S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 12, стр. 552–561 (2015) UDC 512.54
DOI 10.17377/semi.2015.12.045 MSC 20E08, 20F10

ADMISSIBLE SLIDES FOR GENERALIZED BAUMSLAG–SOLITAR GROUPS

F.A.DUDKIN

Abstract. A generalized Baumslag-Solitar group (GBS group) is a finitely generated group G which acts on a tree with all edge and vertex stabilizers infinite cyclic. Any GBS group is isomorphic to fundamental group $\pi_1(\mathbb{A})$ of some labeled graph $\mathbb{A}$. Slide is a transformation of labeled graphs. Slides play an important role in isomorphism problem for GBS groups. Given an edge e with label λ and $\alpha \in \mathbb{Q}$. In this paper we describe an algorithm that checks if there exists a cycle p such that after slide e over p label λ multiplies by α or not. If such cycle exists then the algorithm finds one of them.

Keywords: isomorphism problem, generalized Baumslag–Solitar group, labeled graph.

1. Introduction

Call a finitely generated group G a *generalized Baumslag-Solitar group* or a GBS group if G can act on a tree so that the stabilizers of vertices and edges are infinite cyclic groups. By the Bass-Serre theorem, G is representable as $\pi_1(\mathbb{A})$, the fundamental group of a graph of groups $\mathbb{A}$ (see [1]) with infinite cyclic edge and vertex groups.

Given a GBS group G, we can present the corresponding graph of groups $\mathbb{A}$ by a labeled graph (A, λ), where A is a finite connected graph (with endpoint functions $\partial_0, \partial_1 \colon E(A) \to V(A)$) and $\lambda \colon E(A) \to \mathbb{Z} \setminus \{0\}$ labels the edges of A. The label λ_e of an edge e with the source vertex v defines an embedding $\alpha_e \colon e \to v^{\lambda_e}$ of the cyclic edge group $\langle e \rangle$ into the cyclic vertex group $\langle v \rangle$ (for more details see [5])

Dudkin, F.A., Admissible slides for generalized Baumslag–Solitar groups.

The work is supported by Russian Science Foundation (project 14-21-00065).
Received May, 15, 2015, published September, 14, 2015.

Recently GBS groups have been quite actively studied [8], [4], [3]. In particular, the isomorphism problem for GBS groups has been discussed: to determine algorithmically when two given labeled graphs define isomorphic GBS groups. Despite that, the isomorphism problem is solved only in several special cases [2], [5], [6], but the general solution is not established.

If two labeled graphs $\mathbb{A}$ and $\mathbb{B}$ define isomorphic GBS groups $\pi_1(\mathbb{A}) \cong \pi_1(\mathbb{B})$ and $\pi_1(\mathbb{A})$ is not isomorphic to $\mathbb{Z}, \mathbb{Z}^2$ or Klein bottle group then there exists a finite sequence of *expansion* and *collapse* (see fig.1) moves connecting $\mathbb{A}$ and $\mathbb{B}$ [7]. A labeled graph is called *reduced* if it admits no collapse move (equivalently, the labeled graph contains no edges with distinct endpoints and labels ± 1).

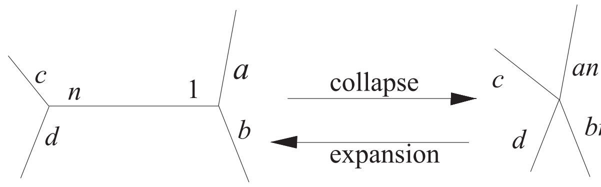

FIG. 1. Expansion and collapse moves.

The following three types of transformations of labeled graphs play an important role for GBS groups:

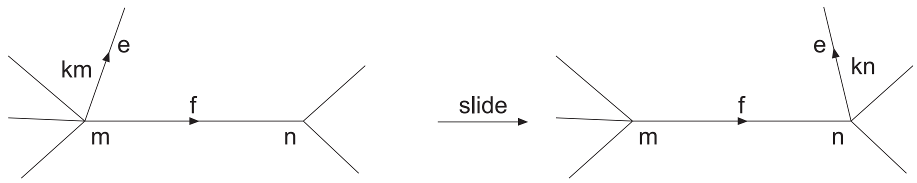

FIG. 2. Slide e/f.

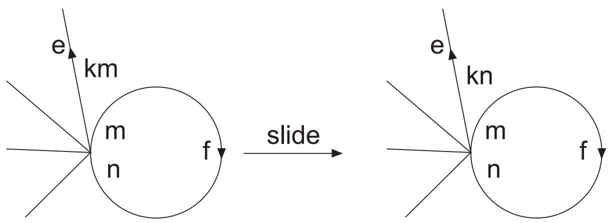

FIG. 3. Slide e/f.

Given a labeled graph $\mathbb{A}$ (a GBS group G), denote the set of reduced labeled graphs with the fundamental group isomorphic to $\pi_1(\mathbb{A})$ (resp. G) by $R(\mathbb{A})$ (resp. $R(G)$).

Theorem (M. Clay, M. Forester [8]**)** *Given GBS group G and $\mathbb{A}, \mathbb{B} \in R(G)$ then labeled graphs $\mathbb{A}$ and $\mathbb{B}$ are related by a finite sequence of slides, inductions, and $\mathscr{A}^{\pm 1}$–moves, with all intermediate labeled graphs reduced.*

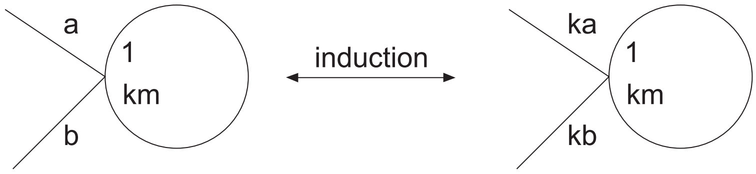

FIG. 4. Induction.

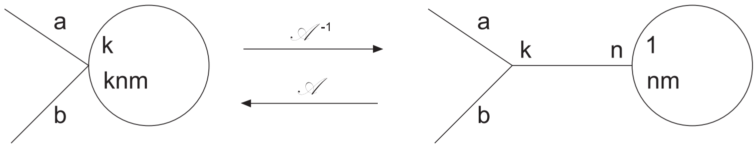

FIG. 5. $\mathscr{A}$ and $\mathscr{A}^{-1}$ moves.

An *edge path* $p = (e_1, e_2, \ldots, e_n)$ is a sequence of edges such that $\partial_0(e_l) = \partial_1(e_{l-1})$ for $l = 1, \ldots, n$, there $n = |p|$ is a *length* of p. A *loop* is an edge e such that $\partial_0(e) = \partial_1(e)$. A *cycle* is an edge path such that $\partial_0(e_1) = \partial_1(e_n)$. A geometric edge is a pair of the form $\{e, \overline{e}\}$.

Given a labeled graph $\mathbb{A} = (A, \lambda)$ and an edge path in A $p = (e_1, e_2, \ldots, e_n)$, put

$$\Delta(p) = \prod_{i=1}^{n} \frac{\lambda_{\bar{e}_i}}{\lambda_{e_i}}.$$

If the sequences of slides e/e_i for $i = 1, \ldots, n$ is admissible, then the slide of e over the path $p = (e_1, \ldots, e_n)$ is called *admissible* and denoted by e/p. If $e \in E(A)$ is fixed then path p is called *admissible* if e/p is admissible.

Given a graph A, denote by A_0 the undirected graph obtained from A by identifying the edges e and $\bar{e}$. If T_{A_0} some maximal subtree of the A_0, then denote by T_A subgraph of A such that $(T_A)_0$ coincides with T_{A_0}.

Suppose that $\{f_1, \ldots, f_k\} = E(A \setminus T_A)$. Given $a \in V(A)$, denote by $p_i, i = 1, \ldots, k$ cycles in A with endpoints a such that p_i one time passes through f_i and doesn't pass through f_j or $\overline{f}_j$ for $j \neq i$. Cycles $p_1, \ldots, p_k$ is called a *basic cycles with endpoints* a. A subgroup $\langle \Delta p_1, \Delta p_2, \ldots, \Delta p_k \rangle$ of the multiplicative group $\mathbb{Q}^*$ is denoted $\Delta\pi_1(\mathbb{A})$. The map $\Delta\colon \pi_1(\mathbb{A}) \to \mathbb{Q}^*$ is defined on GBS groups and called *modular homomorphism* (see for example [6]).

Suppose that there exist an edge e and a cycle p in A such that after slide e/p graph A doesn't change, but the label on e multiplies by $\alpha \in \mathbb{Z} \setminus \{0, \pm 1\}$. In this case slide e/p can be repeated many times. It turns out that the group $\pi_1(\mathbb{A})$ can be represented by infinite numbers of reduced labeled graphs and the theorem of M. Clay and M. Forester can not be used to solve the isomorphism problem without restrictions on graph A.

Known solutions of isomorphism problem was obtained under essential restrictions on $\Delta\pi_1(\mathbb{A})$ or on the structure of graph A. These restrictions usually allowed to describe admissible slides and, as a result, to solve the isomorphism problem. In general case to describe an admissible slides is quite difficult problem. In this paper without assumption on labeled graph $\mathbb{A}$ we try to understand what labels can be written on edge e after all possible slides of e.

Theorem 8 *Given labeled graph $\mathbb{A}$, $e \in E(A)$ and $\alpha \in \Delta\pi_1(\mathbb{A})$. There exists an algorithm that checks if there exists a cycle p such that slide e/p is admissible and $\Delta p = \alpha$ or not. If such cycle exists then the algorithm finds one of them.*

2. Definitions and auxiliary statements

A *graph* A is a set $V(A)$ (vertices), a set $E(A)$ (edges), maps $\partial_0, \partial_1 \colon E(A) \to V(A)$ (endpoints) and a fixed point free involution $e \to \bar{e}$ of $E(A)$ (reversal of orientation) such that $\partial_i \bar{e} = \partial_{1-i} e$. Denote by A_0 the undirected graph corresponding to graph A (we can get A_0 from A by identifying e and $\bar{e}$ for all $e \in E(A)$).

Let $\{p_1, p_2, \ldots, p_r\}$ be the set of prime devisors of $\prod_{e \in E(A)} \lambda_e$. If $\pi_1(\mathbb{C}) \cong \pi_1(\mathbb{A})$ then any label of labeled graph $\mathbb{C}$ is equal to $p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_r^{\alpha_r}$ for some integers $\alpha_i \geqslant 0$ for $i = 1, 2, \ldots, r$. Therefore it is possible to define a map

$$\overline{} \colon \Delta\pi_1(\mathbb{A}) \to \mathbb{Z}^r$$

such that $\overline{p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_r^{\alpha_r}} = (\alpha_1, \alpha_2, \ldots, \alpha_r)$. The multiplication of labels become to the addition of images. Unless otherwise stated, we will use the additive notation for labels and identify labels with elements of $\mathbb{Z}^r$. Let S be a subset of $\{1, 2, \ldots, r\}$. Denote $\{1, 2, \ldots, r\} \setminus S$ by $\overline{S}$.

Given $v, u \in \mathbb{Z}^r_{\geqslant 0}$ denote i-th coordinate of v by $[v]_i \in \mathbb{Z}$. Say that $v \leqslant u (v < u)$ if $[v]_i \leqslant [u]_i$ for all $i = 1, \ldots, r$ (with strict inequality for at least one index). Set $\mathbb{Z}^r_{\geqslant 0}$ with binary relation $\leqslant$ is a partially ordered set. For $a, b \in \mathbb{N}$ it's obvious that $\overline{a} \leqslant \overline{b} (\overline{a} < \overline{b})$ if and only if $a | b$(a is proper divisor of b).

Given an edge path $p = (e_1, e_2, \ldots, e_n)$ in graph A and $\Delta p = v^+ - v^-$ where $v^+, v^- \geqslant 0$. Denote λ_{e_i} by α_i and $\lambda_{\bar{e}_i}$ by β_i. Define γ_p as the minimal element from $\mathbb{Z}^r_{\geqslant 0}$ such that the following inequalities hold

$$v^- + \gamma_p + \sum_{j=1}^{i} (\beta_j - \alpha_j) - \alpha_{i+1} \geqslant 0, i = 0, \ldots, n-1.$$

Or equivalently

$$\gamma_p = \max \left\{ \alpha_{i+1} + \sum_{j=1}^{i} (\alpha_j - \beta_j) - v^-, i = 0, \ldots, n-1 \right\}.$$

Lemma 1. (admissibility criterion) *Slide e/p is admissible if and only if $\lambda_e \geqslant \gamma_p + (\Delta p)^-$.*

Proof. The proof is straightforward.

Proposition 2. *Given a sequence $\widetilde{v} = \{v_i\}_{i=1}^{\infty}$ in $\mathbb{Z}^r_{\geqslant 0}$ such that $|v_i|_1 \leqslant \alpha_i$ (where $|v|_1$ is the L^1-norm). There exists a natural number $L = L(\widetilde{\alpha} = \{\alpha_i\}_{i=1}^{\infty}, r)$ such that there exist $l_1 < l_2 \leqslant L$ such that either $v_{l_1} \geqslant v_{l_2}$ or $v_{l_2} \geqslant v_{l_1}$.*

Proof. Induction on r. If $r = 1$, then $L = 2$. If $r = 2$, then it is easy to see that L is not greater than α_1. Suppose that for dimension $r - 1$ such L exists. Note that if $v_i \ngeqslant v_1$, then for some $k \in \{1, 2, \ldots, r\}$ we have $[v_i]_k < [v_1]_k$. For $n \geqslant 2$ denote the minimal i such that $[v_n]_i < [v_1]_i$ by $\omega(n)$ and the value $[v_n]_{\omega(n)}$ by $\zeta(n)$. It's easy to see that $0 \leqslant \zeta(n) \leqslant \alpha_1 - 1$ and $\omega(n) \in \{1, 2, \ldots, r\}$. Therefore the number of different pairs $(\omega(n), \zeta(n))$ is not greater than $r \cdot \alpha_1$. If it is impossible to find desirable number $L = L(\widetilde{\alpha}, r)$ for dimension r then for some pair (k, l) there exists an infinite subsequence $\{v_{m_i}\}_{i=1}^{\infty}$ such that $\omega(v_{m_i}) = k$ and $\zeta(v_{m_i}) = l$ for all $i \geqslant 1$.

In this case for any natural number L_1 and any $l_1 < l_2 \leqslant L_1$ we have $v_{m_{l_1}} \ngeqslant v_{m_{l_2}}$ and $v_{m_{l_2}} \ngeqslant v_{m_{l_1}}$. This is a contradiction with induction assumption, the number of such elements is not greater than $L(\{\alpha_{m_i}\}_{i=1}^{\infty}, r-1)$. Proposition proved.

Remark 3. *If any element of sequence $\{\alpha_i\}_{i=1}^{\infty}$ can be computed, then the number $L = L(\widetilde{\alpha}, r)$ can be computed.*

Proof. Firstly note that if $\widetilde{\beta} \geqslant \widetilde{\alpha}$ elementwise, then $L(\widetilde{\beta}, r) \geqslant L(\widetilde{\alpha}, r)$. It follows from the fact that if the sequence $\widetilde{v} = \{v_i\}_{i=1}^{\infty}$ in $\mathbb{Z}_{\geqslant 0}^r$ satisfies conditions $|v_i|_1 \leqslant \alpha_i, i \geqslant 1$ then this sequence satisfies conditions $|v_i|_1 \leqslant \beta_i, i \geqslant 1$.

Numerate pairs $(\omega(n), \zeta(n))$ in order of appearing in sequence $\{(\omega(n), \zeta(n))\}_{n\geqslant 2}$ such that the first pair p_1 is equal to $(\omega(2), \zeta(2))$ and the last pair $p_J = (\omega(n_J), \zeta(n_J))$ has number J and $J \leqslant r \cdot \alpha_1$. The number n_k corresponds to the first element v_{n_k} such that $[v_{n_k}]_{\omega(n_k)} = \zeta(n_k)$ and $\omega(n_k)$ is minimal number with condition $[v_{n_k}]_{\omega(n_k)} < [v_1]_{\omega(n_k)}$.

Without lost of generality we can assume that $\widetilde{\alpha}$ is nondecreasing sequence. If it's not true, replace $\{\alpha_i\}_{i=1}^{\infty}$ by $\{\max_{1\leqslant j\leqslant i} \alpha_j\}_{i=1}^{\infty}$ and find upper bound for $L(\widetilde{\alpha}, r)$ using new nondecreasing sequence.

Firstly assume that $r = 3$, then $J < 3 \cdot \alpha_1$. Since $\widetilde{\alpha}$ is nondecreasing sequence the maximal value of $L(\widetilde{\alpha}, 3)$ can be obtained if number n_2 is the largest possible. Therefore for calculation of upper bound of $L(\widetilde{\alpha}, 3)$ we can get $n_2 = L(\{\alpha_i\}_{i\geqslant 2}, 3-1) = \alpha_2$. Then the number of elements v_i corresponding to the pair p_1 is not greater than α_2, and for pair p_2 is not greater than α_{α_2+1}. Thus $n_3 \leqslant L(\{\alpha_i\}_{i\geqslant 1+\alpha_2+\alpha_{\alpha_2+1}}, 2) = \alpha_{1+\alpha_2+\alpha_{\alpha_2+1}}$ and so on. For arbitrary i we have $n_i = \alpha_{n_{i-1}+n_{i-1}+\cdots+n_1+1}$. Since $J \leqslant 3 \cdot \alpha_1$, $n_{3\alpha_1} = \alpha_{(\sum_{j=1}^{3\alpha_1-1} n_j+1)}$ and the whole number $L(\{\alpha_i\}_{i\geqslant 1}, 3)$ is not greater than $\sum_{j=1}^{3\alpha_1} n_j + 1$.

The same reasons for arbitrary r. Since $n_1 = 2$, $n_2 \leqslant L(\{\alpha_i\}_{i\geqslant 2}, r-1)$, $n_3 \leqslant L(\{\alpha_i\}_{i\geqslant n_2+1}, r-1)$,..., $n_{r\cdot\alpha_1} \leqslant L(\{\alpha_i\}_{i\geqslant \sum_{j=1}^{r\alpha_1-1} n_j+1}, r-1)$. Hence $L(\{\alpha_i\}_{i\geqslant 1}, r-1) \leqslant \sum_{j=1}^{r\alpha_1} n_j + 1$. Therefore the upper bound is computable. Remark proved.

Proposition 4. *Given a labeled graph $\mathbb{A}$ and $e \in E(A)$ with $\lambda_e = \mu$. If there exists a cycle q in A such that*

1. slide e/q is admissible,

2. $\Delta q \geqslant 0$ and $\Delta q \neq 0$,

then there exists computable constant C and cycle q' satisfying conditions 1, 2 and

3. $|q'| \leqslant C = C(\mathbb{A}, \mu)$.

Proof. Let p be a cycle of minimal length satisfying conditions 1 и 2. We will find restriction on $|p|$ using induction on the number of vertices that p passes through. Number these vertices in order of appearance on p $v_0, v_1, \ldots, v_{k-1}$, $k < |V(A)|$.

Let $k = 0$, then $p = (e_1, e_2, \ldots, e_{|p|})$. Denote $\tau_l = \mu \cdot \Delta(e_1, e_2, \ldots, e_l)$ for $l = 0, 1, \ldots, |p|$. Then $\tau_{i+1} \leqslant |\tau_i| + M$ where $M = \max_{e\in E(A)} |\Delta e|$. If there exist $i < j < [\frac{|p|-1}{2}]$ such that $\tau_i \leqslant \tau_j$ or $\tau_j \leqslant \tau_i$ then one of the following holds

1. $\tau_i \leqslant \tau_j$ and $\tau_i \neq \tau_j$ then the path $e_1 \cdot \dots \cdot e_j \cdot \overline{e}_i \cdot \overline{e}_{i-1} \cdot \dots \cdot \overline{e}_1$ satisfies conditions 1, 2 and shorter than p, contradiction,

2. $\tau_i \geqslant \tau_j$ and $\tau_i \neq \tau_j$ then the path $e_1 \cdot \dots \cdot e_i \cdot e_{j+1} \cdot \dots \cdot e_{|p|}$ satisfies conditions 1, 2 and shorter than p, contradiction,

3. $\tau_i = \tau_j$ then the path $e_1 \cdot \dots \cdot e_i \cdot e_{j+1} \cdot \dots \cdot e_{|p|}$ satisfies conditions 1, 2 and shorter than p, contradiction.

Therefore there no such i and j, then by prop.2 we have $|p| \leqslant 2L(\{|\mu| + iM\}_{i=1}^{\infty}, r) + 1 =: L_0(\mathbb{A}, |\mu|)$.

To be clear we describe the case $k = 1$. Let p is equal to $p_1 \cdot p_2 \cdot \dots \cdot p_T$ where endpoints of p_i coincides with v_0 and another vertices are equal to v_1. Denote $\tau_l = \mu \cdot \Delta(p_1 \cdot \dots \cdot p_l)$ for $l = 0, \dots, T$, then use above argumentation we can prove that $|\tau_0| = |\mu|, |\tau_1| \leqslant |\mu| + 2M + L_0(\mathbb{A}, |\mu| + M), |\tau_2| \leqslant |\tau_1| + 2M + L_0(\mathbb{A}, |\tau_1| + M), \dots, |\tau_l| \leqslant |\tau_{l-1}| + 2M + L_0(\mathbb{A}, |\tau_{l-1}| + M)$. Again, using ideas of the case $k = 0$ we can obtain restriction $T \leqslant 2L(\{|\tau_i|\}_{i=1}^{\infty}, r) + 1 = L_1(\mathbb{A}, |\mu|)$.

Step of induction $s - 1 \to s$.

Let $k = s$ and path p is equal to $p_1 \cdot p_2 \cdot \dots \cdot p_T$, where endpoints of p_i coincides with v_0 and another vertices belong to $\{v_1, v_2, \dots, v_{s-1}\}$. Denote $\tau_l = \mu \cdot \Delta(p_1 \cdot \dots \cdot p_l)$ for $l = 0, \dots, T$, then use above argumentation we can prove that $|\tau_0| = |\mu|, |\tau_1| \leqslant |\mu| + 2M + L_{s-1}(\mathbb{A}, |\mu| + M), |\tau_2| \leqslant |\tau_1| + 2M + L_{s-1}(\mathbb{A}, |\tau_1| + M), \dots, |\tau_l| \leqslant |\tau_{l-1}| + 2M + L_{s-1}(\mathbb{A}, |\tau_{l-1}| + M)$. Again, using ideas of the case $k = 1$ we can obtain restriction $T \leqslant 2L(\{|\tau_i|\}_{i=1}^{\infty}, r) + 1 = L_s(\mathbb{A}, |\mu|)$. Proposition proved.

Example. Let us construct graph such that the path q' from prop.4 is very long even with respect to $|E(A)|$. We start from graph $\mathbb{A}_1$ and then construct labeled graphs $\mathbb{A}_k$ for $k \in \mathbb{N}$, see fig.6. Labeled graph $\mathbb{A}_i$ obtained by attaching of two copies of labeled graph $\mathbb{A}_{i-1}$ (using vertex v_{i-1}) to the labeled graph with two vertices and two edges. Fix $k \in \mathbb{N}$, define labeled graph $\mathbb{B}_k$ by attaching a loop with labels p_{k+1} and $p_{k+1} \cdot q$ and an edge e with labels p_1^s where $s = s_1 + s_2 + \dots + s_{k-1} + 1$ (the second label is not important) to the vertex v_k (see fig.7).

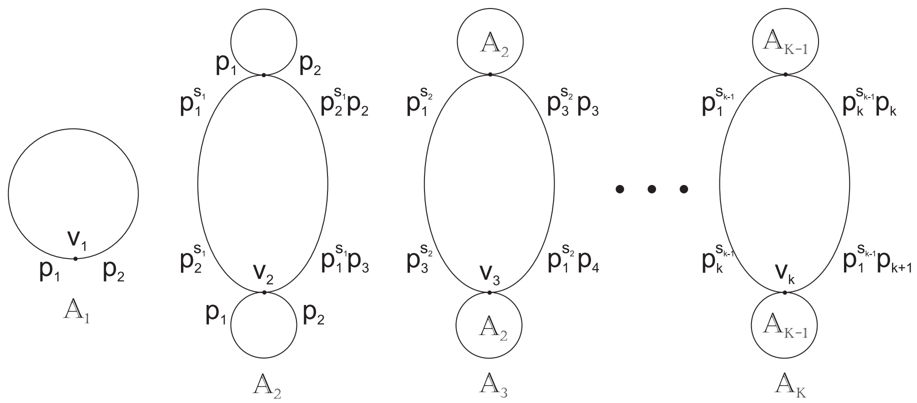

FIG. 6. Labeled graphs $\mathbb{A}_k$.

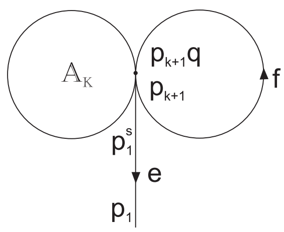

FIG. 7. Labeled graphs $\mathbb{B}_k$.

Then graph B_k has $2^{k-1}+1$ vertices and $2^k+2^{k-1}-1$ edges. The main property of $\mathbb{B}_k$ is following: to change label p_1 to p_2 on e you need one slide, to change $p_1 \to p_3$ requires $\beta_2 = 2s_1 + 3$ slides, to change $p_1 \to p_4$ requires $\beta_3 = s_2 \cdot \beta_2 + 1 + (s_2+1)\beta_2 + 1 = (2s_2+1)\beta_2 + 2$ slides, ..., to change $p_1 \to p_k + 1$ requires $\beta_k = (2s_{k-1}+1)\beta_{k-1} + 2$. Therefore if path p satisfies conditions 1 and 2 of prop. 4, then it's length is not less than $2^k s_1 \cdot s_2 \cdot \dots \cdot s_{k-1}$. Take $s_i = 5$ for all i then $2 \cdot 10^{k-1}$ is the lower bound on $|p|$.

For $v \in \mathbb{Z}^r$ denote the set $\{i | [v]_i \neq 0\}$ by *supp* v.

Proposition 5. *Given a labeled graph* $\mathbb{A}$ *and* $e \in E(A)$ *with label* $\lambda_e = \mu$. *Let* $q_1, q_2, \dots, q_k$ *be admissible cycles in* A *and* $\Delta q_i = v_i \geqslant 0, i = 1, 2, \dots, k$. *If there exists such admissible cycle* p *that*

1. $\Delta p \geqslant 0$,

2. $supp \Delta p \setminus \bigcup_{i=1}^{k} supp\, v_i \neq \emptyset$,

then there exist computable constant C_1 *and admissible cycle* q *satisfying 1, 2 and*

3. $|q| \leqslant C_1 = C_1(\mathbb{A}, \mu, q_1, \dots, q_k)$.

Proof. Numerate coordinates such that $\bigcup_{i=1}^{k} supp\, v_i = \{1, 2, \dots, l\}$. Since cycles $q_1, q_2, \dots, q_k$ are admissible and $\Delta q_i = v_i \geqslant 0$, for any element $n = (n_1, n_2, \dots, n_l, 0, \dots, 0) \in \mathbb{Z}^r$ we can find constants $t_1, t_2, \dots, t_k$ such that $\mu + \sum_{i=1}^{k} t_i v_i \geqslant n$.

Denote by $\mathbb{A}(q_1, \dots, q_k)$ the labeled graph obtained from $\mathbb{A}$ as follows: graph $A(q_1, \dots, q_k)$ coincides with A, if $\lambda_f = (\lambda_1, \lambda_2, \dots, \lambda_r)$ the label in $\mathbb{A}$ then $\delta_f = (\lambda_{l+1}, \lambda_{l+2}, \dots, \lambda_r)$ is a corresponding label in $\mathbb{A}(q_1, \dots, q_k)$.

Since p satisfies condition 2, we can use prop. 4. Therefore, there is a cycle q' with $\Delta q'|_{\{l+1,\dots,r\}} \geqslant 0$ and $\neq 0$. Denote $\Delta(q')$ by v. If $v \geqslant 0$, then q' is a desirable cycle and $C_1 = C$, in other case $v^- \geqslant 0, \neq 0$. But $supp\, v^- \subseteq \bigcup_{i=1}^{k} supp\, v_i$. Since $|v| \leqslant C$, $[v^-]_i \leqslant C$ for $i = 1, 2, \dots, l$. Take $t_1, t_2, \dots, t_k \in \mathbb{Z}$ such that $\mu + \sum_{i=1}^{k} t_i v_i \geqslant v^-$. Constants $t_1, t_2, \dots, t_k \in \mathbb{Z}$ bounded depending on C. Then it's easy to see that $q = q_1^{t_1} \cdot q_2^{t_2} \cdot \dots \cdot q_k^{t_k} \cdot q'$ is the path we looking for and $|q|$ bounded depending on $|q_i|, i = 1, 2, \dots, k$ and C. Proposition proved.

Corollary 6. *Given a labeled graph* $\mathbb{A}$ *and* $e \in E(A)$ *then either any admissible cycle* p *has* $\Delta p \ngeqslant 0$ *or there exists admissible cycle* q *and computable constant* T *such that*

1. $\Delta q \geqslant 0, \neq 0$,

2. If p *is admissible and* $\Delta p \geqslant 0$ *then* $supp \Delta p \subseteq supp \Delta q$,

3. $|q| \leqslant T$.

Such cycle we will call *maximal nonnegative admissible* (or just *maximal*) *cycle* in labeled graph $\mathbb{A}$ and denote $supp \Delta q$ by $S(\mathbb{A}, e)$.

Proof. Using proposition 5 several times we can obtain cycles $q_1, q_2, \dots, q_k$ such that if p is admissible and $\Delta p \geqslant 0$ then $supp \Delta p \subseteq \bigcup_{i=1}^{k} supp \Delta q_i$. Denote $q_1 \cdot q_2 \cdot \dots \cdot q_k$ by q, then $\Delta q \geqslant 0$, cycle q is admissible, $supp \Delta q = \bigcup_{i=1}^{k} supp \Delta q_i$ and $|q| \leqslant \sum_{j=2}^{l} C_1'(\mathbb{A}, \mu, q_1, \dots, q_j) + C = T$. Corollary proved.

3. Main results

Given a labeled graph $\mathbb{A}$ and $e \in E(A)$. Define a graph $\mathfrak{A} = \mathfrak{A}(\mathbb{A}, e)$ in following way: vertices of graph $\mathfrak{A}$ is a labeled graphs obtained from $\mathbb{A}$ by sliding of edge e, two such labeled graphs $\mathbb{B}_1$ and $\mathbb{B}_2$ joined by edge in $\mathfrak{A}$ if there is an edge f in A such that $\mathbb{B}_1$ is obtained from $\mathbb{B}_2$ by slide e/f. It can be established (see prop. 3.10

[5]), that if $\mathbb{A}$ has no admissible cycle p such that $\Delta(p) \geqslant 0, \neq 0$, then graph $\mathfrak{A}$ is finite.

Lemma 7. *Given a labeled graph $\mathbb{A}$ and an edge $e \in E(A)$. There exists an algorithm that finds constant D and admissible nonnegative cycles $q_1, \dots, q_k$ such that if p is admissible cycle with $\Delta p \geqslant 0$, then $\Delta p = \sum_{i=1}^{k} x_i \Delta q_i$ for some integers $x_1, \dots, x_k$ and $|q_i| \leqslant D$ for $i = 1, \dots, k$.*

We will call the set of cycles $q_1, \dots, q_k$ *basis of admissible nonnegative cycles* (or just *basis*) *of* $\mathbb{A}$ *with respect to* e.

Proof. The corollary 6 implies that if p is admissible and $\Delta p \geqslant 0$, then $supp\Delta p \subseteq S(\mathbb{A}, e)$. Furthermore p is admissible in $\mathbb{A}(q)$ and $\Delta p|_{\overline{S(\mathbb{A},e)}} = 0$. It means that in $\mathfrak{A} = \mathfrak{A}(\mathbb{A}(q), e)$ cycle p induce cycle with endpoints $\mathbb{A}(q)$. Since labeled graph $\mathbb{A}(q)$ has no nonnegative admissible cycles, we can see that $\mathfrak{A}$ is a finite graph and the number of it's vertexes and edges is bounded depending on $\mathbb{A}$. Suppose that $f_1^*, \dots, f_k^*$ are basic cycles in graph $\mathfrak{A}$ with endpoints $\mathbb{A}(q)$, then they induce cycles $f_1, \dots, f_k$ in A such that $|f_i| < 2|E(\mathfrak{A})|$, $0 = \Delta f_i|_{\overline{S(\mathbb{A},e)}}$ (therefore $supp\Delta f_i \subseteq S(\mathbb{A}, e)$) and $\Delta p \in \langle \Delta(f_1), \dots, \Delta(f_k)\rangle_{\mathbb{Z}}$.

Then for any $i = 1, \dots, k$ we can find $y_i \in \mathbb{Z}$ such that cycle $q_i = q^{y_i} \cdot f_i$ is admissible in $\mathbb{A}$. We can calculate y_i in following way: after admissible slide e/q^{y_i} the label on edge e is equal to $\mu + y_i \cdot \Delta(q)$. Then by lemma 1 cycle f_i is admissible if and only if

$$\Delta(f_i^-) + \gamma_{f_i} \leqslant \mu + y_i \cdot \Delta(q)$$

This inequality is equivalent to the system of two inequalities

$$\Delta(f_i^-)|_{S(\mathbb{A},e)} + \gamma_{f_i}|_{S(\mathbb{A},e)} \leqslant \mu|_{S(\mathbb{A},e)} + y_i \cdot \Delta(q)|_{S(\mathbb{A},e)},$$

$$\Delta(f_i^-)|_{\overline{S(\mathbb{A},e)}} + \gamma_{f_i}|_{\overline{S(\mathbb{A},e)}} \leqslant \mu|_{\overline{S(\mathbb{A},e)}} + y_i \cdot \Delta(q)|_{\overline{S(\mathbb{A},e)}}.$$

The second inequality holds because f_i is admissible in $\mathbb{A}(q)$ for $i = 1, 2, \dots, k$. In the first inequality left side is bounded. Therefore the value of y_i satisfying conditions can be founded.

We have

$$\langle \Delta q_1, \dots, \Delta q_k, \Delta q\rangle_{\mathbb{Z}} = \langle y_1 \Delta q + \Delta f_1, \dots, y_{k-1}\Delta q + \Delta f_k, \Delta q\rangle_{\mathbb{Z}} =$$
$$= \langle \Delta f_1, \dots, \Delta f_k, \Delta q\rangle_{\mathbb{Z}} = \langle \Delta f_1, \dots, \Delta f_k\rangle_{\mathbb{Z}}.$$

The last equality holds because $q \in \langle \Delta f_1, \dots, \Delta f_k\rangle$ (it's true because q is admissible in $\mathbb{A}$ and in $\mathbb{A}(q)$ too).

The following algorithm finds the basis:

1. Construct $\mathfrak{A}$.
2. Find f_i^* (using a maximal subtree in $\mathfrak{A}$).
3. Induce $f_1, \dots, f_k$.
4. Find y_i using system of inequations.
5. Cycles $q, q_i = q^{y_i} \cdot f_i, i = 1, \dots, k$ is the basis.

Lemma proved.

Theorem 8. *Given a labeled graph $\mathbb{A}$, an edge $e \in E(A)$ and $v \in \Delta\pi_1(\mathbb{A})$. Then there exists an algorithm that either finds a path p such that e/p is admissible and $\Delta p = v$ or says that there no such a path.*

Proof. Case 1. Let $v \geqslant 0$. If a path p exists then by lemma 7 we have $\Delta p = \sum_{i=1}^{k} x_i \Delta q_i$. If $supp\, v \not\subseteq S(\mathbb{A}, e)$ then the answer "no" because corollary 6 implies $supp \Delta p \subseteq S(\mathbb{A}, e)$. But if $supp\, v \subseteq S(\mathbb{A}, e)$ then we can find solution of the system

$$v = \sum_{i=1}^{k} x_i q_i.$$

If there is no solution then the answer "no" (because of lemma 7). Otherwise cycle $q_1^{x_1} \cdot \dots \cdot q_k^{x_k}$ is admissible and desirable.

Case 2. Let $v \leqslant 0$ and $v|_{\overline{S(\mathbb{A},e)}} = 0$. Denote by $\mathbb{B}$ the labeled graph obtained from $\mathbb{A}$ by replacing of label μ by label $\mu + v$ ($\mu + v \geqslant 0$ otherwise the answer is "no"). Initial question is equivalent to the following: to find a cycle p_0 admissible in $\mathbb{B}$ such that $\Delta p_0 = -v \geqslant 0$. This is the case 1. If the answer "no" then we give the same answer, otherwise $p = p_0^{-1}$.

General case. Let q be a maximal cycle in $\mathbb{A}$. Construct a labeled graph $\mathbb{A}(q)$ and finite graph $\mathfrak{A} = \mathfrak{A}(\mathbb{A}(q), \mu)$. Looking for the path in $\mathfrak{A}$ that connects $\mathbb{A}(q)$ and labeled graph $\mathbb{B}$ obtained from $\mathbb{A}(q)$ by replacing of label $\mu|_{\overline{S(\mathbb{A},e)}}$ to the label $(\mu + v)|_{\overline{S(\mathbb{A},e)}}$. If there no such a path (more precisely, there no such vertex $\mathbb{B}$ in graph $\mathfrak{A}$) then the answer is "no" (since p is admissible in $\mathbb{A}$, p have to be admissible in $\mathbb{A}(q)$ and, therefore, p induces a path in $\mathfrak{A}$), otherwise we get path p^*. Let p_1 be a cycle in A that induces p^*. Then

$$\Delta p_1|_{\overline{S(\mathbb{A},e)}} = v|_{\overline{S(\mathbb{A},e)}}.$$

If path p exists, then it's induces in $\mathfrak{A}$ the path that differs from p^* by some cycle. Therefore either the system

$$\Delta(p_1) - v = -\sum_{i=1}^{k} t_1 \Delta(q_i)$$

is solvable over $\mathbb{Z}$, or the answer is "no". Denote the cycle $q_1^{t_i} \cdot \dots \cdot q_k^{t_k} \cdot p_1$ by p_2. If p_2 is admissible, then $p = p_2$.

If p_2 is not admissible, then there is integer x (see proof of lemma 6) such that cycle $q^x \cdot p_2$ is admissible. Denote by y the maximal integer such that cycle $p_3 = q^x \cdot p_2 \cdot q^{-y}$ is admissible. If $y = x$, then $p = p_3$.

Let's note, that the path p exists if and only if there is a path q^* admissible in labeled graph $\mathbb{B}$ (obtained from $\mathbb{A}$ by replacing of label μ by label $\mu + \Delta(p_3)$) such that $\Delta(q^*) = (y - x)\Delta(q) \leqslant 0$. This is the case 2. If the answer in case 2 is negative then we give the same answer. Otherwise we have $p = p_3 \cdot q^*$. Theorem proved.

References

[1] J. P. Serre, *Trees*, Berlin/Heidelberg/New York: Springer, 1980. MR0607504

[2] M. Forester, *Splittings of generalized Baumslag-Solitar groups*, Geometriae Dedicata, **121** (2006), 43–59. MR2276234

[3] M. Clay, *Deformation spaces of G–trees and automorphisms of Baumslag–Solitar groups*, Groups Geom. Dyn., **3**:1 (2009), 39–69. MR2466020

[4] M. Forester, *On uniquenes of JSJ decomposition of finitely generated groups*, Comm. Math. Helv., **78** (2003), 740–751. MR2016693

[5] M. Clay, M. Forester, *On the isomorphism problem for generalized Baumslag–Solitar groups*, Algebraic & Geometric Topology, **8**:4 (2008), 2289—2322. MR2465742

[6] G. Levitt, *On the automorphism group of generalized Baumslag-Solitar groups*, Geometry & Topology, **11** (2007), 473—515. MR2302496

[7] M. Forester, *Deformation and rigidity of simplicial group actions on trees*, Geometry & Topology, **6** (2002), 219–267. MR1914569

[8] M. Clay, M. Forester, *Whitehead moves for G-trees*, Bull. London Math. Soc., **41**:2 (2009), 205–212. MR2496498

Fedor Anatol'evich Dudkin
Sobolev Institute of Mathematics,
pr. Koptyuga, 4,
630090, Novosibirsk, Russia
Novosibirsk State University,
Pirogova str., 2,
630090, Novosibirsk, Russia
E-mail address: DudkinF@ngs.ru