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ON THE NUMBER OF MAXIMUM INDEPENDENT SETS IN DOOB GRAPHS

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Abstract. The Doob graph $D(m,n)$ is a distance-regular graph with the same parameters as the Hamming graph $H(2m{+}n,4)$. The maximum independent sets in the Doob graphs are analogs of the distance-2 MDS codes in the Hamming graphs. We prove that the logarithm of the number of the maximum independent sets in $D(m,n)$ grows as $2^{2m+n-1}(1{+}o(1))$. The main tool for the upper estimation is constructing an injective map from the class of maximum independent sets in $D(m,n)$ to the class of distance-2 MDS codes in $H(2m+n,4)$.

Keywords: Doob graph, independent set, MDS code, latin hypercube.

1. Introduction

The Cartesian product $D(m,n) \stackrel{\mathrm{def}}{=} \mathrm{Sh}^m \times K_4^n$ of m copies of the Shrikhande graph Sh (see Figure 1) and n copies of the complete graph K_q of order $q = 4$ is called a *Doob graph* if $m > 0$, while $D(0,n)$ is the *Hamming graph* $H(n,4)$ (in general $H(n,q) \stackrel{\mathrm{def}}{=} K_q^n$). The Doob graph $D(m,n)$ is a distance-regular graph with the same parameters as $H(2m+n,4)$, see e.g. [1, §9.2.B]. It is easy to see that the independence number of this graph is 4^{2m+n-1}. The maximum independent sets in the Hamming graphs are known as the distance-2 MDS codes (below, simply *MDS codes*), or the latin hypercubes (in the last case, one of the coordinates is usually considered as a function of the others). It is naturally to call the maximum independent sets in Doob graphs by the same notion, the *MDS codes*. Indeed, the

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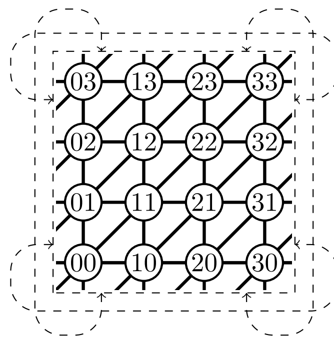

FIGURE 1. The Shrikhande graph drown on a torus; the vertices are identified with the elements of Z_4^2

maximum independent sets in $D(m,n)$ and the MDS codes in $H(2m+n,4)$ have the same parameters being considered as error-correcting codes (see [5] for the background on error-correcting codes) and as completely regular codes (see e.g. [1, §11.3]). (The concept of the latin hypercubes can also be generalized to $D(m,n)$; however, to do this, we need at least one K_4 coordinate to treat as dependent, i.e., $n > 0$.) There are 4 trivial MDS codes in $D(0,1)$; 24 equivalent MDS codes in $D(0,2)$ (16 of them can be found in Figure 2); 16 MDS codes in $D(1,0)$ (see Figure 2), which form two equivalence classes (with 4 and 12 representatives, respectively).

The main result of the current correspondence is the following.

Theorem 1. *The number of the maximum independent sets (distance-2 MDS codes) in the Doob graph $D(m,n)$ grows as $2^{2^{2m+n-1}(1+o(1))}$ as $(2m+n) \to \infty$.*

The statement of the theorem is straightforward from Corollaries 1 (an upper bound) and 2 (a lower bound) proven in the next two sections.

2. An upper bound

In this section, we describe a rather simple recursive way to map injectively the set $\mathrm{MDS}_{m,n}$ of MDS codes in $D(m,n)$ into $\mathrm{MDS}_{0,2m+n}$. At first, we define the map ξ from $\mathrm{MDS}_{1,0}$ into $\mathrm{MDS}_{0,2}$, see Figure 2.

For arbitrary m, $n \geq 0$, the action of $\kappa : \mathrm{MDS}_{m+1,n} \to \mathrm{MDS}_{m,n+2}$ is defined as follows:

$$\kappa M \stackrel{\mathrm{def}}{=} \big\{(x_1,...,x_m,z_1,z_2,y_1,...,y_n) \in D(m,n+2) \,\big|\, (z_1,z_2) \in \xi M_{x_1,...,x_m,y_1,...,y_n}\big\},$$

where

$$M_{x_1,...,x_m,y_1,...,y_n} \stackrel{\mathrm{def}}{=} \{v \in \mathrm{Sh} \mid (x_1,...,x_m,v,y_1,...,y_n) \in M\}$$

Lemma 1. *For every MDS code in $D(m+1,n)$, the set κM is an MDS code in $D(m,n+2)$.*

Proof. The map ξ has the following important property, which can be checked directly, see Figure 2: two MDS codes M' and M'' in $D(1,0)$ intersect if and only if their images $\xi M'$ and $\xi M''$ intersect. Since M is an independent set, $M_{x_1,...,x_m,y_1,...,y_n}$ and $M_{u_1,...,u_m,w_1,...,w_n}$ (and hence, also $\xi M_{x_1,...,x_m,y_1,...,y_n}$ and $\xi M_{u_1,...,u_m,w_1,...,w_n}$) are disjoint for any two neighbor vertices $(x_1,...,x_m,y_1,...,y_n)$ and $(u_1,...,u_m,w_1,...,w_n)$

ξ :

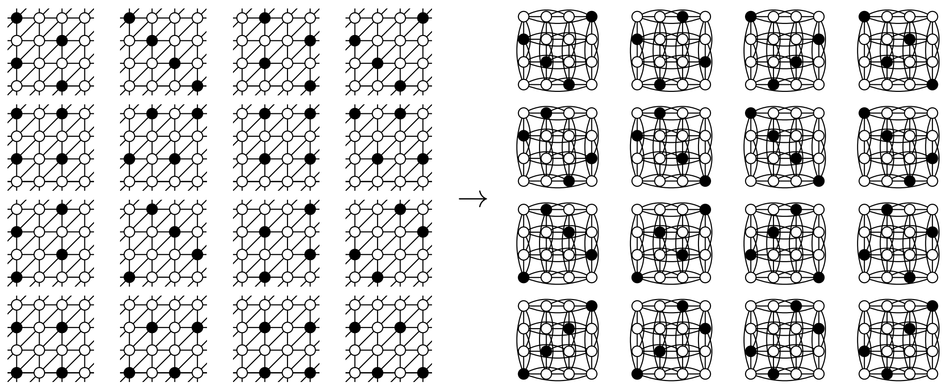

FIGURE 2. The 16 MDS codes in Sh and the corresponding MDS codes in K_4^2

of $D(m,n)$. It follows that κM is also an independent set. Moreover, it has the same cardinality as M, i.e., 4^{2m+n+1}. □

Then, κ^m, the mth iteration of κ, maps $\mathrm{MDS}_{m,n}$ into $\mathrm{MDS}_{0,2m+n}$.

Corollary 1. *The number of the MDS codes in $D(m,n)$ does not exceed*

$$2^{2^{2m+n-1}(1+o(1))}.$$

Proof. Since κ obviously maps different MDS codes to different MDS codes, the statement of the corollary in the general case can be inductively reduced to the partial case $m=0$, which was proven in [8], see also [2]. □

3. A lower bound

In this section, we consider a simple way to construct doubly exponential (with respect to the graph diameter $2m+n$) number of MDS codes in the Doob graph $D(m,n)$.

The vertices of Sh will be identified with the pairs ab (considered as a short notation for (a,b)), where a, $b \in \{0,1,2,3\}$, see Figure 1. The vertices of K_4 will be identified with the pairs ab, where a, $b \in \{0,1\}$. For every function λ from $\{0,1,2,3\}^m \times \{0,1\}^n$ to $\{0,1\}$, we define the set

$$M_\lambda \overset{\mathrm{def}}{=} \Big\{ (x'_1x''_1,\ldots,x'_{m+n}x''_{m+n}) \in D(m,n) \ \Big| \ \sum_{i=1}^{m+n} x'_i \equiv 0 \bmod 2, \ \sum_{i=1}^{m+n} x''_i \equiv \lambda(x'_1,\ldots,x'_{m+n}) \bmod 2 \Big\}.$$

Lemma 2. *For any function $\lambda : \{0,1,2,3\}^m \times \{0,1\}^n \to \{0,1\}$, the set M_λ is an MDS code in $D(m,n)$.*

Proof. It is easy to see that if $m < i \le m+n$, then for any values of $x'_1x''_1$, $\ldots$, $x'_{i-1}x''_{i-1}$, $x'_{i+1}x''_{i+1}$, $\ldots$, $x'_{m+n}x''_{m+n}$ there is a unique pair $x'_ix''_i$ such that $(x'_1x''_1,\ldots,x'_{m+n}x''_{m+n}) \in M_\lambda$. If $1 \le i \le m$, then there are four such pairs (two possibilities for x'_i, of the same parity, and for each choice of x'_i, two possibilities for

x''_i), but they correspond to pairwise independent vertices of the Shrikhande graph. Consequently, at first, $|M_\lambda| = 4^{2m+n-1}$, and at second, M_λ is an independent set. □

Corollary 2. *There are at least $2^{2^{2m+n-1}}$ different MDS codes in $D(m,n)$.*

Proof. We will say that two functions from $\{0,1,2,3\}^m \times \{0,1\}^n$ to $\{0,1\}$ are essentially different if their values are different in at least one point $(x'_1,\ldots,x'_{m+n})$ satisfying $x'_1+\ldots+x'_{m+n} \equiv 0 \bmod 2$. The number of essentially different functions is $2^{2^{2m+n-1}}$. Obviously, essentially different functions λ lead to different MDS codes M_λ. □

4. Conclusion

We have established the asymptotics of $\log|\mathrm{MDS}_{m,n}|$, generalizing the similar result for the MDS codes in the Hamming graph $H(n,4)$ [2], [8]. Note that the case $q=4$ is the only nontrivial case when the asymptotics of the double logarithm of the number of MDS codes is known ($n\to\infty$, q is fixed). Known bounds for the other cases can be found in [4], [9]; the exact values for small q and n, in [6], [7], [9].

A constructive characterization of the class $\mathrm{MDS}_{0,n}$ can be found in [3]. A possibility to relate the MDS codes (maximum independent sets) in $D(m,n)$ with MDS codes in $D(2m+n,4)$ using the map κ^m suggests that a similar characterization might be possible for $\mathrm{MDS}_{m,n}$ with arbitrary m. However, it is not completely clear if the map κ^m itself can be helpful for a reasonable proof of such characterization. Since the map κ is not point-to-point, the result of the mth iteration of κ can depend on the order of coordinates. As a result, it is not easy to track which subclass of $\mathrm{MDS}_{0,2m+n}$ we obtain as the image of $\mathrm{MDS}_{m,n}$ under κ^m and to describe this subclass in terms of the known characterization of $\mathrm{MDS}_{0,2m+n}$. In any case, finding a characterization of the class $\mathrm{MDS}_{m,n}$, using κ or not, will be a natural continuation of the current research.

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