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ON A MORPHISM OF COMPACTIFICATIONS OF MODULI SCHEME OF VECTOR BUNDLES

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Abstract. A morphism of nonreduced Gieseker – Maruyama functor (of semistable coherent torsion-free sheaves) on the surface to the nonreduced functor of admissible semistable pairs with the same Hilbert polynomial, is constructed. This leads to the morphism of moduli schemes with possibly nonreduced scheme structures. As usually, we study subfunctors corresponding to main components of moduli schemes.

Keywords: moduli space, semistable coherent sheaves, moduli functor, algebraic surface.

To the blessed memory of my Mum

Introduction

The purpose of the present paper is to construct a morphism of main components of Gieseker – Maruyama moduli scheme $\overline{M}$ of semistable (in the sense of Gieseker – cf. sect. 1, definition 1) torsion-free coherent sheaves of fixed rank and with fixed Hilbert polynomial on a smooth projective surface, to main components of the moduli scheme $\widetilde{M}$ of semistable admissible pairs (cf. sect. 1, definitions 3 and 4) with same rank and Hilbert polynomial, which were built up in the series of papers of the author [1] - [5]. In [4] the construction of a morphism $\kappa_{\rm red} : \overline{M}_{\rm red} \to \widetilde{M}_{\rm red}$ of same schemes was done but both schemes were considered with reduced scheme structures. This restriction (the absence of nilpotent elements in structure sheaves) is essential for the construction performed in the cited paper. In the present article we remove this restriction and prove the existence of a morphism $\kappa : \overline{M} \to \widetilde{M}$. The morphism $\kappa_{\rm red}$ from [4] is the reduction of κ (in the cited paper the morphism $\kappa_{\rm red}$ was denoted by κ, $\overline{M}_{\rm red}$ and $\widetilde{M}_{\rm red}$ by $\overline{M}$ and $\widetilde{M}$ respectively). In this way we give an answer to the open question announced in [5, remark 3].

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We work on a smooth irreducible projective algebraic surface S over a field $k = \overline{k}$ of characteristic zero. On S an ample invertible sheaf L is chosen and fixed. It is used as a polarization. The Hilbert polynomial of the coherent sheaf E of $\mathcal{O}_S$-modules having rank r, is denoted as $rp(n)$ and is defined by the formula $rp(n) = \chi(E \otimes L^n)$. Reduced Hilbert polynomial $p(n)$ of the sheaf E is a polynomial with rational coefficients. These coefficients depend on the geometry of the surface S, the polarization L and on Chern classes c_1, c_2 of the sheaf E.

The scheme of moduli of Gieseker – Maruyama $\overline{M}$ being a Noetherian projective algebraic scheme of finite type [7],[8], is a classical way to compactify the moduli space M_0 of stable vector bundles with same rank and Hilbert polynomial. The scheme M_0 is reduced and quasi-projective (it is a quasi-projective algebraic variety – [8]). In the construction of the scheme $\overline{M}$ families of locally free stable sheaves are completed by (possibly, nonlocally free) semistable coherent torsion-free sheaves with same rank and Hilbert polynomial on the surface S with polarization L.

In cited papers [1, 2, 4] we developed a procedure to transform a flat family $\mathbb{E}_T$ of semistable coherent torsion-free sheaves on the surface S, parametrized by irreducible and reduced scheme T, to the family $((\pi : \widetilde{\Sigma} \to \widetilde{T}, \widetilde{\mathbb{L}}), \widetilde{\mathbb{E}})$ of semistable admissible pairs, parameterized by reduced irreducible scheme $\widetilde{T}$ which is birational to T. Since we are interested namely in compactifications of moduli space of stable vector bundles, the family $\mathbb{E}_T$ is thought to contain at least one locally free sheaf. Since the requirement of local freeness (as well as requirement of Gieseker-stability – cf. [9, propos. 2.3.1]) is open in flat families, the base scheme T contains an open subscheme T_0 whose closed points correspond to locally free sheaves. In the compactification we built, nonlocally free sheaves in points of closed subscheme $T \setminus T_0$ are replaced by pairs $((\widetilde{S}, \widetilde{L}), \widetilde{E})$ in points of closed subscheme $\widetilde{T} \setminus \widetilde{T}_0$, $\widetilde{T}_0 \cong T_0$, where $\widetilde{S}$ is projective algebraic scheme of certain form with appropriate polarization $\widetilde{L}$, and $\widetilde{E}$ is appropriate locally free sheaf. As shown in [3], schemes $\widetilde{S}$ are connected, and hence rank r of a locally free sheaf $\widetilde{E}$ on such scheme is well-defined. It is equal to the rank of restriction of the sheaf $\widetilde{E}$ to each component of $\widetilde{S}$. The Hilbert polynomial of $\mathcal{O}_{\widetilde{S}}$-sheaf $\widetilde{E}$ is defined in usual way: $rp(n) = \chi(\widetilde{E} \otimes \widetilde{L}^n)$. The precise description of pairs $((\widetilde{S}, \widetilde{L}), \widetilde{E})$ will be given below, sect.1, definitions 3 and 4.

The mentioned procedure of transformation of a flat family of semistable torsion-free sheaves to a flat family of admissible semistable pairs is called a standard resolution. It gives rise to a birational morphism of base schemes $\widetilde{T} \to T$ which becomes an isomorphism when restricted to the preimage $\widetilde{T}_0$ of open subscheme T_0 of locally free sheaves. To perform a standard resolution as it developed in articles [1, 2, 4], one needs irreducibility and reducedness of the base scheme T.

It is known [10, 11, 12] that for arbitrary surface S the scheme $\overline{M}$ is asymptotically (in particular, for big values of c_2) reduced, irreducible and of expected dimension. Although under arbitrary choice of numerical invariants of sheaves this scheme can be nonreduced.

In this article we develop the version of the standard resolution for the family of semistable coherent torsion-free sheaves for the case when the base scheme is nonreduced. For our considerations it is enough to restrict ourselves by the class of schemes T such that their reductions T_{red} are irreducible schemes.

Using our version of standard resolution we construct the natural transformation of the Gieseker – Maruyama functor (sect. 1, (1.1),(1.2)) to the functor $\mathfrak{f}$ of admissible semistable pairs (sect. 1, (1.4), (1.5)). The natural transformation leads to the morphism of moduli schemes.

The article consists of four sections. In sect. 1 we recall the basic notions which are necessary for the further considerations. Therein we give a standard description how the morphism of moduli functors determines morphism of their moduli schemes. Sect. 2 is devoted to procedure of standard resolution of a family of coherent sheaves with nonreduced base. Sect. 3 contains the construction of the natural transformation $\mathfrak{f}^{GM} \to \mathfrak{f}$

using standard resolution from sect. 2. In addition, in sect. 4 we obtain the morphism of moduli schemes of functors of our interest induced by the natural transformation $\mathfrak{f}^{GM} \to \mathfrak{f}$, "by hands without category-theoretical constructions.

In the present article we prove the following result.

Theorem 1. *The Gieseker – Maruyama functor $\mathfrak{f}^{GM}$ of semistable torsion-free coherent sheaves of rank r and with Hilbert polynomial $rp(n)$ on the surface (S, L), has a natural transformation to the functor $\mathfrak{f}$ of admissible semistable pairs $((\widetilde{S}, \widetilde{L}), \widetilde{E})$ where the locally free sheaf $\widetilde{E}$ on the projective scheme $(\widetilde{S}, \widetilde{L})$ has same rank and Hilbert polynomial. In particular, there is a morphism of moduli schemes $\overline{M} \to \widetilde{M}$ associated with this natural transformation.*

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1. Objects and functors

Throughout in this paper we identify locally free $\mathcal{O}_X$-sheaf on a scheme X with corresponding vector bundle and use both terms as synonyms.

We use the classical definition of semistability due to Gieseker [7].

Definition 1. Coherent $\mathcal{O}_S$-sheaf E is *stable* (resp., *semistable*) if for any proper subsheaf $F \subset E$ of rank $r' = \operatorname{rank} F$ for $n \gg 0$

$$\frac{\chi(E \otimes L^n)}{r} > \frac{\chi(F \otimes L^n)}{r'}, \text{ (resp., } \frac{\chi(E \otimes L^n)}{r} \geq \frac{\chi(F \otimes L^n)}{r'} \text{).}$$

Consider the Gieseker – Maruyama functor

$$\mathfrak{f}^{GM} : (Schemes_k)^o \to Sets \tag{1.1}$$

attaching to any scheme T the set of equivalence classes of families of the form $\mathfrak{F}_T^{GM}/\sim$ where

$$\mathfrak{F}_T^{GM} = \left\{ \begin{array}{l} \mathbb{E}\text{– sheaf of } \mathcal{O}_{T\times S} - \text{modules, flat over } T; \\ \mathbb{L}\text{– invertible sheaf of } \mathcal{O}_{T\times S} - \text{modules, ample relative to } T \\ \text{and such that } L|_{t\times S} \cong L \text{ for any } t \in T; \\ E_t := \mathbb{E}|_{t\times S}\text{– torsion-free and semistable due to Gieseker;} \\ \chi(E_t \otimes L_t^n) = rp(n). \end{array} \right\} \tag{1.2}$$

The equivalence relation $\sim$ is defined as follows. Families $(\mathbb{E}, \mathbb{L})$ and $(\mathbb{E}', \mathbb{L}')$ from the class $\mathfrak{F}_T^{GM}$ are said to be equivalent (notation: $(\mathbb{E}, \mathbb{L}) \sim (\mathbb{E}', \mathbb{L}')$) if there are line bundles L', L'' on the scheme T such that $\mathbb{E}' = \mathbb{E} \otimes pr_1^* L'$, $\mathbb{L}' = \mathbb{L} \otimes pr_1^* L''$ where $pr_1 : T \times S \to T$ is projection to the first factor.

Remark 1. Since $\operatorname{Pic}(T \times S) = \operatorname{Pic} T \times \operatorname{Pic} S$, our definition for moduli functor $\mathfrak{f}^{GM}$ is equivalent to the standard definition as formulated, for example, in [9]: the difference in the choice of polarizations $\mathbb{L}$ and $\mathbb{L}'$ with isomorphic restrictions on fibres over T, is eliminated by tensoring by the inverse image of appropriate invertible sheaf L'' from the base T.

Remark 2. Throughout all the text we consider instead of the whole of the functor $\mathfrak{f}^{GM}$ its subfunctor where each T has irreducible reduction and each $\mathbb{E}$ contains at least one locally free sheaf E_t. This leads to the union of those components of Gieseker – Maruyama moduli that contain locally free sheaves.

To proceed further, recall the definition of a sheaf of zeroth Fitting ideals which is known from commutative algebra [13, ch. III, sect. 20.2]. Let X be a scheme, F – $\mathcal{O}_X$-module with finite presentation $F_1 \xrightarrow{\varphi} F_0 \to F$. Here $\mathcal{O}_X$-modules F_0 and F_1 are assumed to be locally free. Without loss of generality we suppose that $\operatorname{rank} F_1 \geq \operatorname{rank} F_0$.

Definition 2. *The sheaf of zeroth Fitting ideals* for $\mathcal{O}_X$-module F is defined as $\mathcal{F}itt^0 F = \mathrm{im}\,(\bigwedge^{\mathrm{rank}\,F_0} F_1 \otimes \bigwedge^{\mathrm{rank}\,F_0} F_0^\vee \xrightarrow{\varphi'} \mathcal{O}_X)$, where φ' is a morphism of $\mathcal{O}_X$-modules induced by the morphism φ.

Definition 3. [3, 4] Polarized algebraic scheme $(\widetilde{S}, \widetilde{L})$ is *admissible* if the scheme $(\widetilde{S}, \widetilde{L})$ satisfies one of the conditions

i) $(\widetilde{S}, \widetilde{L}) \cong (S, L)$,

ii) $\widetilde{S} \cong \mathrm{Proj}\, \bigoplus_{s\geq 0}(I[t]+(t))^s/(t^{s+1})$, where $I = \mathcal{F}itt^0 \mathcal{E}xt^2(\varkappa, \mathcal{O}_S)$ for Artinian quotient sheaf $q : \bigoplus^r \mathcal{O}_S \twoheadrightarrow \varkappa$ of length $l(\varkappa) \leq c_2$. There is a morphism $\sigma : \widetilde{S} \to S$ (which is called *canonical morphism*) and $\widetilde{L} = L \otimes (\sigma^{-1} I \cdot \mathcal{O}_{\widetilde{S}})$ — ample invertible sheaf on $\widetilde{S}$; this polarization $\widetilde{L}$ is called the *distinguished* polarization.

Remark 3. We require the sheaf $\widetilde{L} = L \otimes (\sigma^{-1} I \cdot \mathcal{O}_{\widetilde{S}})$ to be ample on the scheme $\widetilde{S}$. If it is not true, replace $\mathcal{O}_S$-sheaf L by its big enough tensor power (which can be chosen common for all $\widetilde{S}$, as shown in [5, claim 1]). We redenote this tensor power again as L.

Remark 4. The canonical morphism $\sigma : \widetilde{S} \to S$ is determined by the structure of $\mathcal{O}_S$-algebra on $\bigoplus_{s\geq 0}(I[t] + (t))^s/(t^{s+1})$.

As shown in [5, Introduction], the scheme $\widetilde{S}$ consists of several components unless it is isomorphic to S. The main component $\widetilde{S}_0$ is the initial surface S blown up in the sheaf of ideals I. The ideal I defines some zero-dimensional subscheme in S with structure sheaf $\mathcal{O}_X/I$. Main component $\widetilde{S}_0$ corresponds to an algebraic variety which can be singular. Additional components $\widetilde{S}_i$, $i > 0$ can carry nonreduced scheme structures. Number of additional components, i.e. number of components of the scheme $\overline{\widetilde{S} \setminus \widetilde{S}_0}$, equals to the number of maximal ideals of the quotient algebra $\mathcal{O}_S/I$.

Definition 4. [4, 5] *S-stable* (resp.,*semistable*) *pair* $((\widetilde{S}, \widetilde{L}), \widetilde{E})$ is the following data:

- $\widetilde{S} = \bigcup_{i\geq 0} \widetilde{S}_i$ – admissible scheme, $\sigma : \widetilde{S} \to S$ — its canonical morphism, $\sigma_i : \widetilde{S}_i \to S$ — restrictions of σ to components $\widetilde{S}_i$, $i \geq 0$;
- $\widetilde{E}$ — vector bundle on the scheme $\widetilde{S}$;
- $\widetilde{L} \in \mathrm{Pic}\,\widetilde{S}$ — distinguished polarization;

such that

- $\chi(\widetilde{E} \otimes \widetilde{L}^n) = rp(n)$, the polynomial $p(n)$ and the rank r of the sheaf $\widetilde{E}$ are fixed;
- on the scheme $\widetilde{S}$ the sheaf $\widetilde{E}$ is *stable* (resp., *semistable*) *due to Gieseker* i.e. for any proper subsheaf $\widetilde{F} \subset \widetilde{E}$ for $n \gg 0$
$$\frac{h^0(\widetilde{F} \otimes \widetilde{L}^n)}{\mathrm{rank}\,F} < \frac{h^0(\widetilde{E} \otimes \widetilde{L}^n)}{\mathrm{rank}\,E},$$
$$(\text{resp.,}\ \frac{h^0(\widetilde{F} \otimes \widetilde{L}^n)}{\mathrm{rank}\,F} \leq \frac{h^0(\widetilde{E} \otimes \widetilde{L}^n)}{\mathrm{rank}\,E});$$
- on any of additional components $\widetilde{S}_i, i > 0$, the sheaf $\widetilde{E}_i := \widetilde{E}|_{\widetilde{S}_i}$ is *quasi-ideal,* i.e. it has a description
$$\widetilde{E}_i = \sigma_i^* \ker q_0/\mathrm{tors}\,_i. \tag{1.3}$$
for some $q_0 \in \bigsqcup_{l\leq c_2} \mathrm{Quot}\,^l \bigoplus^r \mathcal{O}_S$. The epimorphism $q_0 : \bigoplus^r \mathcal{O}_S \twoheadrightarrow \varkappa$ is common for all components $\widetilde{S}_i$ of the scheme $\widetilde{S}$ as well as $l = \mathrm{length}\,\varkappa$.

Subsheaf $\mathrm{tors}\,_i$ is defined as a restriction $\mathrm{tors}\,_i = \mathrm{tors}\,|_{\widetilde{S}_i}$ where the role of tors in our considerations is analogous to the role of torsion subsheaf on reduced scheme. Let U be Zariski open subset in one of components $\widetilde{S}_i, i \geq 0$ and $\sigma^* \ker q_0|_{\widetilde{S}_i}(U)$ be the corresponding group of sections carrying structure of $\mathcal{O}_{\widetilde{S}_i}(U)$-module. Sections $s \in \sigma^* \ker q_0|_{\widetilde{S}_i}(U)$ annihilated by prime ideals of positive codimensions in $\mathcal{O}_{\widetilde{S}_i}(U)$, form a submodule in $\sigma^*\ker q_0|_{\widetilde{S}_i}(U)$. We denote this submodule as $\mathrm{tors}'_i(U)$. The correspondence $U \mapsto \mathrm{tors}'_i(U)$ defines a subsheaf $\mathrm{tors}'_i \subset \sigma^*\ker q_0|_{\widetilde{S}_i}$. Note that associated primes of positive codimensions annihilating sections $s \in \sigma^*\ker q_0|_{\widetilde{S}_i}(U)$, correspond to subschemes supported in the preimage $\sigma^{-1}(\mathrm{Supp}\,\varkappa) = \bigcup_{i>0} \widetilde{S}_i$. Since by the construction the scheme $\widetilde{S} = \bigcup_{i\geq 0} \widetilde{S}_i$ is connected, subsheaves $\mathrm{tors}'_i, i > 0$, allow to form a subsheaf $\mathrm{tors} \subset \sigma^*\ker q_0$. The former is defined as follows. Section $s \in \sigma^*\ker q_0|_{\widetilde{S}_i}(U)$ satisfies $s \in \mathrm{tors}|_{\widetilde{S}_i}(U)$ if and only if

- there exists a section $y \in \mathcal{O}_{\widetilde{S}_i}(U)$ such that $ys = 0$,
- at least one of following conditions is satisfied: either $y \in \mathfrak{p}$ where $\mathfrak{p}$ is prime ideal of positive codimension, or there exist Zariski-open subset $V \subset \widetilde{S}$ and a section $s' \in \sigma^*\ker q_0(V)$ such that $V \supset U$, $s'|_U = s$, and $s'|_{V\cap \widetilde{S}_0} \in \mathrm{tors}\,(\sigma^*\ker q_0|_{\widetilde{S}_0})(V \cap \widetilde{S}_0)$. In the former expression the subsheaf of torsion $\mathrm{tors}\,(\sigma^*\ker q_0|_{\widetilde{S}_0})$ is understood in usual sense.

As shown in [4], there is a map taking any semistable coherent torsion-free sheaf E to the admissible semistable pair $((\widetilde{S}, \widetilde{L}), \widetilde{E})$ as follows. If E is locally free then $((\widetilde{S}, \widetilde{L}), \widetilde{E}) = ((S, L), E)$. Otherwise $\widetilde{S} = \mathrm{Proj} \bigoplus_{s\geq 0}(I[t] + (t))^s/(t^{s+1})$, where $I = \mathcal{F}itt^0\mathcal{E}xt^1(E, \mathcal{O}_S)$, $\widetilde{L} = L \otimes (\sigma^{-1}I \cdot \mathcal{O}_{\widetilde{S}})$ and $\widetilde{E} = \sigma^*E/\mathrm{tors}$ where tors is understood as described. This map corresponds to the morphism $\kappa_{\mathrm{red}} : \overline{M}_{\mathrm{red}} \to \widetilde{M}_{\mathrm{red}}$.

Let T be a scheme over a field k, $\pi : \widetilde{\Sigma} \to T$ a morphism of k-schemes. We introduce the following

Definition 5. The family of schemes $\pi : \widetilde{\Sigma} \to T$ is *birationally S-trivial* if there exist isomorphic open subschemes $\widetilde{\Sigma}_0 \subset \widetilde{\Sigma}$ and $\Sigma_0 \subset T \times S$ and there is a scheme equality $\pi(\widetilde{\Sigma}_0) = T$.

The former equality means that all fibres of the morphism π have nonempty intersections with the open subscheme $\widetilde{\Sigma}_0$.

In particular, if $T = \mathrm{Spec}\,k$ then π is a constant morphism and $\widetilde{\Sigma}_0 \cong \Sigma_0$ is open subscheme in S.

Since in the present paper we consider only S-birationally trivial families, they will be referred to as *birationally trivial* families.

Also we consider families of semistable pairs

$$\mathfrak{F}_T = \left\{ \begin{array}{l} \pi : \widetilde{\Sigma} \to T \ \text{ birationally } S\text{-trivial },\\ \widetilde{\mathbb{L}} \in \mathrm{Pic}\,\widetilde{\Sigma} \ \text{ flat over } T,\\ \text{for } m \gg 0\ \widetilde{\mathbb{L}}^m \text{ very ample relatively } T,\\ \forall t \in T\ \widetilde{L}_t = \widetilde{\mathbb{L}}|_{\pi^{-1}(t)} \ \text{ ample;}\\ (\pi^{-1}(t), \widetilde{L}_t) \text{ admissible scheme with distinguished polarization;}\\ \chi(\widetilde{L}_t^n) \ \text{ does not depend on } t,\\ \widetilde{\mathbb{E}} \ \text{ locally free } \mathcal{O}_{\Sigma} - \text{sheaf flat over } T;\\ \chi(\widetilde{\mathbb{E}} \otimes \widetilde{\mathbb{L}}^n)|_{\pi^{-1}(t)}) = rp(n);\\ ((\pi^{-1}(t), \widetilde{L}_t), \widetilde{\mathbb{E}}|_{\pi^{-1}(t)}) - \text{semistable pair} \end{array} \right\} \tag{1.4}$$

and a functor

$$\mathfrak{f} : (Schemes_k)^o \to (Sets) \tag{1.5}$$

from the category of k-schemes to the category of sets. It attaches to a scheme T the set of equivalence classes of families of the form $\mathfrak{F}_T/\sim$.

The equivalence relation $\sim$ is defined as follows. Families $((\pi : \widetilde{\Sigma} \to T, \widetilde{\mathbb{L}}), \widetilde{\mathbb{E}})$ and $((\pi' : \widetilde{\Sigma} \to T, \widetilde{\mathbb{L}}'), \widetilde{\mathbb{E}}')$ from the class $\mathfrak{F}_T$ are said to be equivalent (notation: $((\pi : \widetilde{\Sigma} \to T, \widetilde{\mathbb{L}}), \widetilde{\mathbb{E}}) \sim ((\pi' : \widetilde{\Sigma} \to T, \widetilde{\mathbb{L}}'), \widetilde{\mathbb{E}}'))$ if

1) there exists an isomorphism $\iota : \widetilde{\Sigma} \xrightarrow{\sim} \widetilde{\Sigma}'$ such that the diagram

$$\begin{array}{ccc} \widetilde{\Sigma} & \xrightarrow[\sim]{\iota} & \widetilde{\Sigma}' \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi'} & \\ & T & \end{array}$$

commutes.
2) There exist line bundles L', L'' on T such that $\iota^*\widetilde{\mathbb{E}}' = \widetilde{\mathbb{E}} \otimes \pi^* L'$, $\iota^*\widetilde{\mathbb{L}}' = \widetilde{\mathbb{L}} \otimes \pi^* L''$.

Remark 5. The definition of the functor of semistable admissible pairs given here differs from definition from preceding papers [4] – [6]: we added new requirement of birational triviality. This requirement was missed by the author before. Indeed, without this requirement, the consideration of "twisted" families of schemes is allowed. One can take even twisted families with all fibres isomorphic to the surface S. This makes sets $\mathfrak{F}_T/\sim$ *too big* in such a sense that non-isomorphic families of schemes over a same base T arise where corresponding fibres are isomorphic. For a simple example take a projective plane $S = \mathbb{P}^2$. Not any $\mathbb{P}^2$-bundle over the base T is isomorphic to the product $T \times \mathbb{P}^2$. This circumstance does not allow to conclude the isomorphism of open subfunctor of Gieseker-semistable vector bundles, to an open subfunctor corresponding to semistable S-pairs if the former is considered without requirement of birational triviality. All the results of articles [4] – [6] become true together with their proofs when the requirement of birational triviality is added.

Now discuss what is the "size" of the maximal under inclusion of those open subschemes $\widetilde{\Sigma}_0$ in a family of admissible schemes $\widetilde{\Sigma}$, which are isomorphic to appropriate open subschemes in $T \times S$ in the definition 5. The set $F = \widetilde{\Sigma} \setminus \widetilde{\Sigma}_0$ is closed. If T_0 is open subscheme in T whose points carry fibres isomorphic to S, then $\widetilde{\Sigma}_0 \supsetneqq \pi^{-1}T_0$ (inequality is true because $\pi(\widetilde{\Sigma}_0) = T$ in the definition 5). The subscheme Σ_0 which is open in $T \times S$ and isomorphic to $\widetilde{\Sigma}_0$, is such that $\Sigma_0 \supsetneqq T_0 \times S$. If $\pi : \widetilde{\Sigma} \to T$ is family of admissible schemes then $\widetilde{\Sigma}_0 \cong \widetilde{\Sigma} \setminus F$, and F is (set-theoretically) the union of additional components of fibres which are non-isomorphic to S.

Remark 6. We also consider instead of the whole of the functor $\mathfrak{f}$ its subfunctor where each T has irreducible reduction and each $\pi : \widetilde{\Sigma} \to T$ contains a fibre isomorphic to S. This leads to the union of those components of the moduli space that contain pairs with $\widetilde{S} \cong S$.

Following [9, ch. 2, sect. 2.2] we recall some definitions. Let $\mathcal{C}$ be a category, $\mathcal{C}^o$ its dual category, $\mathcal{C}' = \mathcal{F}unct(\mathcal{C}^o, Sets)$ a category of functors to the category of sets. By Yoneda's lemma, the functor $\mathcal{C} \to \mathcal{C}' : F \mapsto (\underline{F} : X \mapsto \mathrm{Hom}_{\mathcal{C}}(X, F))$ includes $\mathcal{C}$ as full subcategory in $\mathcal{C}'$.

Definition 6. [9, ch. 2, definition 2.2.1] The functor $\mathfrak{f} \in \mathcal{O}b\,\mathcal{C}'$ is *corepresented by the object* $F \in \mathcal{O}b\,\mathcal{C}$ if there exists a $\mathcal{C}'$-morphism $\psi : \mathfrak{f} \to \underline{F}$ such that any morphism $\psi' : \mathfrak{f} \to \underline{F}'$ factors through the unique morphism $\omega : \underline{F} \to \underline{F}'$.

Definition 7. The scheme M is a *coarse moduli space* for the functor $\mathfrak{f}$ if $\mathfrak{f}$ is corepresented by M.

Now let we are given two functors $\mathfrak{f}, \widetilde{\mathfrak{f}} : \mathcal{C}^o \to Sets$, and a natural transformation $\underline{\kappa} : \mathfrak{f} \to \widetilde{\mathfrak{f}}$ where for any $T \in \mathcal{O}b\,\mathcal{C}$ there is a commutative diagram

$$\begin{array}{ccc} T & \overset{\mathfrak{f}}{\longmapsto} & \mathfrak{f}(T) \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \underline{\kappa}(T)} \\ T & \overset{\widetilde{\mathfrak{f}}}{\longmapsto} & \widetilde{\mathfrak{f}}(T) \end{array}$$

Let the functor $\mathfrak{f}$ have coarse moduli space M and the functor $\widetilde{\mathfrak{f}}$ have coarse moduli space $\widetilde{M}$. In the situation if definition 6 the following diagram commutes:

$$\begin{array}{ccccc}
T & & \xrightarrow{\quad = \quad} & T & \\
\downarrow & & & \downarrow & \\
\mathfrak{f}(T) & & \xrightarrow{\quad \underline{\kappa}(T) \quad} & \widetilde{\mathfrak{f}}(T) & \\
& \searrow^{\psi(T)} & & & \searrow^{\widetilde{\psi}(T)} \\
\downarrow{\scriptstyle \psi'(T)} & & \mathrm{Hom}(T,M) \overset{\epsilon}{\cdots\cdots\to} & \downarrow{\scriptstyle \widetilde{\psi}'(T)} & \mathrm{Hom}(T,\widetilde{M}) \\
& \swarrow_{\omega(T)} & & & \swarrow_{\widetilde{\omega}(T)} \\
\mathrm{Hom}(T,F') & & \xrightarrow{\quad = \quad} & \mathrm{Hom}(T,F') &
\end{array}$$

The dash arrow ϵ is defined since the functor $\mathfrak{f}$ is corepresented by the object M. Setting $T = M$ we have $\kappa := \epsilon(\mathrm{id}) : M \to \widetilde{M}$.

In this article $\overline{M}$ is moduli space for $\mathfrak{f}^{GM}$, and $\widetilde{M}$ is moduli space for $\mathfrak{f}$. We construct a natural transformation of functors $\mathfrak{f}^{GM} \to \mathfrak{f}$, expressed by the diagram

$$\begin{array}{ccc}
T & \overset{\mathfrak{f}^{GM}}{\longmapsto} & \mathfrak{F}_T^{GM}/\sim \\
\| & & \downarrow \\
T & \overset{\mathfrak{f}}{\longmapsto} & \mathfrak{F}_T/\sim
\end{array}$$

This natural transformation yields a morphism of moduli schemes $\kappa : \overline{M} \to \widetilde{M}$.

2. Standard resolution for families with nonreduced base

In this section we develop the analogue of the procedure of standard resolution from articles [1] – [4].

Let T be arbitrary (possibly nonreduced) k-scheme of finite type. We assume that its reduction T_{red} is irreducible. If $\mathbb{E}$ is a family of coherent torsion-free sheaves on the surface S having reduced base T then homological dimension of $\mathbb{E}$ as $\mathcal{O}_{T\times S}$-module is not greater then 1. The proof of this fact for reduced equidimensional base can be found, for example, in [14, proposition 1].

Now we need the following simple lemma concerning homological dimension of the family $\mathbb{E}$ with nonreduced base.

Lemma 1. *Let coherent $\mathcal{O}_{T\times S}$-module $\mathbb{E}$ of finite type is T-flat and its reduction $\mathbb{E}_{\mathrm{red}} := \mathbb{E} \otimes_{\mathcal{O}_T} \mathcal{O}_{T_{\mathrm{red}}}$ has homological dimension not greater then 1: $\mathrm{hd}_{T_{\mathrm{red}}\times S}\mathbb{E}_{\mathrm{red}} \leq 1$. Then $\mathrm{hd}_{T\times S}\mathbb{E} \leq 1$.*

Proof. Assume the opposite; let in the exact triple of $\mathcal{O}_{T\times S}$-modules

$$0 \to E_1 \to E_0 \to \mathbb{E} \to 0$$

E_1 be nonlocally free as $\mathcal{O}_{T\times S}$-module while E_0 is locally free. Passing to reductions (this means tensoring by $\otimes_{\mathcal{O}_T}\mathcal{O}_{T_{\mathrm{red}}}$) we come to the exact triple

$$0 \to E_{1\mathrm{red}} \to E_{0\mathrm{red}} \to \mathbb{E}_{\mathrm{red}} \to 0.$$

Here we took into account that since $\mathbb{E}$ is T-flat, then $\mathcal{T}or_1^{\mathcal{O}_T}(\mathbb{E}, \mathcal{O}_{T_{\mathrm{red}}}) = 0$, and left-exactness is preserved. Since $\mathrm{hd}_{T_{\mathrm{red}}\times S}\mathbb{E}_{\mathrm{red}} \leq 1$, then $E_{1\mathrm{red}}$ is locally free as $\mathcal{O}_{T_{\mathrm{red}}\times S}$-module. Also since $\mathbb{E}$ and E_0 are T-flat and of finite type then E_1 is also T-flat and of finite type. It rests to conclude that E_1 is locally free. Apply the sheaf-theoretic version of the following result from A. Grothendieck's SGA [15, ch. IV, corollaire 5.9]:

Proposition 1. *If $X \xrightarrow{g} Y \xrightarrow{f} T$ are morphisms of Noetherian schemes and f is flat morphism then a coherent $\mathcal{O}_X$-sheaf $\mathcal{E}$ is flat over Y if and only if it is flat over T and for any closed point $t \in T$ $\mathcal{E}|_{(f\circ g)^{-1}(t)}$ is flat over $\mathcal{O}_{f^{-1}(t)}$.*

Set $X = Y = T \times S$, $g : X \to Y$ to be an identity isomorphism, $\mathcal{E} := E_1$, $f = p : T \times S \to T$ a projection, $p^{-1}(t) = t \times S$. Then for any closed point $t \in T$ (keeping in mind that sets of closed points are equal and residue fields of corresponding closed points are isomorphic for a scheme T and for its reduction T_{red}) $E_1|_{t\times S} = E_{1\mathrm{red}}|_{t\times S}$, and $E_{1\mathrm{red}}|_{t\times S}$ is flat over $\mathcal{O}_{t\times S}$ because of local freeness. From this we conclude that E_1 is locally free as $\mathcal{O}_{T\times S}$-module. This completes the proof of the lemma. □

We do computations as in [1]. Choose and fix locally free $\mathcal{O}_{T\times S}$-resolution of the sheaf $\mathbb{E}$:

$$0 \to E_1 \to E_0 \to \mathbb{E} \to 0. \tag{2.1}$$

Form a sheaf of 0-th Fitting ideals

$$\mathbb{I} = \mathcal{F}itt^0 \mathcal{E}xt^1(\mathbb{E}, \mathcal{O}_{T\times S}). \tag{2.2}$$

Let $\boldsymbol{\sigma} : \widehat{\Sigma} \to \Sigma$ be a morphism of blowing up the scheme $\Sigma := T \times S$ in the sheaf of ideals $\mathbb{I}$. Apply inverse image $\boldsymbol{\sigma}^*$ to the dual sequence of (2.1):

$$\boldsymbol{\sigma}^*\mathbb{E}^\vee \longrightarrow \boldsymbol{\sigma}^* E_0^\vee \longrightarrow \boldsymbol{\sigma}^*\mathcal{W} \longrightarrow 0,$$

$$\boldsymbol{\sigma}^*\mathcal{W} \longrightarrow \boldsymbol{\sigma}^* E_1^\vee \longrightarrow \boldsymbol{\sigma}^*\mathcal{E}xt^1(\mathbb{E}, \mathcal{O}_\Sigma) \longrightarrow 0. \tag{2.3}$$

The symbol $\mathcal{W}$ stands for the sheaf

$$\ker (E_1^\vee \to \mathcal{E}xt^1(\mathbb{E}, \mathcal{O}_\Sigma)) = \mathrm{coker}\, (\mathbb{E}^\vee \to E_0^\vee).$$

In (2.3) denote $\mathcal{N} := \ker (\boldsymbol{\sigma}^* E_1^\vee \to \boldsymbol{\sigma}^*\mathcal{E}xt^1(\mathbb{E}, \mathcal{O}_\Sigma))$. The sheaf $\mathcal{F}itt^0(\boldsymbol{\sigma}^*\mathcal{E}xt^1(\mathbb{E}, \mathcal{O}_\Sigma))$ is invertible by functorial property of $\mathcal{F}itt$:

$$\mathcal{F}itt^0(\boldsymbol{\sigma}^*\mathcal{E}xt^1_{\mathcal{O}_\Sigma}(\mathbb{E}, \mathcal{O}_\Sigma)) = (\boldsymbol{\sigma}^{-1}\mathcal{F}itt^0(\mathcal{E}xt^1_{\mathcal{O}_\Sigma}(\mathbb{E}, \mathcal{O}_\Sigma))) \cdot \mathcal{O}_{\widehat{\Sigma}} = (\boldsymbol{\sigma}^{-1}\mathbb{I}) \cdot \mathcal{O}_{\widehat{\Sigma}}.$$

Although 0th Fitting ideal sheaf is invertible, non-reducedness of the scheme $\widehat{\Sigma}$ makes Tikhomirov's lemma [16, lemma 1] nonapplicable in its initial formulation. However it can be slightly generalized as follows.

Lemma 2. *Let X be Noetherian scheme such that its reduction X_{red} is irreducible, $\mathcal{F}$ nonzero coherent $\mathcal{O}_X$-sheaf supported on a subscheme of codimension ≥ 1. Then the sheaf of 0-th Fitting ideals $\mathcal{F}itt^0(\mathcal{F})$ is invertible $\mathcal{O}_X$-sheaf if and only if $\mathcal{F}$ has homological dimension equal to 1:* $\mathrm{hd}\,_X\mathcal{F} = 1$.

Proof. is almost identical to the proof in [16, Lemma 1], except some details. The part "if" is obvious. For opposite implication it is necessary to prove that $\mathrm{hd}\,_{\mathcal{O}_{X,x}}\mathcal{F}_x = 1$ for any point $x \in X$. Irreducibility of reduction X_{red} means that the local ring $\mathcal{O}_{X,x}$ contains no zero-divisors except nilpotents.

Consider finite presentation of $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$:

$$M \xrightarrow{f} N \to \mathcal{F}_x \to 0, \tag{2.4}$$

where M and N are free $\mathcal{O}_{X,x}$-modules of the form $M = \bigoplus_{i=1}^{n+r} \mathcal{O}_{X,x} e_i$ and $N = \bigoplus_{j=1}^{n} \mathcal{O}_{X,x} e'_j$ respectively. Let $A = (a_{ij})$ be a matrix of $\mathcal{O}_{X,x}$-linear map f with respect to systems of generating elements $(e_i)_{i=1}^{n+r}$, $(e'_j)_{j=1}^{n}$ where $a_{ij} \in \mathcal{O}_{X,x}$. Since $Fitt^0\mathcal{F}_x$ is principal ideal in $\mathcal{O}_{X,x}$ generated by all $n \times n$-minors of the matrix A, we can put (possibly after re-ordering of generating elements) that $Fitt^0\mathcal{F}_x = (a)$,

$$a = \det \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \in \mathcal{O}_{X,x}.$$

The element a is non-zerodivizor in $\mathcal{O}_{X,x}$, and by our restriction on the form of the ring $\mathcal{O}_{X,x}$ this element is not nilpotent.

Assume that $r \geq 1$ (the case $r = 0$ will be considered separately). For $1 \leq i \leq n$, $n+1 \leq k \leq n+r$ denote by Δ_{ik} $n \times n$-minor of the matrix A. This minor is obtained by replacing of i-th column by k-th column of the matrix A. For $n+1 \leq k \leq n+r$ consider r similar systems each of n linear equations (one system for each k) in n variables:

$$\sum_{j=1}^{n} a_{ij}x_{jk} = a_{ik}, \quad 1 \leq i \leq n, \quad n+1 \leq k \leq n+r.$$

By Cramer's rule following relations are true

$$\sum_{j=1}^{n} a_{ij}\Delta_{jk} = a_{ik}a, \quad 1 \leq i \leq n, \quad n+1 \leq k \leq n+r. \tag{2.5}$$

Since ideal $Fitt^0\mathcal{F}_x$ is 1-generated by the element a, there exist such $\lambda_{ik} \in \mathcal{O}_{X,x}$, that $\Delta_{ik} = \lambda_{ik}a$. Their substitution in (2.5) yields

$$(\sum_{j=1}^{n} a_{ij}\lambda_{jk})a = a_{ik}a, \quad 1 \leq i \leq n, \quad n+1 \leq k \leq n+r.$$

Since a is non-zerodivisor, then $(\sum_{j=1}^{n} a_{ij}\lambda_{jk}) = a_{ik}, \quad 1 \leq i \leq n, \quad n+1 \leq k \leq n+r$. By the definition of the matrix A this means that $f(e_k) \in f(\bigoplus_{i=1}^{n} \mathcal{O}_{X,x}e_i)$ for $n+1 \leq k \leq n+r$. Hence $f(M) = f(\bigoplus_{i=1}^{n} \mathcal{O}_{X,x}e_i)$. Then replacing in (2.4) M by $\bigoplus_{i=1}^{n} \mathcal{O}_{X,x}e_i$ we have $r = 0$.

It rests to prove injectivity of the homomorphism f for $r = 0$. Consider the system of linear equations

$$\sum_{i=1}^{n} a_{li}x_i = 0,$$

defining $\ker f$. Denoting by X the column of indeterminates $x_1, \dots, x_n$ we obtain a matrix equation. Let A^* be an adjoint matrix for A. Elements a^*_{kl} of A^* equal to algebraic complements of elements of A: $a^*_{kl} = A_{lk}$. Then $A^*A = E \det A$ and hence $A^*AX = (\det A)X = aX = 0$. Since a is non-zerodivisor, then $X = 0$. This completes the proof of the lemma. □

Applying the lemma we conclude that $\operatorname{hd} \boldsymbol{\sigma}^*\mathcal{E}xt^1_{\mathcal{O}_\Sigma}(\mathbb{E}, \mathcal{O}_\Sigma) = 1$.

Hence the sheaf $\mathcal{N} = \ker(\boldsymbol{\sigma}^*E_1^\vee \to \boldsymbol{\sigma}^*\mathcal{E}xt^1_{\mathcal{O}_\Sigma}(\mathbb{E}, \mathcal{O}_\Sigma)$ is locally free. Then there is a morphism of locally free sheaves $\boldsymbol{\sigma}^*E_0^\vee \to \mathcal{N}$. Let Q be a sheaf of $\mathcal{O}_\Sigma$-modules which factors the morphism $E_0^\vee \to E_1^\vee$ into the composite of epimorphism and monomorphism. By the definition of the sheaf $\mathcal{N}$ it also factors the morphism $\boldsymbol{\sigma}^*Q \to \boldsymbol{\sigma}^*E_1^\vee$ in the composite of epimorphism and monomorphism and $\boldsymbol{\sigma}^*E_0^\vee \to \boldsymbol{\sigma}^*Q$ is an epimorphism. From this we conclude that the composite $\boldsymbol{\sigma}^*E_0^\vee \to \boldsymbol{\sigma}^*Q \to \mathcal{N}$ is an epimorphism of locally free sheaves. Then its kernel is also locally free sheaf. Now set $\widehat{\mathbb{E}} := \ker(\boldsymbol{\sigma}^*E_0^\vee \to \mathcal{N})^\vee$. Consequently we have an exact triple of locally free $\mathcal{O}_{\widehat{\Sigma}}$-modules

$$0 \to \widehat{\mathbb{E}}^\vee \to \boldsymbol{\sigma}^*E_0^\vee \to \mathcal{N} \to 0.$$

Its dual is also exact.

Now there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{N}^\vee & \longrightarrow & \boldsymbol{\sigma}^*E_0 & \longrightarrow & \widehat{\mathbb{E}} & \longrightarrow & 0 \\ & & \uparrow & & \| & & \uparrow & & \\ & & \boldsymbol{\sigma}^*E_1 & \longrightarrow & \boldsymbol{\sigma}^*E_0 & \longrightarrow & \boldsymbol{\sigma}^*\mathbb{E} & \longrightarrow & 0 \end{array} \tag{2.6}$$

where the right vertical arrow is an epimorphism.

Recall the following definition from [17, O_{III}, definition 9.1.1].

Definition 8. The continuous mapping $f : X \to Y$ is called *quasi-compact* if for any open quasi-compact subset $U \subset Y$ its preimage $f^{-1}(U)$ is quasi-compact. Subset Z is called *retro-compact in* X if the canonical injection $Z \hookrightarrow X$ is quasi-compact, and if for any open quasi-compact subset $U \subset X$ the intersection $U \cap Z$ is quasi-compact.

Let $f : X \to S$ be a scheme morphism of finite presentation, $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_X$-module of finite type.

Definition 9. [18, part 1, definition 5.2.1] $\mathcal{M}$ is *S-flat in dimension* $\geq n$ if there exist a retro-compact open subset $V \subset X$ such that $\dim(X \setminus V)/S < n$ and if $\mathcal{M}|_V$ is S-flat module of finite presentation.

If $\mathcal{M}$ is S-flat module of finite presentation and schemes X and S are of finite type over the field, then any open subset $V \subset X$ fits to be used in the definition. Setting $V = X$ we have $X \setminus V = \emptyset$ and $\dim(X \setminus V)/S = -1 - \dim S$. Consequently, S-flat module of finite presentation is flat in dimension $\geq -\dim S$.

Conversely, let $\mathcal{O}_X$-module $\mathcal{M}$ be S-flat in dimension $\geq -\dim S$. Then there is an open retro-compact subset $V \subset X$ such that $\dim(X \setminus V)/S < -\dim S$ and such that $\mathcal{M}|_V$ is S-flat module. By the former inequality for dimensions we have $\dim(X \setminus V) < 0$, what implies $X = V$, and $\mathcal{M}|_V = \mathcal{M}$ is S-flat.

Definition 10. [18, part 1, definition 5.1.3] Let $f : S' \to S$ be a morphism of finite type, U be an open subset in S. The morphism f is called *U-admissible blowup* if there exist a closed subscheme $Y \subset S$ of finite presentation which is disjoint from U and such that f is isomorphic to the blowing up a scheme S in Y.

Theorem 2. [18, theorem 5.2.2] *Let S be a quasi-compact quasi-separated scheme, U be open quasi-compact subscheme in S, $f : X \to S$ of finite presentation, $\mathcal{M}$ $\mathcal{O}_X$-module of finite type, n an integer. Assume that $\mathcal{M}|_{f^{-1}(U)}$ is flat over U in dimension $\geq n$. Then there exist U-admissible blowup $g : S' \to S$ such that $g^*\mathcal{M}$ is S'-flat in dimension $\geq n$.*

Recall the following

Definition 11. [19, definition 6.1.3] the scheme morphism $f : X \to Y$ is *quasi-separated* if the diagonal morphism $\Delta_f : X \to X \times_Y X$ is quasi-compact. The scheme X is *quasi-separated* if it is quasi-separated over $\operatorname{Spec} \mathbb{Z}$.

If the scheme X is Noetherian, then any morphism $f : X \to Y$ is quasi-compact. Since we work in the category of Noetherian schemes, all morphisms of our interest and all arising schemes are quasi-compact.

Due to theorem 2, there exist a T_0-admissible blowing up $g : \widetilde{T} \to T$ such that inverse images of sheaves $\mathcal{O}_{\widehat{\Sigma}}$ and $\widehat{\mathbb{E}}$ are $\widetilde{T}$-flat. Namely, in the notation fixed by the following fibred square

$$\begin{array}{ccc} \widetilde{\Sigma} & \xrightarrow{\widetilde{g}} & \widehat{\Sigma} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle f} \\ \widetilde{T} & \xrightarrow{g} & T \end{array} \tag{2.7}$$

$\mathcal{O}_{\widetilde{\Sigma}} = \widetilde{g}^*\mathcal{O}_{\widehat{\Sigma}}$ and $\widetilde{\mathbb{E}} := \widetilde{g}^*\widehat{\mathbb{E}}$ are flat $\mathcal{O}_{\widetilde{T}}$-modules.

Remark 7. Since $\widehat{\mathbb{E}}$ is locally free as $\mathcal{O}_{\widehat{\Sigma}}$-module, it is sufficient to achieve that $\mathcal{O}_{\widetilde{\Sigma}} = \widetilde{g}^*\mathcal{O}_{\widehat{\Sigma}}$ be flat as $\mathcal{O}_{\widetilde{T}}$-module. Then the locally free $\mathcal{O}_{\widetilde{\Sigma}}$-module $\widetilde{g}^*\widehat{\mathbb{E}}$ is also flat over $\widetilde{T}$.

The epimorphism

$$\widetilde{g}^*\boldsymbol{\sigma}^*\mathbb{E} \twoheadrightarrow \widetilde{\mathbb{E}} \tag{2.8}$$

induced by the right vertical arrow in (2.6), provides quasi-ideality on closed fibres of the morphism π.

The transformation of families we constructed, has a form

$$(T, \mathbb{L}, \mathbb{E}) \mapsto (\pi : \widetilde{\Sigma} \to \widetilde{T}, \widetilde{\mathbb{L}}, \widetilde{\mathbb{E}})$$

and is defined by the commutative diagram

$$\begin{array}{ccc} T & \longmapsto & \{(T, \mathbb{L}, \mathbb{E})\} \\ {\scriptstyle g^o}\downarrow & & \downarrow \\ \widetilde{T} & \longmapsto & \{(\pi : \widetilde{\Sigma} \to \widetilde{T}, \widetilde{\mathbb{L}}, \widetilde{\mathbb{E}})\} \end{array}$$

where left vertical arrow is a morphism in the category $(Schemes_k)^o$. This morphism is dual to the blowup morphism $g : \widetilde{T} \to T$. The right vertical arrow is the map of sets. Their elements are families of objects to be parametrized. The map is determined by the procedure of resolution as it developed in this section.

Remark 8. The morphism g is defined by the structure of the concrete family of coherent sheaves under resolution but not by the set of (equivalence classes of) families which is an image of the scheme T under the functor $\mathfrak{f}^{GM}$. Then the transformation as it is constructed now does not define a morphism of functors.

Remark 9. The resolution we constructed for families with nonreduced base is not applicable to families without locally free sheaves, because in the suggested procedure the result from [18] is involved (theorem 2). This result operates with the notion of flatness in dimension $\geq n$. In particular, the construction is not applicable to families of nonlocally free sheaves with zero-dimensional base, as well as to investigate such components of Gieseker – Maruyama moduli scheme which do not contain locally free sheaves.

3. Construction of morphism of functors

To obtain the natural transformation of interest it is necessary to show that flatness of the family $(\pi : \widetilde{\Sigma} \to \widetilde{T}, \widetilde{\mathbb{L}}, \widetilde{\mathbb{E}})$ over $\widetilde{T}$ implies that the family $(pr_1 \circ \boldsymbol{\sigma} : \widehat{\Sigma} \to T, \widehat{\mathbb{L}}, \widehat{\mathbb{E}})$ is also flat over T.

To solve the descent problem for the property of flatness of a constructed family $(\pi : \widetilde{\Sigma} \to \widetilde{T}, \widetilde{\mathbb{L}}, \widetilde{\mathbb{E}})$ along the morphism g we need the following

Definition 12. The scheme morphism $h : X \to Y$ *has infinitesimal sections* if for any closed point $y \in Y$ and for any zero-dimensional subscheme $Z_Y \subset Y$ supported at y there is a zero-dimensional subscheme $Z_X \subset X$ such that the induced morphism $h|_{Z_X} : Z_X \to Z_Y$ is an isomorphism.

It is clear that the morphism having infinitesimal sections is surjective. Any blowup morphism has infinitesimal sections. Indeed, for any ring A, ideals $I \subset A \supset I_Y$ where I_Y is an ideal of a zero-dimensional subscheme in Spec A, and for a canonical morphism onto zeroth component $h^\sharp : A \to \bigoplus_{s \geq 0} I^s$, the following diagram of A-modules commutes

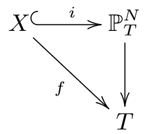
$$\begin{array}{ccc} A & \xrightarrow{h^\sharp} & \bigoplus_{s \geq 0} I^s \\ \downarrow & & \downarrow \\ A/I_Y & = & A/I_Y \end{array}$$

The right vertical arrow in this diagram is a projection on the quotient module of zeroth component.

Now we need the analogue of the well known criterion of flatness involving Hilbert polynomial, for the case of nonreduced base scheme.

If $t \in T$ is closed point of the scheme T corresponding to the sheaf of maximal ideals $\mathfrak{m}_t \subset \mathcal{O}_X$, then we denote mth infinitesimal neighborhood of the point $t \in T$ by the symbol $t^{(m)}$. It is a subscheme defined in T by the sheaf of ideals $\mathfrak{m}_t^{m+1}$.

Proposition 2. [20, theorem 3] *Let a projective morphism of Noetherian schemes of finite type $f : X \to T$ is include in the commutative diagram*



where i is closed immersion. The coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is flat with respect to f (i.e. flat as a sheaf of $\mathcal{O}_T$-modules) if and only if for an invertible $\mathcal{O}_X$-sheaf $\mathcal{L}$ which is very ample relatively to T and such that $\mathcal{L} = i^*\mathcal{O}(1)$, for any closed point $t \in T$ the function

$$\varpi_t^{(m)}(\mathcal{F}, n) = \frac{\chi(\mathcal{F} \otimes \mathcal{L}^n|_{f^{-1}(t^{(m)})})}{\chi(\mathcal{O}_{t^{(m)}})}$$

does not depend of the choice of $t \in T$ and of $m \in \mathbb{N}$.

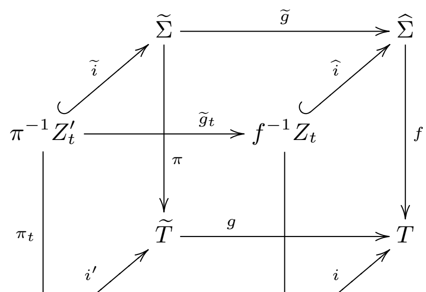
Remark 10. For $m = 0$ we have $\varpi_t^{(m)}(\mathcal{F}, n) = \chi(\mathcal{F} \otimes \mathcal{L}^n|_{f^{-1}(t)})$. This is Hilbert polynomial of the restriction of the sheaf $\mathcal{F}$ to the fibre at the point t.

Remark 11. The proof done in [20, theorem 3] implies that all possible zero-dimensional subschemes supported at the closed point t can be considered instead of infinitesimal neighborhoods of this point.

Proposition 3. *Let we are given fibred diagram (2.7) of Noetherian schemes where f is projective morphism and the morphism g has infinitesimal sections. Let also the scheme $\widehat{\Sigma}$ carry an invertible sheaf $\widehat{\mathbb{L}}$ which is very ample relatively to T, and its inverse image $\widetilde{\mathbb{L}} = \widetilde{g}^*\widehat{\mathbb{L}}$ is flat relatively to the morphism π. Then the morphism f is also flat and the sheaf $\widehat{\mathbb{L}}$ is flat relatively to f.*

Proof. To verify flatness we use the criterion of the proposition 2. Choose a closed point t in T and a zero-dimensional subscheme Z_t, $\operatorname{Supp} Z_t = t$.

Now choose a zero-dimensional subscheme $Z'_t \subset \widetilde{T}$ in the preimage $g^{-1}Z_t$ such that this zero-dimensional subscheme maps isomorphically to Z_t under the morphism g. Consider a following commutative diagram



where all skew arrows are closed immersions, left and right parallelograms and the rectangle containing T are fibred. Usual verifying of universality shows that the rectangle with Z_t is also fibred. This implies that $\widetilde{g}_t := \widetilde{g}|_{\pi^{-1}Z'_t}$ is an isomorphism. Then

$$h^0(f^{-1}Z_t, \widehat{\mathbb{L}}^n|_{f^{-1}Z_t}) = h^0(f^{-1}Z_t, \widehat{i}^*\widehat{\mathbb{L}}^n) = h^0(\pi^{-1}Z'_t, \widetilde{g}_t^*\widehat{i}^*\widehat{\mathbb{L}}^n)$$

$$= h^0(\pi^{-1}Z'_t, \widetilde{i}^*\widetilde{g}^*\widehat{\mathbb{L}}^n) = h^0(\pi^{-1}Z'_t, \widetilde{i}^*\widetilde{\mathbb{L}}^n) = h^0(\pi^{-1}Z'_t, \widetilde{\mathbb{L}}^n|_{\pi^{-1}Z'_t}).$$

In particular if $Z_t = t$ and $Z'_t = \widetilde{t}$, $g(\widetilde{t}) = t$ are reduced points, then Hilbert polynomials of fibres $\chi(\widehat{\mathbb{L}}^n|_{f^{-1}(t)})$ and $\chi(\widetilde{\mathbb{L}}^n|_{\pi^{-1}(\widetilde{t})})$ coincide.

Then for $m \gg 0$ we have

$$\chi(\widehat{\mathbb{L}}^m|_{f^{-1}Z_t}) = h^0(f^{-1}Z_t, \widehat{\mathbb{L}}^m|_{f^{-1}Z_t}) = h^0(\pi^{-1}Z'_t, \widetilde{\mathbb{L}}^m|_{\pi^{-1}Z'_t}) = \chi(\widetilde{\mathbb{L}}^m|_{\pi^{-1}Z'_t}).$$

By the proposition 2 in view of the remark 11, since π is flat morphism and $\widetilde{\mathbb{L}}$ provides equal Hilbert polynomials on its fibres, then

$$\chi(\widetilde{\mathbb{L}}^m|_{\pi^{-1}Z'_t}) = \chi(\widetilde{\mathbb{L}}^m|_{\pi^{-1}(\widetilde{t})})\text{length}\,(Z'_t) = \chi(\widehat{\mathbb{L}}^m|_{f^{-1}(t)})\text{length}\,(Z_t) = \chi(\widehat{\mathbb{L}}^m|_{f^{-1}Z_t}).$$

Hence the morphism f is also flat what completes the proof of the proposition. □

Then $(T, f : \widehat{\Sigma} \to T, \widehat{\mathbb{L}}, \widehat{\mathbb{E}})$ is the required family of semistable admissible pairs with base T. The performed construction defines the natural transformation of the functor of semistable torsion-free coherent sheaves to the functor of admissible semistable pairs and hence it completes the proof of the theorem 1.

4. Morphism of moduli schemes

The developed procedure of standard resolution for a family of semistable torsion-free coherent sheaves with possibly nonreduced base, allows to construct the morphism of the Gieseker – Maruyama moduli scheme to the moduli scheme of admissible semistable pairs without categorical considerations.

According to the classical construction of Gieseker – Maruyama moduli scheme $\overline{M}$, choose an integer $m \gg 0$ such that for each semistable coherent sheaf E the morphism $H^0(S, E \otimes L^m) \otimes L^{-m} \to E$ is surjective, $H^0(S, E \otimes L^m)$ is a k-vector space of dimension $rp(m)$ and $H^i(S, E \otimes L^m) = 0$ for all $i > 0$.

Then take a k-vector space V of dimension $rp(m)$ and form a Grothendieck scheme $\text{Quot}^{\,rp(n)}(V \otimes L^{-m})$ of coherent $\mathcal{O}_S$-quotient sheaves of the form $V \otimes L^{-m} \twoheadrightarrow E$. Consider its quasi-projective subscheme $Q \subset \text{Quot}^{\,rp(n)}(V \otimes L^{-m})$, corresponding to those E which are Gieseker-semistable and torsion-free.

The scheme $\text{Quot}^{\,rp(n)}(V \otimes L^{-m})$ is acted upon by the algebraic group $PGL(V)$ by linear transformations of the vector space V. The scheme $\overline{M}$ is obtained as a (good) GIT-quotient of the subscheme Q (which is immersed $PGL(V)$-equivariantly in $\text{Quot}^{\,rp(n)}(V \otimes L^{-m})$), by the action of $PGL(V) : \overline{M} = Q//PGL(V)$.

By [5], the scheme $\widetilde{M}$ is built up in analogous way. For an admissible semistable pair $((\widetilde{S}, \widetilde{L}), \widetilde{E})$ for $m \gg 0$ there is a closed immersion $j : \widetilde{S} \hookrightarrow G(V, r)$, determined by the epimorphism of locally free sheaves $H^0(\widetilde{S}, \widetilde{E} \otimes \widetilde{L}^m) \boxtimes \widetilde{L}^{-m} \to \widetilde{E}$. Here $G(V, r)$ is Grassmann variety of r-dimensional quotient spaces of the vector space V.

Let $\mathcal{O}_{G(V,r)}(1)$ be a positive generator of the group $\text{Pic}\, G(V, r)$, then $P(n) := \chi(j^*\mathcal{O}_{G(V,r)}(n))$ is a Hilbert polynomial of the closed subscheme $j(\widetilde{S})$. Fix the polynomial $P(n)$ and consider the Hilbert scheme $\text{Hilb}^{\,P(n)}G(V, r)$ of subschemes in $G(V, r)$ with Hilbert polynomial equal to $P(n)$. Let H_0 be quasi-projective subscheme in $\text{Hilb}^{\,P(n)}G(V, r)$ whose points correspond to admissible semistable pairs. The Grassmann variety $G(V, r)$ and in induced way the Hilbert scheme $\text{Hilb}^{\,P(n)}G(V, r)$ are acted upon by the group $PGL(V)$. This action, as in the case with Gieseker – Maruyama compactification, is inspired by linear transformations of the vector space V. The scheme $\widetilde{M}$ is obtained as a (good) GIT-quotient of the scheme H_0 by the action of the group $PGL(V)$: $\widetilde{M} = H_0//PGL(V)$.

To construct a morphism $\overline{M} \to \widetilde{M}$ we recall that the Grothendieck' scheme $\text{Quot} := \text{Quot}^{\,rp(n)}(V \otimes L^{-m})$ carries a universal family of quotient sheaves

$$V \boxtimes \mathcal{O}_{\text{Quot}\,\times S} \twoheadrightarrow \mathbb{E}_{\text{Quot}}\,.$$

Restrict it to the subscheme $Q \times S \subset \text{Quot} \times S$:

$$V \boxtimes \mathcal{O}_{Q\times S} \twoheadrightarrow \mathbb{E}_Q.$$

The sheaf of $\mathcal{O}_{Q\times S}$-modules $\mathbb{E}_Q := \mathbb{E}_{\text{Quot}}|_{Q\times S}$ provides a family of Q-based coherent semistable torsion-free $\mathcal{O}_S$-sheaves.

Applying the procedure of standard resolution as developed in this article, to $(\Sigma = Q \times S, \mathbb{L} = \mathcal{O}_Q \boxtimes L, \mathbb{E} = \mathbb{E}_Q)$, we come to the collection of data $((\widehat{\Sigma}_Q, \widehat{\mathbb{L}}_Q), \widehat{\mathbb{E}}_Q)$. By the universal property of the Hilbert scheme $\text{Hilb}^{\,P(n)}G(V,r)$, the family $((\widehat{\Sigma}_Q, \widehat{\mathbb{L}}_Q), \widehat{\mathbb{E}}_Q)$ induces a morphism $\widehat{\Sigma}_Q \to \text{Univ}^{\,P(n)}G(V,r)$ into the universal subscheme

$$\text{Univ}^{\,P(n)}G(V,r) \subset \text{Hilb}^{\,P(n)}G(v,r) \times S,$$

and a morphism of the base scheme $\mu : Q \to \text{Hilb}^{\,P(n)}G(V,r)$. These morphisms are included into the following commutative diagram with fibred square

$$\begin{array}{ccccc}
\widehat{\Sigma}_Q & \longrightarrow & \text{Univ}^{\,P(n)}G(V,r) & \hookrightarrow & \text{Hilb}^{\,P(n)}G(V,r) \times G(V,r) \\
\downarrow f & & \downarrow & \swarrow pr_1 & \\
Q & \xrightarrow{\mu} & \text{Hilb}^{\,P(n)}G(V,r) & &
\end{array}$$

Hence the morphism μ decomposes as

$$Q \twoheadrightarrow \mu(Q) \hookrightarrow \text{Hilb}^{\,P(n)}G(V,r).$$

Now insure that this composite factors through the subscheme $H_0 \subset \text{Hilb}^{\,P(n)}G(V,r)$ of admissible semistable pairs.

Firstly, existence and structure of a morphism $\boldsymbol{\varpi} : \widehat{\Sigma}_Q \to Q \times S$ as a blowing up morphism in the sheaf of Fitting ideals (2.2) guarantees that the family $\widehat{\Sigma}_Q \to Q$ is formed by admissible schemes.

Secondly, resolution of the sheaf $\mathbb{E}_Q$ into the sheaf $\widehat{\mathbb{E}}_Q$ provides Gieseker's semistability of locally free sheaves in the family $\widetilde{\mathbb{E}}_Q$. Indeed, in [4] it is proven that this is true in the case when the base Q of the family is considered with reduced scheme structure: $Q = Q_{\text{red}}$. Since Gieseker's semistability of a coherent sheaf is open condition in flat families, then if the image of Q_{red} under the morphism $\mu : Q \to \text{Hilb}^{\,P(n)}G(V,r)$ belongs to the open subset of semistable sheaves, the same is true for the whole of the scheme Q.

Thirdly, the epimorphism (2.8) provides quasi-ideality of sheaves in the family $\widehat{\mathbb{E}}_Q$. Applying resolution from section 2 to the case $T = Q$, we come to quasi-ideality on the image of the scheme Q under the resolution.

The procedure of resolution constructed in Section 2 and applied to the quasi-projective scheme Q, leads to the family of admissible semistable pairs $((f : \widehat{\Sigma} \to Q, \widehat{\mathbb{L}} := \boldsymbol{\varpi}^*\mathbb{L} \otimes \boldsymbol{\varpi}^{-1}I \cdot \mathcal{O}_{Q\times S}), \widehat{\mathbb{E}}))$. This family fixes a subscheme $\mu(\widetilde{Q}) \subset H_0$ in $\text{Hilb}^{\,P(n)}G(V,r)$. It defines $PGL(V)$-equivariant composite $Q \twoheadrightarrow \mu(Q) \subset H_0$. This composite leads to the morphism of GIT-quotients $\kappa : \overline{M} \to \widetilde{M}$.

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