

$$\int_{\mathbb{R}^n} \|S_\omega(\cdot,\, \varepsilon v)\|_\Phi \tau(v). \tag{5}$$

Now, turn to estimating $\|S_\omega(x,\, \varepsilon v)\|_\Phi$. Take $\theta \in L^\Psi(U, \Lambda^k)$ with $\rho_\Psi(\theta) \leq 1$. We have

$$\begin{aligned} &\left| \int_U (S_\omega(x,\, \varepsilon v), \theta(x))_x \rho(x)\, dx \right| \leq \int_U |(S_\omega(x,\, \varepsilon v), \theta(x))_x| \rho(x)\, dx \\ &\leq \int_U |S_\omega(x,\, \varepsilon v)|_x |\theta(x)|_x \rho(x)\, dx \leq \int_U C(\varepsilon v) |\omega(s_{\varepsilon v} x)|_{s_{\varepsilon v} x} |\theta(x)|_x \rho(x)\, dx \\ &\leq C(\varepsilon v) \int_U |\omega(y)|_y |\theta(s_{-\varepsilon v} y)|_{s_{-\varepsilon v} y} J_{\varepsilon v}(y) \rho(s_{-\varepsilon v} y)\, dy \\ &\leq C(\varepsilon v) C_2(\varepsilon v) \int_U |\omega(y)|_y |\theta(s_{-\varepsilon v} y)|_{s_{-\varepsilon v} y} \rho(y)\, dy, \end{aligned} \tag{6}$$

where $C_2(v)$ stands for the function

$$\sup_{y \in U} \frac{\rho(s_{-v} y) J_v(y)}{\rho(y)}.$$

Given a function $g(x)$ with $\rho_\Psi(g) \leq 1$, we have

$$\rho_\Psi(g \circ s_{-\varepsilon v}) = \int_U \Psi(g(s_{-\varepsilon v} y)) dy = \int_U \Psi(g(z)) |J_{\varepsilon v}(z)| dz \leq L_0(\varepsilon v) \int_U \Psi(g(z)) dz$$

where $L_0(v) = \sup_{y\in U} |J_v(y)|$, and hence $\rho_\Psi(g \circ s_{-\varepsilon v}) \le L_0(\varepsilon v)\rho_\Psi(g)$.

For a function $\lambda(x)$ with $\rho_\Psi(\lambda) < \infty$, put

$$\tilde\lambda(y) = \begin{cases} \frac{\lambda(s_{-\varepsilon v}y)}{\rho_\Psi(\lambda\circ s_{-\varepsilon v})} & \text{if } \rho_\Psi(\lambda\circ s_{-\varepsilon v}) > 1; \\ \lambda(s_{-\varepsilon v}y) & \text{if } \rho_\Psi(\lambda\circ s_{-\varepsilon v}) \le 1. \end{cases} \tag{7}$$

We also set

$$L(v) = \max\{L_0(v), 1\}.$$

In the case of $\rho_\Psi(\lambda\circ s_{-\varepsilon v}) > 1$, we have

$$\begin{aligned}\rho_\Psi(\tilde\lambda) &= \int_U \Psi\left(\frac{|\lambda(s_{-\varepsilon v}y)|}{\rho_\Psi(\lambda\circ s_{-\varepsilon v})}\right) dy \\ &\le \frac{1}{\rho_\Psi(\lambda\circ s_{-\varepsilon v})}\int_U \Psi(|\lambda(s_{-\varepsilon v}y)|)dy \le \frac{\rho_\Psi(\lambda\circ s_{-\varepsilon v})}{\rho_\Psi(\lambda\circ s_{-\varepsilon v})} = 1.\end{aligned}$$

Continuing (6) and putting $K_\theta := |\theta|$, we can now infer

$$\begin{aligned} &C(\varepsilon v)C_2(\varepsilon v)\int_U |\omega(y)|_y |\theta(s_{-\varepsilon v}y)|_{s_{-\varepsilon v}y}\rho(y)\,dy \\ &\le C(\varepsilon v)C_2(\varepsilon v)\max\{\rho_\Psi(K_\theta\circ s_{-\varepsilon v}),1\}\int_U |\omega(y)|_y\widetilde{K_\theta}(y)\rho(y)\,dy \\ &\le C(\varepsilon v)C_2(\varepsilon v)\max\{L_0(\varepsilon v)\rho_\Psi(\theta),1\}\int_U |\omega(y)|_y\widetilde{K_\theta}(y)\rho(y)\,dy \\ &\le C(\varepsilon v)C_2(\varepsilon v)L(\varepsilon v)\int_U |\omega(y)|_y\widetilde{K_\theta}(y)\rho(y)\,dy \le \|\omega\|_\Phi\int_{\mathbb{R}^n} C(\varepsilon v)C_2(\varepsilon v)L(\varepsilon v)\tau(v).\end{aligned}$$

where $\tilde\theta$ is defined from θ by (7). Here the last inequality is obtained with account taken of Lemma 2.1. Taking the supremum over $\theta \in L^\Psi(U,\Lambda^k)$ with $\rho_\Psi(\theta) \le 1$ in

$$\left|\int_U (S_\omega(x,\,\varepsilon v),\theta(x))_x\rho(x)\,dx\right|$$

and involving (5), we get

$$\int_{\mathbb{R}^n} \|S_\omega(x,\,\varepsilon v)\|_\Phi\tau(v) \le \|\omega\|_\Phi\int_{\mathbb{R}^n} C(\varepsilon v)C_2(\varepsilon v)L(\varepsilon v)\tau(v).$$

Thus, by Lemma 2.2,

$$\|R^*_\varepsilon\omega\|_\Phi \le C_1(\varepsilon)\|\omega\|_\Phi,$$

where $C_1(\varepsilon) = \int_{\mathbb{R}^n} C(\varepsilon v)C_2(\varepsilon v)L(\varepsilon v)\tau(v)$.

It is not hard to see that $C(\varepsilon v)C_2(\varepsilon v)L(\varepsilon v) \to 1$ as $\varepsilon \to 0$. Thus, if

$$|C(\varepsilon v)C_2(\varepsilon v)L(\varepsilon v) - 1| < \delta \quad \text{for } \varepsilon < \varepsilon(\delta)$$

then

$$\begin{aligned}\left|\int_{\mathbb{R}^n} C(\varepsilon v)C_2(\varepsilon v)L(\varepsilon v)\tau(v) - 1\right| &= \left|\int_{\mathbb{R}^n} C(\varepsilon v)C_2(\varepsilon v)L(\varepsilon v)\tau(v) - \int_{\mathbb{R}^n}\tau(v)\right| \\ &\le \int_{\mathbb{R}^n} |C(\varepsilon v)C_2(\varepsilon v)L(\varepsilon v) - 1|\,\tau(v) \le \int\limits_{\mathbb{R}^n}\delta\tau(v) \le \delta \quad \text{for } \varepsilon < \varepsilon(\delta).\end{aligned}$$

Now we conclude that

$$\int_{\mathbb{R}^n} C(\varepsilon v)C_2(\varepsilon v)L(\varepsilon v)\tau(v) \to 1 \quad \text{as } \varepsilon \to 0.$$

□

Lemma 3.3. *For any $\varepsilon > 0$, the operator A_ε^* maps $L^\Phi(U, \Lambda^k)$ into $L^\Phi(U, \Lambda^k)$ and*

$$\|A_\varepsilon^*\|_\Phi \leq M_1(\varepsilon), \tag{8}$$

where $M_1(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Proof. As above, by Lemma 2.2, it suffices to estimate

$$\int\limits_{\mathbb{R}^n} \int\limits_0^1 \|G_\omega(x,\, t,\, \varepsilon v)\|_\Phi dt\, \tau(v).$$

Take $\theta \in L^\Psi(U, \Lambda^k)$ with $\rho_\Psi(\theta) \leq 1$. We now infer

$$\left| \int_U (G_\omega(x,\, t,\, \varepsilon v), \theta(x))_x \rho(x)\, dx \right| \leq \int_U |(G_\omega(x,\, t,\, \varepsilon v), \theta(x))_x| \rho(x)\, dx$$
$$\leq \int_U |G_\omega(x,\, t,\, \varepsilon v)|_x |\theta(x)|_x \rho(x)\, dx \leq \int_U M(\varepsilon v) |\omega(s_{t\varepsilon v}x)|_{s_{t\varepsilon v}x} |\theta(x)|_x \rho(x)\, dx$$
$$= \int_U M(\varepsilon v) |\omega(y)|_y |\theta(s_{-t\varepsilon v}y)|_x \rho(s_{-t\varepsilon v}y) |J_{s_{t\varepsilon v}}(y)|\, dy$$

Using Lemma 3.1, like in the proof of Lemma 3.2, we can deduce that

$$\int_{\mathbb{R}^n} \int\limits_0^1 \|G_\omega(x,\, t,\, \varepsilon v)\|_\Phi dt\, \tau(v) \leq \|\omega\|_\Phi \int_{\mathbb{R}^n} M(\varepsilon v) \int\limits_0^1 C_2(t\varepsilon v) L(t\varepsilon v) dt\, \tau(v).$$

Put

$$M_1(\varepsilon) = \int_{\mathbb{R}^n} M(\varepsilon v) \int\limits_0^1 C_2(t\varepsilon v) L(t\varepsilon v) dt\, \tau(v).$$

Since $M(v) \to 0$ as $v \to 0$, also $M_1(v) \to 0$ as $v \to 0$. □

Remark 3.1. Initially in [1], de Rham defined R_ε^* and A_ε^* by (2) and (3) on the space of smooth forms of compact support. The operators R_ε and A_ε are defined on the space of currents as adjoints to R_ε^* and A_ε^*. De Rham proved that, for a symmetric kernel τ (which is our case as well),

$$R_\varepsilon^* = R_\varepsilon \text{ and } A_\varepsilon^* = A_\varepsilon \tag{9}$$

on the space of smooth forms with compact support. In [2], equalities (9) were established in $L^p(U, \Lambda^k)$. The same argument as in [2] shows that equalities (9) also hold for an Orlicz space $L^\Phi(U, \Lambda^k)$.

4. De Rham Regularization Operators on a Riemannian manifold

Choose a locally finite atlas $\{h_\nu : U_\nu \to X\}$ on X such that the domain $U_\nu \subset \mathbb{R}^n$ of each chart $h_\nu \colon U_\nu \to X$ contains the unit ball B. For each ν, any current T on X admits a decomposition of the form $T = T' + T''$, where the support of T' lies in $h_\nu(U_\nu)$ while the support of T'' lies in $X \setminus h_\nu(B)$. Put

$$R_{\nu,\,\varepsilon} T = h_\nu^{-1} R_\varepsilon h_\nu T' + T'', \quad A_{\nu,\,\varepsilon} T = h_\nu^{-1} A_\varepsilon h_\nu T'$$

For every sequence $\lambda = \{\varepsilon_1,\, \varepsilon_2, \dots\}$ of positive numbers, consider the operators

$$R_\lambda^{(\nu)} = R_{1,\,\varepsilon_1} R_{2,\,\varepsilon_2} \dots R_{\nu,\,\varepsilon_\nu}, \quad A_\lambda^{(\nu)} = R_\lambda^{(\nu-1)} A_{\nu,\,\varepsilon_\nu}$$

$$R_\lambda = \lim_{\nu\to\infty} R_\lambda^{(\nu)}, \quad A_\lambda = \sum_{\nu=1}^{\infty} A_\lambda^{(\nu)}$$

Theorem 4.1. *Let X be a Riemannian manifold. There is a sequence of operators $R_{(i)}$, $A_{(i)}$ on the space of currents on X such that*

(1) *$A_{(i)}$, $R_{(i)}$ take $L^\Phi(X,\Lambda^k)$ into $L^\Phi(X,\Lambda^k)$ in such a way that*

$$\|R_{(i)}\|_\Phi \le 1+\frac{1}{i}, \ \|A_{(i)}\|_\Phi \le \frac{1}{i};$$

(2) *if ω is a smooth form then so is $R_{(i)}\omega$;*

(3) *if Φ is Δ_2-regular then $R_{(i)}\omega \to \omega$ as $i\to\infty$ for any $\omega \in L^\Phi(X,\Lambda^k)$;*

(4) *if Φ_1 and Φ_2 are Δ_2-regular and $\omega \in \Omega^k_{\Phi_1,\Psi_2}(X)$ then $R_{(i)}\omega \in \Omega^k_{\Phi_1,\Phi_2}(X)$, $dR_{(i)}\omega = R_{(i)}d\omega$, and $R_{(i)}\omega \to \omega$ in $\Omega^k_{\Phi_1,\Phi_2}(X)$;*

(5) *if Φ is Δ_2-regular and $\omega \in \Omega^k_\Phi(X)$ then $A_{(i)}\omega \in \Omega^{k-1}_\Phi(X)$ and $R_{(i)}\omega - \omega = dA_{(i)}\omega + A_{(i)}d\omega$.*

Proof. (1) In view of estimates (4) and (8), we can choose $\varepsilon_{i,j}$ so that

$$\prod_{\nu=1}^{N} \|R_{\nu,\,\varepsilon_{i,\,\nu}}\|_\Phi \le 1+\frac{1}{i}, \quad \varepsilon_{i,\,j} < \frac{1}{i},$$

$$\sum_{\nu=1}^{N} \|R_{1,\,\varepsilon_{i,\,1}}\|_\Phi \|R_{2,\,\varepsilon_{i,\,2}}\|_\Phi \dots \|R_{\nu-1,\,\varepsilon_{i,\,\nu-1}}\|_\Phi \le \frac{1}{i}$$

Put $\lambda_i = \{\varepsilon_{i,\,1},\, \varepsilon_{i,\,2}, \dots\}$, $R_{(i)} = R_{\lambda_i}$, $A_{(i)} = A_{\lambda_i}$. By the above, the operators $R_{(i)}$ and $A_{(i)}$ map $L^\Phi(X,\Lambda^k)$ into $L^\Phi(X,\Lambda^k)$ and $\|R_{(i)}\|_\Phi \le 1+\frac{1}{i}$, $\|A_{(i)}\|_\Phi \le \frac{1}{i}$.

(2) was proved by de Rham for all currents ω on X.

(3) Property (5) of the de Rham operators R given in the introduction implies the convergence $R_{(i)}\omega \to \omega$ in the space of all compactly-supported forms for every smooth form ω with compact support on X. Consequently, for such forms, we have $R_{(i)}\omega \to \omega$ in $L^\Phi(X,\Lambda^k)$ for any N-function Φ. Since Φ is Δ_2-regular, smooth forms with compact support are dense in $L^\Phi(X,\Lambda^k)$ (cf. [6, p. 65]). Let ω be an arbitrary form in $L^\Phi(X,\Lambda^k)$ and let $\varepsilon > 0$. Choose a smooth k-form $\overline{\omega}$ with compact support on X such that $\|\omega - \overline{\omega}\| < \frac{\varepsilon}{6}$. Since by the above $R_{(i)}\overline{\omega} \to \overline{\omega}$ in $L^\Phi(X,\Lambda^k)$ as $i\to\infty$, there is $i_0 \in \mathbb{N}$ such that $\|R_{(i)}\overline{\omega} - \overline{\omega}\|_\Phi < \frac{\varepsilon}{2}$ for $i \ge i_0$. Thus, for $i \ge i_0$,

$$\begin{aligned}\|R_{(i)}\omega - \omega\|_\Phi &\le \|R_{(i)}\omega - R_{(i)}\overline{\omega}\|_\Phi + \|R_{(i)}\overline{\omega} - \overline{\omega}\|_\Phi + \|\overline{\omega} - \omega\|_\Phi \\ &\le (\|R_{(i)}\|+1)\|\omega - \overline{\omega}\|_\Phi + \|R_{(i)}\overline{\omega} - \overline{\omega}\| < 3\cdot\frac{\varepsilon}{6} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Hence, $R_{(i)}\omega \to \omega$ in $L^\Phi(X,\Lambda^k)$ as $i\to\infty$.

(4) Let ω be a form in $\Omega^k_{\Phi_1,\Phi_2}(X)$. We have $dR_{(i)}\omega = R_{(i)}d\omega$. Since $d\omega \in L^{\Phi_2}(X,\Lambda^{k+1})$, it is clear that $R_{(i)}d\omega \in L^\Psi(X,\Lambda^{k+1})$ and $dR_{(i)}\omega \in L^{\Phi_2}(X,\Lambda^{k+1})$. Therefore, $R_{(i)}\omega \in \Omega^k_{\Phi_1,\Phi_2}(X)$. Since from claim (3) we have $R_{(i)}\omega \to \omega$ in $L^{\Phi_1}(X,\Lambda^k)$ and $R_{(i)}d\omega \to d\omega$ in $L^{\Phi_2}(X,\Lambda^{k+1})$, we obtain $R_{(i)}\omega \to \omega$ in $\Omega^k_{\Phi_1,\Phi_2}(X)$.

(5) As was shown in [1],

$$R_{(i)}\omega - \omega = dA_{(i)}\omega + A_{(i)}d\omega$$

for any form ω on X. Since $R_{(i)}\omega,\ \omega,\ A_{(i)}d\omega \in L^\Phi(X,\Lambda^k)$ for $\omega \in \Omega^k_\Phi(X)$, we have $dA_{(i)}\omega \in L^\Phi(X,\Lambda^k)$. By assertion (1), $A_{(i)}\omega \in L^\Phi(X,\Lambda^k)$. Thus, $A_{(i)}\omega \in \Omega^k_\Phi(X)$.

□

Corollary 4.1. *For any smooth Riemannian manifold X and any Δ_2-regular N-functions Φ_1 and Φ_2, smooth forms lying in $\Omega^k_{\Phi_1,\Phi_2}(X)$ constitute a dense subset in $\Omega^k_{\Phi_1,\Phi_2}(X)$.*

In the cochain complex $\{\Omega^*_\Phi(X), d\}$, consider its subcomplex $\{\Omega^*_{\Phi,\mathrm{smooth}}(X), d\}$ consitituted by all smooth forms in $\{\Omega^*_\Phi(X), d\}$ and denote its cohomology by $H^k_{\Phi,\mathrm{smooth}}(X)$. We have

Corollary 4.2. *If Φ is a Δ_2-regular N-function then the natural transformation $H^k_{\Phi,\mathrm{smooth}}(X) \to H^k_\Phi(X)$ is an isomorphism.*

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