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SOME PROBLEMS ON KNOTS, BRAIDS, AND
AUTOMORPHISM GROUPS

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ABSTRACT. We present and discuss some open problems formulated by participants of the International Workshop “Knots, Braids, and Automorphism Groups” held in Novosibirsk, 2014. Problems are related to palindromic and commutator widths of groups; properties of Brunnian braids and two-colored braids, corresponding to an amalgamation of groups; extreme properties of hyperbolic 3-orbifold groups, relations between inner and quasi-inner automorphisms of groups; and Staic’s construction of symmetric cohomology of groups.

Keywords: Knot, braid, faithful representation, palindromic width, symmetric cohomology.

INTRODUCTION

The aim of this paper is to present some problems formulated by participants of the International Workshop «Knots, Braids and Automorphism Groups» held at Sobolev Institute of Mathematics (Novosibirsk, Russia) in July, 21–25, 2014. The list of participants, the program, and abstracts of talks are available on the workshop homepage [1].

Various open problems were presented and discussed during the problem session. They were motivated by talks given during the workshop as well as research interests of the participants. We present some of those problems in this paper. For some problems, we give preliminary discussions and useful references. In section 1, we discuss palindromic and commutator widths of groups, and give estimates of these

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widths for some classes of groups. In section 2, we formulate problems on braids with some special properties: Brunnian braids and two-colored braids, corresponding to an amalgamation of groups. In section 3, we discuss intersecting subgroups of link groups, extreme properties of hyperbolic 3-orbifold groups, and relations between inner and quasi-inner automorphisms of groups. In section 4, we present Staic's construction of symmetric cohomology of groups and formulate some problems related to it.

1. PALINDROMIC WIDTHS OF GROUPS

Let G be a group with a set of generators X . A reduced word in the alphabet $X^{\pm 1}$ is a *palindrome* if it reads the same forwards and backwards. The palindromic length $l_{\mathcal{P}}(g)$ of an element g in G is the minimum number k such that g can be expressed as a product of k palindromes. The *palindromic width* of G with respect to X is defined to be

$$\text{pw}(G, X) = \sup_{g \in G} l_{\mathcal{P}}(g).$$

When there is no confusion about the underlying generating set X , we simply denote the palindromic width with respect to X by $\text{pw}(G)$. Palindromes in groups have already proved useful in studying various aspects of combinatorial group theory and geometry, for example see [2]–[13]. It was proved in [2] that the palindromic width of a non-abelian free group is infinite. This result was generalized in [3] where the authors proved that almost all free products have infinite palindromic width; the only exception is given by the free product of two cyclic groups of order two, when the palindromic width is two. Piggot [13] studied the relationship between primitive words, i.e. words which can be include in a basis, and palindromes in free groups of rank two. It follows from [2, 13] that up to conjugacy, a primitive word can always be written as either a palindrome or a product of two palindromes and that certain pairs of palindromes will generate the group.

Bardakov and Gongopadhyay [14] initiated the investigation of palindromic width of a finitely generated group that is free in some variety of groups. They demonstrated finiteness of palindromic widths of free nilpotent and certain solvable groups. The following results were established.

Theorem 1. [14, 15] *Let $N_{n,r}$ be the free r -step nilpotent group of rank $n \geq 2$. Then the following holds:*

- (1) *The palindromic width $\text{pw}(N_{n,1})$ of a free abelian group of rank n is equal to n .*
- (2) *For $r \geq 2$, $2(n-1) \leq \text{pw}(N_{n,r}) \leq 3n$.*
- (3) *$2(n-1) \leq \text{pw}(N_{n,2}) \leq 3(n-1)$.*

Problem 1. (V. Bardakov – K. Gongopadhyay)

- (1) *For $n \geq 3$, $r \geq 2$, find $\text{pw}(N_{n,r})$.*
- (2) *Construct an algorithm that determines $l_{\mathcal{P}}(g)$ for arbitrary $g \in N_{n,r}$.*

We recall that a group G is said to satisfy *the maximal condition for normal subgroups* if every normal subgroup of G is the normal closure of a finite subset of G . A group G is said to be *abelian-by-nilpotent-by-nilpotent* if G is an extension of a group H by a nilpotent group, where H is an extension of an abelian group by a nilpotent group.

Theorem 2. [16]

- (1) Let A be a normal abelian subgroup of a finitely generated solvable group $G = \langle X \rangle$ such that G/A satisfies the maximal condition for normal subgroups. Then $\text{pw}(G, X) < \infty$.
- (2) Let G be a finitely generated free abelian-by-nilpotent-by-nilpotent group. Then the palindromic width of G is finite.

As a corollary to (2) of the above theorem the following result holds.

Corollary 1. [16] *Every finitely generated 3-step solvable group has finite palindromic width with respect to any finite generating set.*

Riley and Sale [17] studied palindromic width of solvable and metabelian groups using another technique. They proved the special case of Theorem 2(1) when A is the trivial subgroup: the palindromic width of a finitely generated solvable group that satisfies the maximal condition for normal subgroups has finite palindromic width. All mentioned results concerning solvable groups established finiteness of palindromic widths in these groups. However, the techniques that have been used to prove these results do not provide a bound for the widths. This poses the following problem.

Problem 2 (V. Bardakov – K. Gongopadhyay). *Let G be a group in Theorem 2. Find the lower and upper bounds on the palindromic width $\text{pw}(G)$.*

It would be interesting to understand palindromic width of wreath products. It was proved in [16] and [17] independently that $\text{pw}(\mathbb{Z} \wr \mathbb{Z}) = 3$. The proof from [17] is based on the estimate of $\text{pw}(G \wr \mathbb{Z}^r)$, where G is a finitely generated group. The proof from [16] relies on the fact that any element in the commutator subgroup of $\mathbb{Z} \wr \mathbb{Z}$ is a commutator. Palindromic width in wreath products has also been investigated by Fink [18] who has also obtained an estimate of $\text{pw}(G \wr \mathbb{Z}^r)$. All these results are related to the following general problem.

Problem 3 (V. Bardakov – K. Gongopadhyay). *Let $G = A \wr B$ be a wreath product of group $A = \langle X \rangle$ and $B = \langle Y \rangle$ such that $\text{pw}(A, X) < \infty$ and $\text{pw}(B, Y) < \infty$. Is it true that $\text{pw}(G, X \cup Y) < \infty$?*

Riley and Sale [17] have proved that if B is a finitely generated abelian group then the answer is positive. Fink [18] has also proved the finiteness of palindromic width of wreath product for some more cases. However, the general case when B is a finitely generated non-abelian group is still open.

If \mathcal{C} is the set of all commutators in some group G then the commutator subgroup G' is generated by \mathcal{C} . The *commutator length* $l_{\mathcal{C}}(g)$ of an element $g \in G'$ is the minimal number k such that g can be expressed as a product of k commutators. The *commutator width* of G is defined to be

$$\text{cw}(G) = \sup_{g \in G'} l_{\mathcal{C}}(g).$$

It is well known [19] that the commutator width of a free non-abelian group is infinite, but the commutator width of a finitely generated nilpotent group is finite (see [20, 21]). An algorithm of the computation of the commutator length in free non-abelian groups can be found in [22].

Bardakov and Gongopadhyay [14, Problem 2] asked to understand a connection between commutator width and palindromic width. In [16] they investigated this

question and obtained further results in this direction. But still, the exact connection is subject of further investigation.

Let us also note the following problem concerning palindromic width in solvable groups.

Problem 4 (V. Bardakov – K. Gongopadhyay). *Is it true that the palindromic width of a finitely generated solvable group of step $r \geq 4$ is finite?*

The same problem is also open for the commutator width of solvable groups [23, Question 4.34].

It is proved in [16] that the palindromic width of a solvable Baumslag – Solitar group is equal 2. For the non-solvable Baumslag – Solitar groups we formulate the following.

Problem 5 (V. Bardakov – K. Gongopadhyay). *Let*

$$BS(m, n) = \langle a, t \mid t^{-1}a^m t = a^n \rangle, \quad m, n \in \mathbb{Z} \setminus \{0\}$$

be a non solvable Baumslag – Solitar group. Is it true that $\text{pw}(BS(m, n), \{a, t\})$ is infinite?

Let us formulate the general question on the palindromic width of HNN–extensions.

Problem 6 (V. Bardakov – K. Gongopadhyay). *Let $G = \langle X \rangle$ be a group and A and B are proper isomorphic subgroups of G and $\varphi : A \rightarrow B$ be an isomorphism. Is it true that the HNN-extension*

$$G^* = \langle G, t \mid t^{-1}At = B, \varphi \rangle$$

of G with associated subgroups A and B has infinite palindromic width with respect to the generating set $X \cup \{t\}$?

In [3] palindromic widths of free products were investigated. For the generalized free products we formulate:

Problem 7 (V. Bardakov – K. Gongopadhyay). *Let $G = A *_C B$ be a free product of A and B with amalgamated subgroup C and $|A : C| \geq 3$, $|B : C| \geq 2$. Is it true that $\text{pw}(G, \{A, B\})$ is infinite?*

Finally, we note the following problem raised by Riley and Sale.

Problem 8. [17] *Is there a group G with finite generating sets X and Y such that $\text{pw}(G, X)$ is finite but $\text{pw}(G, Y)$ is infinite?*

2. BRUNNIAN WORDS, BRUNNIAN AND OTHER SPECIAL BRAIDS

2.1 Brunnian words. Let G be a group generated by a finite set X . An element $g \in G$ is called *Brunnian* if there exists a word $w = w(X)$ on X with $w = g$ such that, for each $x \in X$, $g = w$ becomes a trivial element in G by replacing all entries x in the word $w = w(X)$ to be 1.

Example. Let $X = \{x_1, \dots, x_n\}$. Then $w = [[x_1, x_2], x_3, \dots, x_n]$ is a Brunnian word in G . Any products of iterated commutators with their entries containing all elements from X are Brunnian words.

Problem 9 (J. Wu). *Given a group G and a (finite) generating set X , find an algorithm for detecting a Brunnian word that can NOT be given as a product of iterated commutators with their entries containing all elements from X .*

If G is a free group with X a basis, then all Brunnian words are given as products of iterated commutators with their entries containing all elements from X .

Most interesting Example. Let $G = \langle x_0, x_1, \dots, x_n \mid x_0 x_1 \cdots x_n = 1 \rangle$, where $X = \{x_0, x_1, \dots, x_n\}$. For $n = 2$, $[x_1, x_2]$ is Brunnian word in G . The solution of the question for this example may imply a combinatorial determination of homotopy groups of the 2-sphere [24].

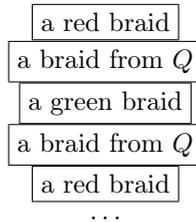
2.2 Brunnian braids. The Brunnian braids over general surfaces have been studied in [25].

Problem 10. (J. Wu)

- (1) Find a basis for Brunnian braid group.
- (2) Determine Vassiliev Invariants for Brunnian braids. The relative Lie algebras of the Brunnian braid groups have been studied in [26].
- (3) Classifying Brunnian links obtained from Brunnian braids.
- (4) Classifying links obtained from Cohen braids, where the Cohen braids were introduced in [27].

Note that Question 23 in [28] asks to determine a basis for Brunnian braid group over S^2 . A connection between Brunnian braid groups over S^2 and the homotopy groups $\pi_*(S^2)$ is given in [29].

2.3. Two-colored knot theory. Let Q be a subgroup of P_n . Then the free product $B_n *_Q B_n$ with amalgamation can be described as



Problem 11 (J. Wu). *Develop 2-colored knot theory. What are the links obtained from $B_n *_Q B_n$?*

There is a connection between the groups $P_n *_Q P_n$ and homotopy groups of spheres [30].

2.4. Finite type invariants of Gauss knots and Gauss braids. Gibson and Ito [31] defined finite type invariants of Gauss knots. It is not difficult to define finite type invariants of Gauss braids.

Problem 12 (V. Bardakov). *Is it true that finite type invariants classify Gauss braids (Gauss knots)?*

2.5. Gröbner-Shirshov bases. About Gröbner-Shirshov bases see, for example, [32, 33].

Problem 13 (L. Bokut). *Find Gröbner - Shirshov bases for generalizations of braid groups (Artin groups of types B_n , C_n , D_n , virtual braid groups and so on).*

3. BRAID GROUPS, LINK GROUPS, AND 3-MANIFOLD GROUPS

3.1. Intersecting subgroups of link groups. Let L_n be an n -component link. Let R_i be the normal closure of the i th meridian.

Problem 14 (J. Wu). *Determine the group*

$$\frac{R_1 \cap R_2 \cap \cdots \cap R_n}{[R_1, R_2, \dots, R_n]_S},$$

where $[R_1, R_2, \dots, R_n]_S = \prod_{\sigma \in \Sigma_n} [[R_{\sigma(1)}, R_{\sigma(2)}, \dots, R_{\sigma(n)}]$.

The cases $n = 2, 3$ have been discussed in [34] (see also [35]). A link L_n is called strongly nonsplittable if any nonempty proper sublink of L_n is nonsplittable. In the case that L_n is strongly nonsplittable, the group in the question is isomorphic to the homotopy group $\pi_n(S^3)$.

3.2. Extreme properties of hyperbolic 3-orbifold groups. Fundamental groups of orientable hyperbolic 3-manifolds (in particular, knot complements) and groups of orientable hyperbolic 3-orbifolds have explicit presentations in the group $\mathrm{PSL}(2, \mathbb{C})$, isomorphic to the group of all orientation-preserving isometries of the hyperbolic 3-space. It is one of the key problems in the theory of hyperbolic 3-manifolds and 3-orbifolds to answer the question whether a given subgroup of in $\mathrm{PSL}(2, \mathbb{C})$ is discrete. In 1977 Jørgensen [36] proved that the question of a discreteness of arbitrary groups can be reduced to the question of discreteness of two-generated groups.

In [37] Jørgensen obtained a necessary condition for the discreteness of a two-generated group. The condition is presented by case (1) of theorem 3 and it looks as a nonstrict inequality connecting the trace of one of the generators and the trace of the commutator. Two more necessary discreteness conditions of similar form (see cases (2) and (3) of theorem 3) were later obtained by Tan [38] and independently by Gehring and Martin [39]. Summarizing the mention results we have

Theorem 3. *Let $f, g \in \mathrm{PSL}(2, \mathbb{C})$ generate a discrete group. Then the properties hold:*

(1) *If $\langle f, g \rangle$ is nonelementary, then*

$$|\mathrm{tr}^2(f) - 4| + |\mathrm{tr}[f, g] - 2| \geq 1.$$

(2) *If $\mathrm{tr}[f, g] \neq 1$, then*

$$|\mathrm{tr}^2(f) - 2| + |\mathrm{tr}[f, g] - 1| \geq 1.$$

(3) *If $\mathrm{tr}^2(f) \neq 1$, then*

$$|\mathrm{tr}^2(f) - 1| + |\mathrm{tr}[f, g]| \geq 1.$$

The nonelementary discrete groups, having such a pair of generators that the inequality in the case (1) becomes equality, are called *Jørgensen groups*. It was shown by Callahan [40] that the figure-eight knot group is the only 3-manifold group which is a Jørgensen group. Analogously, the discrete groups, having such a pair of generators where the inequality in the case (2), respectively in the case (3), becomes equality, are called *Tan groups*, respectively, *Gehring – Martin – Tan groups*. It was shown by Masley and Vesnin [41] that the group of the figure-eight orbifold with singularity of order four is a Gehring – Martin – Tan group.

Problem 15 (A. Vesnin). *Find all hyperbolic 3-orbifold groups which are Gehring – Martin – Tan groups or Tan groups.*

3.3. Cabling operations on some subgroups of $\mathrm{PSL}(2, \mathbb{C})$. Let K be a framed knot with its group as a subgroup of $\mathrm{PSL}(2, \mathbb{C})$. Let K_n be the n -component link obtained by naive cabling of K along the frame.

Problem 16 (J. Wu). *Describe a cabling construction for the link group $\pi_1(S^3 \setminus K_n)$ as a subgroup of $\mathrm{PSL}(2, \mathbb{C})$.*

The groups of K_n with $n \geq 1$ admit a canonical simplicial group structure with its geometric realization homotopic to the loop space of S^3 if K is a nontrivial framed knot (see [42]).

3.4. Inner and quasi-inner automorphisms. An automorphism φ of some group G is called *class-preserving automorphism* if for every $g \in G$ its image $\varphi(g)$ is conjugate to g in G . M. Neshchadim proved that for the group $G = \langle a, b \mid a^2 = b^2 \rangle$ the group $\mathrm{Aut}_c(G)$ of class-preserving automorphisms is not equal to the group $\mathrm{Inn}(G)$ of inner automorphisms. This group G is a free product of two cyclic groups with amalgamation $G = \mathbb{Z} *_2 \mathbb{Z} = {}_2\mathbb{Z}$ and is a group of $(2, 2)$ -torus link. On the other hand the group of trefoil knot is isomorphic to the braid group $B_3 = \mathbb{Z} *_3 \mathbb{Z} = {}_3\mathbb{Z}$ and for this group $\mathrm{Aut}_c(B_3) = \mathrm{Inn}(B_3)$. It is known that if K is alternating knot, then its group $\pi_1(S^3 \setminus K)$ has a decomposition $\pi_1(S^3 \setminus K) = F_n *_{F_{2n-1}} F_n$ for some natural n .

Problem 17 (V. Bardakov). *For which knot (link) K its group $\pi_1(S^3 \setminus K)$ has the property:*

$$\mathrm{Aut}_c(\pi_1(S^3 \setminus K)) = \mathrm{Inn}(\pi_1(S^3 \setminus K))?$$

3.5. Free nilpotent groups. Consider a reduced word $w = w(a, b)$ in the free group $F_2 = \langle a, b \rangle$ such that the product $x * y = w(x, y)$ defines a group operation in the free nilpotent group $F_n / \gamma_k(F_n)$ for all $n \geq 2$ and $k \geq 1$.

Problem 18 (M. Neshchadim). *Find all words w having above property. Is it true that $w = ab$ or $w = ba$?*

4. SYMMETRIC COHOMOLOGY OF GROUPS

4.1. Cohomology of groups. Cohomology of groups is a contravariant functor turning groups and covariant functor turning modules over groups into graded abelian groups. It came into being with the fundamental work of Eilenberg and MacLane [43, 44]. The theory was further developed by Hopf, Eckmann, Segal, Serre, and many other mathematicians. It has been studied from different perspectives with applications in various areas of mathematics, and provides a beautiful link between algebra and topology.

We quickly recall the construction of cohomology of groups. Let G be a group and A a G -module. For convenience, A is written additively and G is written multiplicatively. For each $n \geq 0$, let $C^n(G, A)$ be the group of all maps $\sigma : G^n \rightarrow A$. Define

$$\partial^n : C^n(G, A) \rightarrow C^{n+1}(G, A)$$

by

$$\begin{aligned} \partial^n(\sigma)(g_1, \dots, g_{n+1}) &= g_1\sigma(g_2, \dots, g_{n+1}) \\ &\quad + \sum_{i=1}^n (-1)^i \sigma(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) + (-1)^{n+1} \sigma(g_1, \dots, g_n), \end{aligned}$$

for $\sigma \in C^n(G, A)$ and $(g_1, \dots, g_{n+1}) \in G^{n+1}$. It is easy to see that $\partial^{n+1}\partial^n = 0$ making $\{C^*(G, A), \partial^*\}$ a cochain complex. Its cohomology, denoted by $H^*(G, A)$, is called the cohomology of G with coefficients in A .

Let $\mathcal{E}(G, A)$ be the set of equivalence classes of extensions of G by A giving rise to the given action of G on A . Then there is a one-one correspondence

$$\Phi : H^2(G, A) \rightarrow \mathcal{E}(G, A).$$

Under Φ , the trivial element of $H^2(G, A)$ corresponds to the equivalence class of an extension

$$0 \rightarrow A \rightarrow E \rightarrow G \rightarrow 1$$

admitting a section $s : G \rightarrow E$ which is a group homomorphism.

An extension $\mathcal{E} : 0 \rightarrow A \rightarrow E \rightarrow G \rightarrow 1$ of G by A is called a symmetric extension if there exists a section $s : G \rightarrow E$ such that $s(g^{-1}) = s(g)^{-1}$ for all $g \in G$. Such a section is called a symmetric section. Let $\mathcal{S}(G, A) = \{[\mathcal{E}] \in \mathcal{E}(G, A) \mid \mathcal{E} \text{ is a symmetric extension}\}$. Then the following question seems natural.

Question 1. *What elements of $H^2(G, A)$ corresponds to $\mathcal{S}(G, A)$ under Φ ?*

Staic [45, 46] answered the above question for (abstract) groups. Motivated by some questions regarding construction of invariants of 3-manifolds, Staic introduced a new cohomology theory of groups, called the symmetric cohomology, which classifies symmetric extensions in dimension two.

4.2. Symmetric cohomology of groups. In this section, we present Staic's construction of symmetric cohomology of groups. For topological aspects of this construction, we refer the reader to [45, 46]. For each $n \geq 0$, let Σ_{n+1} be the symmetric group on $n + 1$ symbols. In [45], Staic defined an action of Σ_{n+1} on $C^n(G, A)$. Since the transpositions of adjacent elements form a generating set for Σ_{n+1} , it is enough to define the action of these transpositions $\tau_i = (i, i + 1)$ for $1 \leq i \leq n$. For $\sigma \in C^n(G, A)$ and $(g_1, \dots, g_n) \in G^n$, define

$$\begin{aligned} (\tau_1\sigma)(g_1, g_2, g_3, \dots, g_n) &= -g_1\sigma(g_1^{-1}, g_1g_2, g_3, \dots, g_n), \\ (\tau_i\sigma)(g_1, g_2, g_3, \dots, g_n) &= -\sigma(g_1, \dots, g_{i-2}, g_{i-1}g_i, g_i^{-1}, g_i g_{i+1}, g_{i+2}, \dots, g_n) \\ &\quad \text{for } 1 < i < n, \end{aligned}$$

$$(\tau_n\sigma)(g_1, g_2, g_3, \dots, g_n) = -\sigma(g_1, g_2, g_3, \dots, g_{n-1}g_n, g_n^{-1}).$$

It is easy to see that

$$\begin{aligned} \tau_i(\tau_i(\sigma)) &= \sigma, \\ \tau_i(\tau_j(\sigma)) &= \tau_j(\tau_i(\sigma)) \text{ for } j \neq i \pm 1, \\ \tau_i(\tau_{i+1}(\tau_i(\sigma))) &= \tau_{i+1}(\tau_i(\tau_{i+1}(\sigma))). \end{aligned}$$

Thus there is an action of Σ_{n+1} on $C^n(G, A)$. Let $CS^n(G, A) = C^n(G, A)^{\Sigma_{n+1}}$ be the group of invariant n -cochains. If $\sigma \in CS^n(G, A)$, then it can be proved that $\partial^n(\sigma) \in CS^{n+1}(G, A)$. Thus the action is compatible with the coboundary

operators and we obtain a cochain complex $\{CS^*(G, A), \partial^*\}$. Its cohomology, denoted by $HS^*(G, A)$, is called the symmetric cohomology of G with coefficients in A . In [45, 46], Staic gave examples of groups for which the symmetric cohomology is different from the ordinary cohomology.

The inclusion $CS^*(G, A) \hookrightarrow C^*(G, A)$ induces a homomorphism $h^* : HS^*(G, A) \rightarrow H^*(G, A)$. In [46], Staic proved that the map

$$h^* : HS^2(G, A) \rightarrow H^2(G, A)$$

is injective and the composite map

$$\Phi \circ h^* : HS^2(G, A) \rightarrow \mathcal{S}(G, A)$$

is bijective. Thus the symmetric cohomology in dimension two classifies symmetric extensions. This answers Question 1.

When a group under consideration is equipped with a topology or any other structure, it is natural to look for a cohomology theory which also takes the topology or the other structure into account. This led to various cohomology theories of topological groups and Lie groups. Topology was first inserted in the formal definition of cohomology of topological groups in the works of Hu [47], van Est [48] and Heller [49]. In a recent paper [50], Singh studied continuous and smooth versions of Staic's symmetric cohomology. He defined a symmetric continuous cohomology of topological groups and gave a characterization of topological group extensions that correspond to elements of the second symmetric continuous cohomology. He also defined symmetric smooth cohomology of Lie groups and proved similar results. These results answered continuous and smooth analogues of Question 1.

4.3. Some Problems. Symmetric cohomology of groups is a fairly recent construction, and not much is known about its theoretical and computational aspects. In view of this, the following questions seem natural. We hope that answers to these questions will help in better understanding and possible applications of symmetric cohomology.

Problem 19 (M. Singh). *Does there exist a Lyndon – Hochschild – Serre type spectral sequence for symmetric cohomology of groups?*

Problem 20 (M. Singh). *Cohomology of a discrete group can also be defined as the cohomology of its classifying space. Is there a topological way of obtaining symmetric cohomology of discrete groups?*

Problem 21 (M. Singh). *Is it possible to define a symmetric cohomology of Lie algebras? How does this relate to the symmetric cohomology of Lie groups? It seems possible to do so, and Staic suspects that it is equal to the usual cohomology [51].*

Problem 22 (M. Singh). *Fiedorowicz and Loday [52] introduced crossed simplicial groups and a cohomology theory of these objects, which is similar to symmetric cohomology. It would be interesting to explore some connection between the two [45].*

Problem 23 (M. Singh). *Let G be a group and \mathbb{C}^\times a trivial G -module. Then the Schur multiplier of G is defined as $\mathcal{M}(G) := H^2(G, \mathbb{C}^\times)$. It turns out that the Schur multiplier $\mathcal{M}(G)$ of a finite group G is a finite abelian group. Finding bounds on the order of $\mathcal{M}(G)$ is an active area of research and has wide range of applications, particularly in automorphisms and representations of finite groups. We define the symmetric Schur multiplier of G as*

$$\mathcal{MS}(G) := HS^2(G, \mathbb{C}^\times).$$

Clearly, the symmetric Schur multiplier $MS(G)$ of a finite group G is a finite group. It would be interesting to find bounds on the order of $MS(G)$ and $\mathcal{M}(G)/MS(G)$.

Problem 24 (M. Singh). *Let G be a finite group. Then the Bogomolov multiplier of G is defined as*

$$B_0(G) = \ker(\text{res}_A^G : H^2(G, \mathbb{C}^\times) \rightarrow \bigoplus_{A \subset G} H^2(A, \mathbb{C}^\times))$$

where A runs over all abelian subgroups of G and res_A^G is the usual restriction homomorphism. The group $B_0(G)$ is a subgroup of the Schur multiplier $\mathcal{M}(G)$, and appears in classical Noether's problem and birational geometry of quotient spaces of G . See [53] for a recent survey article. It would be interesting to find relation between $MS(G)$ and $B_0(G)$.

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