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NON-REGULAR GRAPH COVERINGS AND LIFTING THE HYPERELLIPTIC INVOLUTION

MAXIM P. LIMONOV

Abstract. In this paper, we prove that there exists a non-regular hyperelliptic covering of any odd degree over a hyperelliptic graph. Also, some properties of a dihedral covering, with a rotation being of odd degree, over a genus two hyperelliptic graph are derived. In the proof, the Bass-Serre theory is employed.

Keywords: Riemann surface, graph, hyperelliptic graph, hyperelliptic involution, fundamental group, automorphism group, harmonic map, branched covering, non-regular covering, graph of groups.

1. Introduction

In [1], R. D. M. Accola proved that for any odd n and any hyperelliptic Riemann surface N, there exists a non-regular smooth degree n covering $M \to N$, where M is hyperelliptic. Also, he established there some properties of a dihedral covering over a genus two Riemann surface. Namely, if the deck transformation group of the covering is generated by the rotation of odd order and by an involution, then the covering surface is 1-hyperelliptic, and the factor of covering surface by the involution is hyperelliptic.

In this paper, we prove discrete analogues of those assertions. We substitute finite connected graphs for Riemann surfaces, and harmonic maps between graphs for holomorphic maps between Riemann surfaces. In this settings, the category of

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graphs, together with harmonic maps between them, closely mirrors the category of Riemann surfaces, together with the holomorphic maps between them.

The existence of a non-regular hyperelliptic covering over a hyperelliptic graph in the particular case, when a covered graph is of genus two and a covering is of degree three, have been derived in [2].

In the proofs of the discrete versions, the Bass-Serre theory is employed to uniformize the coverings of a graph. The approach is suggested by A. Mednykh and I. Mednykh in [3]. Also, we take advantage of some topological aspects of coverings of graphs of groups developed by M. T. Green [4].

2. Preliminaries

2.1. **Graphs.** In this paper, a graph is a finite connected multigraph, possibly with loops. Denote by $V(X)$ and by $E(X)$ the set of vertices and the set of directed edges of X. Following J.-P. Serre [5], we introduce two maps ∂_0, $\partial_1 : E(X) \to V(X)$ (endpoints) and a fixed point free involution $e \to \bar{e}$ of $E(X)$ (reversal of orientation) such that $\partial_i \bar{e} = \partial_{1-i} e$. We put

$$\mathrm{St}(a) = \mathrm{St}^X(a) = \partial_0^{-1}(a) = \{e \in E(X) \,|\, \partial_0 e = a\},$$

the *star* of a, and call $\deg(a) = |St(a)|$ the *degree* (or *valency*) of a. A *morphism* of graphs $\varphi : X \to Y$ carries vertices to vertices, edges to edges, and, for $e \in E(X)$, $\varphi(\partial_i e) = \partial_i \varphi(e)\,(i = 0, 1)$ and $\varphi(\bar{e}) = \overline{\varphi(e)}$. Note that a morphism of graphs maps loops to loops. One of the possible ways to deal with loops in a graph correctly is developed in [6] and is based on the notion of semiedges.

For $a \in X$ we have the local map

$$\varphi_a : \mathrm{St}^X(a) \to \mathrm{St}^Y(\varphi(a)).$$

A map φ is *locally bijective* if φ_a is bijective for all $a \in X$. We call φ a *covering* if φ is surjective and locally bijective. A bijective morphism is called an *isomorphism*, and an isomorphism $\varphi : X \to X$ is called an *automorphism.*

Remark 1. Note that M. Baker and S. Norine in [7] use another definition of a morphism of graphs than we do here. Namely, let $\varphi : X \to Y$ be morphism of graphs and for some edge $e \in E(X)$ let $\varphi(\partial_0 e) = \varphi(\partial_1 e) = b \in V(Y)$. Then morphism φ, in the sense of [7], sends edge e to vertex b. In our case, morphism φ must send edge e to a loop based at vertex b.

2.2. **Harmonic maps and harmonic actions.** In this section, we specify the class of morphisms of graphs, called harmonic maps, that share most properties with holomorphic maps between Riemann surfaces. The notion of harmonic maps between graphs was introduced by H. Urakawa [8] for simple graphs and was generalized by M. Baker and S. Norine [7] for multigraphs.

Definition 1. A morphism $\varphi : X \to Y$ of graphs is said to be a *harmonic map* or *branched covering* if, for all $x \in V(X)$, $y \in V(Y)$ such that $y = \varphi(x)$, the quantity

$$|e \in E(X) : x = \partial_0 e,\ \varphi(e) = e'|$$

is the same for all edges $e' \in E(Y)$ such that $y = \partial_0 e'$.

Note that the composition of two harmonic morphisms is again harmonic, and an arbitrary covering of graphs is a harmonic map.

Let $\varphi : X \to Y$ be harmonic and $x \in V(X)$. We define the *multiplicity* of φ at x by

$$m_\varphi(x) = |e \in E(X) : x = \partial_0 e,\ \varphi(e) = e'|$$

for any edge $e' \in E(X)$ such that $\varphi(x) = \partial_0 e'$. By the definition of a harmonic morphism, $m_\varphi(x)$ is independent of the choice of e'. If $m_\varphi(x) > 1$ for some vertex $x \in V(X)$, such a vertex is called a *ramification point* of φ. The image $\varphi(x)$ of a ramification point is called a *branch point*.

Define the degree of a harmonic map $\varphi : X \to Y$ by the formula

$$\deg(\varphi) := |e \in E(X) : \varphi(e) = e'| \tag{1}$$

for any edge $e' \in E(Y)$. From the definition of a harmonic map of graphs and connectivity of the graphs, it follows that the right-hand side of (1) does not depend on the choice of e' and therefore $\deg(\varphi)$ is well defined.

Let $G < \mathrm{Aut}(X)$ be a group of automorphisms of a graph X. An edge $e \in E(X)$ is called *invertible* if there is $h \in G$ such that $h(e) = \bar{e}$. Let G act without invertible edges. Define the quotient graph X/G so that its vertices and edges are G-orbits of the vertices and edges of X. Note that if the endpoints of an edge $e \in E(X)$ lie in the same G-orbit then the G-orbit of e is a loop in the quotient graph X/G. Following S. Corry [9], we say that the group G acts *harmonically* on a graph X if for all subgroups $H < G$, the canonical projection $\varphi_H : X \to X/H$ is harmonic. If G acts harmonically and without invertible edges, we say that G acts *purely harmonically* on X.

The *genus* of a graph is defined as the rank of the first homology group of the graph (that is, its cyclomatic number). Let X, Y be graphs of genera g and γ respectively, and $\varphi : X \to Y$ be a harmonic map. By the same arguments as in [7], for the graph morphism under consideration we get an analogue of the Riemann-Hurwitz relation:

$$g - 1 = \deg(\varphi)(\gamma - 1) + \sum_{a \in V(X)} (m_\varphi(a) - 1), \tag{2}$$

where $m_\varphi(a)$ is the multiplicity of map φ at vertex a.

Definition 2. A graph X of genus $g \geq 2$ is said to be *hyperelliptic*, if there is a degree 2 harmonic map $F : X \to Y$, where graph Y is a tree (that is, a graph of genus 0) and 1-*hyperelliptic*, if graph Y is of genus 1. Each edge of Y has two pre-images under F and there is an order 2 automorphism τ of X, which swaps these pre-images. This automorphism is called *hyperelliptic* involution (1-*hyperelliptic* involution respectively).

Remark 2. Let X be a hyperelliptic graph and F be the corresponding harmonic map onto a tree. Since at every ramification point $x \in V(X)$ the multiplicity $m_F(x) = 2$, by (2) the number of ramification points of F is equal to $g + 1$.

Definition 3. Suppose that a harmonic map $F : X \to Y$ can be represented as a canonical projection $X \to X/G = Y$, where G acts purely harmonically. Then we call it *regular*, and *non-regular* otherwise.

A finite group G is said to admit a *partition* $\{G_1, \ldots, G_s\}$, where $G_i < G$ and $s \geq 2$, if $G = \bigcup_{i=1}^{s} G_i$ and $G_i \cap G_j = \{1\}$, $i, j = 1, 2, \ldots, s$, $i \neq j$. Let $G < \mathrm{Aut}(X)$ act purely harmonically on a graph X and admit a partition $\{G_1, \cdots, G_s\}$. Recall that the Euler characteristic $\chi(X)$ of a graph X is related to the genus $g(X)$ of X via $\chi(X) = 1 - g(X)$. By Corollary 1 in [10], we have

$$(s-1)\,g(X) + |G|\,g(X/G) = \sum_{i=1}^{s} |G_i|\,g(X/G_i).$$

Suppose a graph X_p of genus p has a group of automorphisms isomorphic to the Klein four-group $V_4 = \{U, C, UC, e\}$ which admits a partition into three subgroups of order two. Let p_1 be the genus of $X_p/\langle U\rangle$; let p_2 be the genus of $X_p/\langle C\rangle$; let p_3 be the genus of $X_p/\langle UC\rangle$; and let p_0 be the genus of X_p/V_4. Then we rewrite the formula above as

$$p + 2p_0 = p_1 + p_2 + p_3. \tag{3}$$

2.3. **Graphs of groups.** The theory of graphs of groups is employed in this paper to uniformize the coverings of a graph. Following [11], we give the definition.

Definition 4. A *graph of groups* $\mathbb{X} = (X, \mathcal{A})$ consists of

(i) a connected graph X;
(ii) an assignment $\mathcal{A}$ to every vertex $a \in V(X)$ a group $\mathcal{A}_a$, and to every edge $e \in E(X)$ a group $\mathcal{A}_e = \mathcal{A}_{\bar{e}}$;
(iii) monomorphisms $\alpha_e : \mathcal{A}_e \to \mathcal{A}_a$, where $a = \partial_0 e$.

Here we use only graphs of groups having trivial groups $\mathcal{A}_e = \{1\}$ for all edges $e \in E(X)$ and finite groups $\mathcal{A}_a$ for all vertices $a \in V(X)$. It will be enough to uniformize the coverings of a graph.

One of the possible ways to define the fundamental group of a graph of groups is as follows. Choose a spanning tree T in X. The fundamental group of $\mathbb{X}$ with respect to T, denoted $\pi_1(\mathbb{X}, T)$, is defined as the quotient of the free product

$$\left[\left(\mathop{*}_{a\in V(X)} \mathcal{A}_a\right) * F(E(X))\right] / R,$$

where $F(E(X))$ denotes the free group with basis $E(X)$ and R is the following set of relations:

(i) $\bar{e} = e^{-1}$ for every e in $E(X)$;
(ii) $e = 1$ for every e in $E(T)$.

Any other fundamental group $\pi_1(\mathbb{X}, T')$, corresponding to a different choice of a spanning tree T', will be isomorphic to $\pi_1(\mathbb{X}, T)$. In what follows, we will use notation $\pi_1(\mathbb{X})$, ignoring the way the fundamental group was constructed.

It follows from the above definition that if X is a graph of genus g then $F_g = F(E(X))/R$ is the free group of rank g. Then

$$\pi_1(\mathbb{X}) = \left(\mathop{*}_{a\in V(X)} \mathcal{A}_a\right) * F_g.$$

2.4. **Coverings of graphs of groups and harmonic maps.** Let us take graph morphisms in the definition of a covering of graphs of groups, given in [4] or [11], to be the class of all harmonic maps. Taking into consideration the fact that a trivial group is assigned to any edge, the definition of a covering of graphs of groups can be formulated as follows.

Definition 5. Let $\mathbb{X} = (X, \mathcal{A})$ and $\mathbb{Y} = (Y, \mathcal{B})$ be graphs of groups with trivial edge groups. A *covering* $\mathbb{F} = (F, \Phi) : \mathbb{X} \to \mathbb{Y}$ of graphs of groups consists of

(i) a harmonic morphism $F : X \to Y$;
(ii) a set Φ of monomorphisms $F_a : \mathcal{A}_a \to \mathcal{B}_{F(a)}$, $a \in V(X)$, such that $m_F(a)|\mathcal{A}_a| = |\mathcal{B}_{F(a)}|$, where $m_F(a)$ is the multiplicity of F at the point a.

This definition was introduced in [3]. A covering $\mathbb{F} : \mathbb{X} \to \mathbb{Y}$ of graphs of groups we call *smooth* if underlying harmonic map F is a covering, and *regular* if F is regular.

According to [4], there is a one-to-one correspondence between coverings of graphs of groups $\mathbb{F} : \mathbb{X} \to \mathbb{Y}$ and subgroups of $\pi_1(\mathbb{Y})$. If a covering is regular, then its deck transformation group is isomorphic to the factor of $\pi_1(\mathbb{Y})$ by the defining subgroup of the covering.

3. Main results

In what follows, subscript q in graph X_q is a genus of the graph. For a regular harmonic map $X \to Y$ of graphs, $G(X, Y)$ denotes its deck transformation group. The following theorem claims the existence of a non-regular hyperelliptic covering of any odd degree over a hyperelliptic graph. In the case when the degree equals 3, and the covered graph has genus 2, one of two possible non-regular hyperelliptic coverings of X_2 is depicted in the Figure 1.

Theorem 1. *Let X_q be a hyperelliptic graph, and let n be an odd positive number. Then there exists a degree n non-regular covering $X_p \to X_q$ where X_p is hyperelliptic.*

Proof. Let n be odd and positive. Let graph X_q be hyperelliptic, that is, there is the order two harmonic automorphism $\tau \in \mathrm{Aut}(X_q)$, such that the factor graph $X_0 = X_q/\langle\tau\rangle$ is a tree. Let $\psi : X_q \to X_0$ be the corresponding harmonic map.

Turn graphs X_q and X_0 into graphs of groups as follows. Let $\mathbb{X}_q = (X, \mathcal{A})$ be a graph of groups based on graph X_q, and where $\mathcal{A}$ assigns a trivial group $\mathcal{A}_z = \{1\}$ to each vertex and each edge z of X_q. Let $\mathbb{X}_0 = (X_0, \mathcal{B})$ be a graph of groups based on tree X_0, and where $\mathcal{B}$ assigns the group $\mathcal{B}_{a_i} = \mathbb{Z}_2$ to each branch point a_i, $i = 1, 2, \ldots, q+1$ of map ψ, and a trivial group $\mathcal{B}_z = \{1\}$ to every other vertex and edge z of X_0.

Let us denote by $\mathcal{F}$ the fundamental group $\pi_1(\mathbb{X}_0)$ of graph of groups $\mathbb{X}_0$. By Section 2.3, group $\mathcal{F}$ is a free product of $q+1$ copies of $\mathbb{Z}_2$ and has the presentation

$$\mathcal{F} = \left\langle \gamma_1, \gamma_2, \ldots, \gamma_{q+1} \mid \gamma_1^2 = \gamma_2^2 = \ldots = \gamma_{q+1}^2 = 1 \right\rangle.$$

We now define a smooth regular covering $\mathbb{X}_{2p-1} \to \mathbb{X}_q$ with covering transformation group D_n, the dihedral group of order $2n$. To do this we define a homomorphism

$$\mu : \mathcal{F} \to \mathbb{Z}_2 \times D_n \quad (\cong D_{2n} \text{ since } n \text{ is odd})$$

as follows. Let $\mathbb{Z}_2 = \langle C \mid C^2 = 1\rangle$, $D_n = \langle V, R \mid V^2 = R^n = (VR)^2 = 1\rangle$. Then $\langle C\rangle$ is the center of D_{2n}, and $\langle CV, R\rangle$ is also isomorphic to D_n.

Each γ_i must correspond to an involution of $\mathbb{X}_{2p-1}$ having a fixed point in $V(X)$. Since the constructed covering is smooth, no one of γ_i can be mapped by μ to involutions of $\langle V, R\rangle$. On the other hand, the images $\mu(\gamma_i)$ must generate all D_{2n}. So let

$$\begin{aligned} \mu(\gamma_i) &= C \quad \text{for } i = 1, 2, \ldots, q-1, \\ \mu(\gamma_q) &= CV, \\ \mu(\gamma_{q+1}) &= CVR. \end{aligned}$$

μ then extends to a homomorphism onto D_{2n}. The kernel F_q of the composition

$$\mathcal{F} \to \mathbb{Z}_2 \times D_n \to \langle C \rangle$$

corresponds to $\mathbb{X}_q$, and the kernel of μ, F_{2p-1}, corresponds to a covering of graphs of groups $\mathbb{X}_{2p-1} \to \mathbb{X}_q$ with deck transformation group isomorphic to D_n ($\cong F_q/F_{2p-1}$) acting on $\mathbb{X}_{2p-1}$ without fixed points. The genus of the underlying graph X_{2p-1} is $2n(q-1)+1$. The deck transformation group of the covering $\mathbb{X}_{2p-1} \to \mathbb{X}_0$, and of the underlying harmonic map $X_{2p-1} \to X_0$, is isomorphic to $\mathbb{Z}_2 \times D_n$. In this group of automorphisms let $\mathbb{Z}_2' = \langle C' \rangle$ and let $D_n' = \langle V', R' \rangle$. (We use primes to distinguish the automorphisms on X_{2p-1} from the elements of the abstract group D_{2n}.)

The central involution has branch points above a_i, $i = 1, 2, \ldots, q-1$, so the ramification for the harmonic map $X_{2p-1} \to X_{2p-1}/\langle C' \rangle$ is $2n(q-1)$. By the Riemann-Hurwitz formula the genus of $X_{2p-1}/\langle C' \rangle$ is one.

According to the homomorphism construction, the deck transformation group of covering $X_{2p-1} \to X_q$ is $\langle V', R' \rangle$. The other D_n is $\langle C'V', R' \rangle$ which contains n reflections, all of whose fixed points lie over the a_q and a_{q+1}. Thus each such reflection has $2(2n)/n$ ($= 4$) fixed points. By the Riemann-Hurwitz formula the genus of $X_{2p-1}/\langle C'V' \rangle$ is $p-2$.

Now consider the Klein four-group $H = \{V', C', C'V', e\}$. The genera of the quotients with respect to $\langle V' \rangle, \langle C' \rangle, \langle C'V' \rangle$ are $p, 1$ and $p-2$. By formula (3) the genus of X_{2p-1}/H is zero. Thus, $X_p = X_{2p-1}/\langle V' \rangle$ is hyperelliptic and is a degree n covering of X_q.

The subgroup of F_q corresponding to the covering of graphs $X_p \to X_q$, group F_p, coincides with $\mu^{-1}(\langle V \rangle)$. So subgroup F_p is not normal in F_q, therefore, the covering $X_p \to X_q$ is non-regular. □

Remark 3. The particular case $n = 3$ and $q = 2$ have been derived in [2] by I. Mednykh.

To prove the next theorem, we need the following lemmas.

Lemma 1 ([2], [12])**.** *Let a graph X_3 be a degree 2 covering of a hyperelliptic graph X_2. Then X_3 is hyperelliptic.*

Lemma 2. *Let $X_p \to X_q$ be a cyclic covering of graphs degree n and X_q is hyperelliptic. Then the harmonic map $X_p \to X_q \to X_0$ is regular with deck transformation group isomorphic to D_n.*

Proof. The harmonic map $X_p \to X_q \to X_0$ is regular if and only if the hyperelliptic involution on X_q lifts to an involution on X_p. The general theory tells us that an automorphism can be lifted to a smooth covering provided the automorphism preserves the defining subgroup of the covering. Our covering $X_p \to X_q$ is cyclic and so abelian. In this case the defining subgroup contains the commutator subgroup of the fundamental group. Therefore, we can consider the action of the hyperelliptic involution on the image of the defining subgroup in the first homology group of the graph X_q which is isomorphic to the fundamental group factored by its commutator subgroup. The hyperelliptic involution acts as minus the identity on the homology group; so all subgroups are preserved.

If U is a lift of the hyperelliptic involution u on X_q and the deck transformation group of $X_p \to X_q$ is $\langle R \mid R^n = 1 \rangle$, then RU is a lift of u as well. Since $(RU)^2 = 1$, the deck transformation group of $X_p \to X_q \to X_0$ generated by R and U is isomorphic to D_n. □

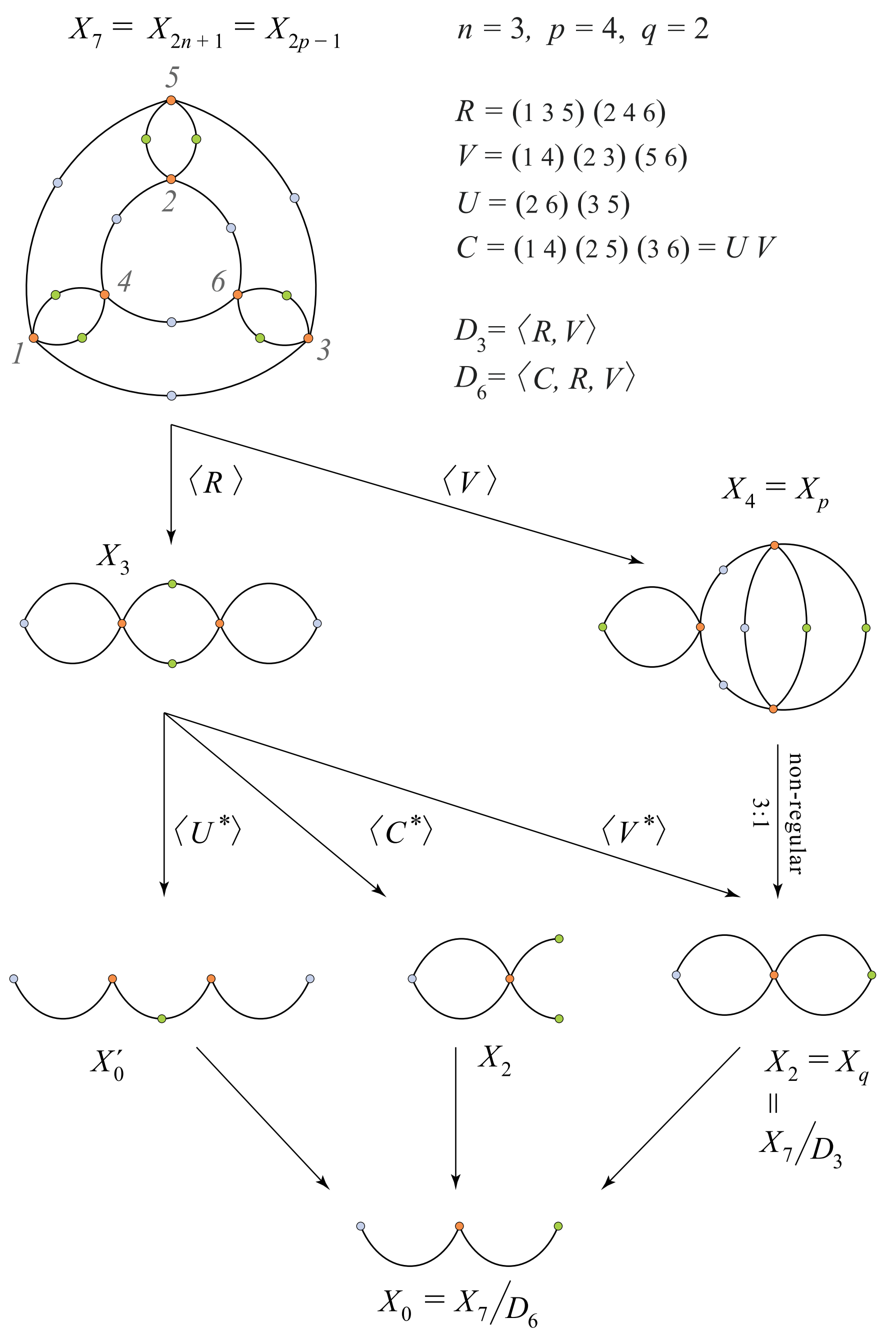
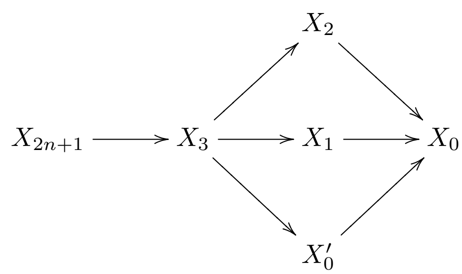


FIGURE 1. An illustration to Theorem 1 and Theorem 2 in the case $q = 2$ and $n = 3$. The notation U^* stand for an involution on X_3 which is a projection of U acting on X_7.

Using the arguments in the proof of Theorem 1 with $q = 2$, we prove the following theorem. A scheme of coverings from the proof of the theorem in case $n = 3$ is depicted in the Figure 1.

Theorem 2. *Let $X_{2n+1} \to X_2$ be a dihedral covering of graphs where n is odd. Then X_{2n+1} is 1-hyperelliptic. If $G(X_{2n+1}, X_2) = \langle V', R' \rangle$ ($\cong D_n$), then the quotient graph $X_{2n+1}/\langle V' \rangle$ is hyperelliptic.*

Proof. By relation (2), the genus of the quotient graph $X_3 = X_{2n+1}/\langle R' \rangle$ is three. By Lemma 1, X_3 is hyperelliptic. We apply Lemma 2 to see that the degree $2n$ harmonic map $X_{2n+1} \to X_3 \to X'_0$ is dihedral. Its covering transformation group $G(X_{2n+1}, X'_0) = \langle U', R' \rangle$, where U' is a lift of the hyperelliptic involution of X_3 to graph X_{2n+1}. We have the following array of coverings:



The hyperelliptic involution of X_2 lifts to X_{2n+1} through X_3, and so the entire array is regular with covering transformation group isomorphic to $Z_2 \times D_n$ ($\cong D_{2n}$ since n is odd). Then central element, C, of order 2 in D_{2n} is unique. We may assume $C' = U'V'$ by replacing V' by $V'R'^{\alpha}$ for suitable α.

We are now in the situation considered in the proof of Theorem 1 with $q = 2$ and $\mu : \mathcal{F} \to D_{2n}$ where $D_{2n} = \langle C, V, R \rangle$ with $CV = U$. Since $\mu(\gamma_i)$ has order 2 for all i and all $\mu(\gamma_i)$'s lie outside one of the D_n's in D_{2n}, say $\langle V, R \rangle$, the possibilities for $\mu(\gamma_i)$ are C and CVR^{α}. Since $\langle CV, R \rangle \cong D_n$ one of $\mu(\gamma_i)$'s must be C to get all of D_{2n}. We have that X_{2n+1} coincides with X_{2p-1} from the proof of Theorem 1. Therefore, in the degree 2 harmonic map $X_{2n+1} \to X_{2n+1}/\langle C' \rangle$, the graph $X_{2n+1}/\langle C' \rangle$ is of genus 1 and so X_{2n+1} is 1-hyperelliptic. Also, $X_{2p-1}/\langle V' \rangle$ is hyperelliptic. □

Remark 4. In the book [13] by W. S. Massey, the regular covering $X_7 \to X_2$ (Example 7.1) and the non-regular covering $X_4 \to X_2$ (Example 7.2) are presented.

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Maxim P. Limonov

Sobolev Institute of Mathematics,
pr. Koptyuga, 4,
630090, Novosibirsk, Russia

Novosibirsk State University,
Pirogova st. 2,
630090, Novosibirsk, Russia

Laboratory of Quantum Topology, Chelyabinsk State University,
Br. Kashirinykh str., 129, room 419, 430
454001, Chelyabinsk, Russia

E-mail address: volsterm@gmail.com