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ON CONTINUOUS LINEAR FUNCTIONALS IN SOME WEIGHTED FUNCTIONAL CLASSES ON PRODUCT DOMAINS

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Abstract. This expository article provides an overview of some new research results of authors and their colleagues regarding the problem of description of continuous linear functionals in weighted spaces of analytic and n-harmonic functions in one dimension case and n-dimensional complex space C^n. New extension is discussed, some new interesting problems in analytic spaces in product domains are also discussed.

Keywords: weighted spaces, linear continuous functionals, conjugate space, analytic functions, polydisk, unit ball, pseudoconvex domains, tubular domains.

1. Introduction

Questions of description of continuous linear functionals in spaces of analytic functions are closely related to the classical questions on complex and functional analysis and have substantial applications in approximation theory and interpolation, operator theory, the theory of differential equations; many other mathematic problems are reduced to problems of the theory of analytic functions via the description of the conjugate spaces.

Interest in the description of the dual space to the spaces of analytic functions appeared after work P. Duren, B. Romberg, A. Shields [1], which give a complete description of continuous linear functionals of various interesting spaces.

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We formulate now one of the main results from [1], which will be given in much more general form in this paper, namely for new mixed norm analytic Hardy spaces in the unit polydisk.

We introduce some notation for the formulation of the result (see [1]).

Let $H(U)$ denote the space of all analytic functions in unit disc U.

For $0 < p \leq +\infty$, H^p is the linear space of functions $f \in H(U)$, such that

$$M_p(r,f) = \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p d\theta \right)^{\frac{1}{p}}, \quad 0 < p < +\infty,$$

or

$$M_\infty(r,f) = \max_{0 \leq \theta < 2\pi} \left| f(re^{i\theta}) \right|$$

remains bounded as $r \to 1$.

Let $F(t)$ be a complex-valued function defined on $-\infty < t < +\infty$. F is said to belong to the Lipschitz class $\Lambda_\alpha (0 < \alpha \leq 1)$ if

$$\sup_{|t-s| \leq h} |F(t) - F(s)| = O(h^\alpha)$$

as $h \to 0$.

Further, a continuous function F is said to belong to the class Λ_* if

$$|F(t+h) - 2F(t) + F(t-h)| = O(h),$$

uniformly in t.

Let X denote the class of functions f analytic in U and continuous in $\overline{U}$. For analytic functions f we write $f \in \Lambda_\alpha(\Lambda_*)$ to indicate that $f \in$ X and the boundary function $F(\theta) = f(e^{i\theta})$ is in $\Lambda_\alpha(\Lambda_*)$.

If $1 < p < +\infty$, $\frac{1}{p} + \frac{1}{q} = 1$, it is well known that every bounded linear functional Φ in H^p has a unique representation

$$\Phi(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) g(e^{-i\theta}) d\theta, g \in H^q.$$

In [1] given extension of this result to $p < 1$:

Theorem A. *Let $\Phi \in (H^p)^*$, $0 < p < 1$. Then there is a unique function $g \in$ X such that*

$$\Phi(f) = \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) g(e^{-i\theta}) d\theta, f \in H^p.(A)$$

If $\frac{1}{n+1} < p < \frac{1}{n} (n = 1, 2, ...)$, then $g^{(n-1)} \in \Lambda_\alpha$, where $\alpha = \frac{1}{p} - n$. Conversely, for any g with $g^{(n-1)} \in \Lambda_\alpha$, the limit (A) exists for all $f \in H^p$ and defines a functional $\Phi \in (H^p)^*$.

In the case $p = \frac{1}{n+1}$, $g^{(n-1)} \in \Lambda_*$; and conversely, any g with $p = \frac{1}{n+1}$, $g^{(n-1)} \in \Lambda_*$ defines through (A) a bounded linear functional on H^p.

This topic has received a further leap after classical work Ch. Fefferman [2], which was formulated following results:

Theorem B. *BMO is the dual of Hardy space $H^1(R^n)$. The inner product is given by $\langle f, g\rangle = \int_{R^n} f(x)g(x)dx$ for $f \in BMO$ and g belonging to the dense subspace of C^∞ rapidly decreasing functions in H^1.*

In connection with it we mention also [3]-[7].

We also note the recent work [8]-[12].

The purpose of this article is to list results obtained by us and our colleagues in the past two decades related to the description of continuous linear functionals in weighted spaces of analytic functions in domains of the complex plane $\mathbb{C}$ and multidimensional complex space $\mathbb{C}^n$ – the unit ball and polydisc.

Note that many of the results we present without proof, referring the reader to the source in the text of the work, but the general methods of obtaining results are briefly discussed.

We mention separately, that properties of r-lattices in the unit ball, tubular domains, polydisk and pseudoconvex domains are serving as basic tools for proofs of some of our theorems.

The special attention in this paper is given to some special weights (see [13], [14]) and related analytic function spaces.

The study of various weighted analytic spaces is a known problem (see [5]). Some results of this paper for spaces with zero smoothness can be also similarly proved in a little bit more general form for Besov-Bergman analytic spaces, where fractional derivative in a definition of the space is involved.

This paper can also serve as a model for further research of analytic spaces in product domains, where unit disk in various expressions below is replaced by more complicated domains of tubular type, or unit ball, or pseudoconvex domain.

Obviously, need of writing of such a review arised since we and our colleagues conducted rather intensive research on this area, but many results are published in not reviewed journals or in dissertations of young scientists.

Our technique, machinery and methods can be seen as far reaching development methods and machinery of some early works [13], [14] of F.A. Shamoyan, where much simpler cases were considered.

It should be noted that integral representations of functions play an important role in the corresponding classes and estimate of integral operators in these classes.

Note some results of the authors is presented with proofs. Some possible extensions to tubular domains and bounded pseudoconvex domains is also presented.

Some results of this paper probably can be extended to more general analytic spaces by similar methods, when then unit disk is replaced by more general domain, there issues will not be discussed in this paper.

We note in conclusion that this work, in our opinion, may attract the attention of various experts working actively in the area of complex and functional analysis, a range of professional interests of which includes such type.

The plan of this paper is the following: first we consider representation of continuous linear functionals in weighted spaces in the polydisc, then - in a unit ball and the polyball, then finally - in simply connected domains.

At the end of paper we consider some possible extentions of our duality theorems to more complicated tubular domains over symmetric cones and bounded strictly pseudoconvex domains with smooth boundary.

2. On representation of continuous linear functionals in weighted spaces in polydisc

In this section we study the above questions in weighted spaces of analytic, n-harmonic, pluriharmonic functions, also well-known analytic classes Hardy-Sobolev in the polydisc.

Consider, to begin with, as more interesting, in our opinion, results related to the description of continuous linear functionals in weighted spaces of analytic functions n -dimensional complex space $\mathbb{C}^n$, namely, the polydisc.

Let

$$U^n = \left\{z \in \mathbb{C}^n, z = (z_1, ..., z_n) : |z_j| < 1, j = \overline{1, n}\right\}$$

be the unit polydisc in n-dimensional complex space C^n; $U = U^1$;

$$T^n = \left\{z \in \mathbb{C}^n, z = (z_1, ..., z_n) : |z_j| = 1, j = \overline{1, n}\right\} -$$

be the Shilov boundary of U^n; $T = T^1$; $H(U^n)$ be the set of all analytic fuctions in U^n; $H^p(U^n)$ be the Hardy class in U^n.

Let us denote by Ω^n the set of all functions $\omega(t) = (\omega_1(t), ..., \omega_n(t))$ (see [15]), for which $\omega_j(t)$, $j = \overline{1, n}$, – positive integrable of the interval $(0; 1)$ functions, such that there exist positive numbers m_ω, M_ω, q_ω, and $m_\omega, q_\omega \in (0,\, 1)$

$$m_\omega \leq \frac{\omega(\lambda t)}{\omega(t)} \leq M_\omega, \forall t \in (0; 1), \lambda \in [q_\omega,\, 1].$$

For convenience, we denote $\alpha_\omega = \dfrac{\ln m_\omega}{\ln \frac{1}{q_\omega}}$, $\beta_\omega = \dfrac{\ln M_\omega}{\ln \frac{1}{q_\omega}}$; $\alpha_\omega > -1$, $0 < \beta_\omega < 1$.

If $z = (z_1, ..., z_n) \in \mathbb{C}^n$, $\zeta = (\zeta_1, ..., \zeta_n) \in \mathbb{C}^n$, $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{R}^n$, then $z^\alpha := z^{\alpha_1} \cdot ... \cdot z^{\alpha_n}$, $|\alpha| := \alpha_1 + ... + \alpha_n$.

In addition, put

$$\left(1 - |z|^2\right)^\alpha := \prod_{j=1}^n \left(1 - |z_j|^2\right)^{\alpha_j}; (1 - \zeta z)^\alpha := \prod_{j=1}^n (1 - \zeta_j z_j)^{\alpha_j};$$

$$\omega_\Pi(1 - |z|) := \prod_{j=1}^n \omega_j(1 - |z_j|); \omega_\Pi^s(1 - |z|) := \prod_{j=1}^n \omega_\Pi^s(1 - |z_j|),\, s \in \mathbb{R}.$$

Let also for $\alpha_j > -1$, $\omega_j \in \Omega$, $1 < p_j < +\infty$, $j = \overline{1, n}$:

$$\omega_{\alpha_j}(t) := \omega_j(t) \left(\frac{t^{\alpha_j}}{\omega_j(t)}\right)^{q_j},\, t \in (0, 1),\, q_j = \frac{p_j}{p_j - 1}, \omega_\alpha(t) = (\omega_{\alpha_1}(t), ..., \omega_{\alpha_n}(t)).$$

If $f \in H(U^n)$:

$$f(z_1, ..., z_n) = \sum_{k_1, ..., k_n = 0}^{+\infty} a_{k_1 ... k_n} z_1^{k_1} ... z_n^{k_n},$$

then we define the fractional derivative in type of the Riemann-Liouville the function

$$D^\alpha f(z) = \sum_{|k|=0}^{+\infty} \frac{\Gamma(\alpha + 1 + k)}{\Gamma(\alpha + 1)\Gamma(k + 1)} a_k z^k,$$

$$\frac{\Gamma(\alpha + 1 + k)}{\Gamma(\alpha + 1)\Gamma(k + 1)} = \prod_{j=1}^n \frac{\Gamma(\alpha_j + 1 + k_j)}{\Gamma(\alpha_j + 1)\Gamma(k_j + 1)},$$

where Γ is the Euler function, $\alpha_j > -1$, $j = \overline{1, n}$.

Clearly, if $f \in H(U^n)$, then $D^\alpha f(z) \in H(U^n)$ for all α.

We denote in what follows c, $c_1, ..., c_n(\alpha, \beta, ...), ...$ will denote arbitrary positive constants, depending on $\alpha, \beta, ...$, specific values are not play no role.

Futher, we will need the following classes of function.

Let $L^{\vec{p}}_\omega(U^n)$, $\omega \in \Omega^n$, $\vec{p} = (p_1, ..., p_n)$, $0 < p_j < +\infty$, $j = \overline{1, n}$, be the set of measurable functions f in U^n, for which

$$\|f\|_{L^{\vec{p}}_\omega(U^n)} = (\int_U \omega_n(1 - |z_n|)(\int_U \omega_{n-1}(1 - |z_{n-1}|)...$$

$$...(\int_U |f(z_1, ..., z_n)|^{p_1} \omega_1(1 - |z_1|) dm_2(z_1))^{\frac{p_2}{p_1}} ...)^{\frac{p_n}{p_{n-1}}} dm_2(z_n))^{\frac{1}{p_n}} < +\infty, \quad (1)$$

here and in the future $d\sigma, d\sigma_n, dm_n, dm_{2n}, d\nu, ...$, this is the Lebesgue measure in the corresponding boundary, unit disc, polydisc, polyball, ...

We denote the subspace $L^{\vec{p}}_\omega(U^n)$ consisting of analytic functions by $A^{\vec{p}}_\omega(U^n)$; the subspace $L^{\vec{p}}_\omega(U^n)$ consisting of n-harmonic functions by $h^{\vec{p}}_\omega(U^n)$; the subspace $h^{\vec{p}}_\omega$ consisting of pluriharmonic functions by $\tilde{h}^{\vec{p}}_\omega(U^n)$.

Recall that a function $u(z_1, ..., z_n)$ is called n-harmonic in U^n, if it is harmonic on each variable separately, i.e. $u(z_1, ..., z_n)$ is a solution of equations

$$\frac{\partial^2 u}{\partial z_j \partial \bar{z}_j} = 0, \quad j = \overline{1, n}.$$

The function $u(z_1, \ldots, z_n)$ is called a pluriharmonic in U^n, if it is a real part of some analytic function, that is

$$\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} = 0, \quad j, k = \overline{1, n}.$$

Obviously, for $\|u\|_{h^{\vec{p}}_\omega}$ and $\|u\|_{\tilde{h}^{\vec{p}}_\omega}$, $1 \le p_j < +\infty$, $j = \overline{1, n}$, spaces $h^{\vec{p}}_\omega(U^n)$, $\tilde{h}^{\vec{p}}_\omega(U^n)$ are Banach, and for $0 < p_j < 1$, $j = \overline{1, n}$, in the corresponding spaces can be provided quasi-norms $\|u\| = \|u\|_{h^{\vec{p}}_\omega(U^n)}$, $\|u\| = \|u\|_{\tilde{h}^{\vec{p}}_\omega(U^n)}$, for which $h^{\vec{p}}_\omega(U^n)$, $\tilde{h}^{\vec{p}}_\omega(U^n)$ transformed into quasi-normed space.

In [16] F.A. Shamoyan, O.V. Yaroslavtseva, and also in [17], received description of linear continuous functionals in weighted spaces with mixed norm analytical, n-harmonic, pluriharmonic in the unit polydisc functions.

So, summing up the results with respect to spaces of analytic functions, we can formulate the following result:

Theorem 1. *Let Φ be continuous linear functional on $A^{\vec{p}}_\omega(U^n)$, $\omega = (\omega_1, ..., \omega_n)$, $\omega_j \in \Omega$, $\vec{p} = (p_1, ..., p_n)$, $1 < p_j < +\infty$, $j = \overline{1, n}$, and $g(z) = \Phi(e_z)$,*

$$e_z(\zeta) = \frac{1}{1 - \zeta z}, \quad z, \zeta \in U^n. \quad (2)$$

Then $g \in H(U^n)$, $D^{\alpha+1} g \in A^{\vec{q}}_{\omega_\alpha}(U^n)$, $\alpha_j > \alpha_{\omega_j}$, $j = \overline{1, n}$, and the functional Φ has the form

$$\Phi(f) = \lim_{\rho \to 1-0} \frac{1}{(2\pi)^n} \int_{T^n} f(\rho\zeta) g(\rho\bar{\zeta}) d\sigma_n(\zeta), \quad (3)$$

and estimates

$$c_1(\vec{p},\alpha)\left\|D^{\alpha+1}g\right\|_{A^{\vec{q}}_{\omega_\alpha}(U^n)} \le \|\Phi\| \le c_2(\vec{p},\alpha)\left\|D^{\alpha+1}g\right\|_{A^{\vec{q}}_{\omega_\alpha}(U^n)}, \tag{4}$$

are valid.

The converse is also valid: any g such that $D^{\alpha+1}g \in A^{\vec{q}}_{\omega_\alpha}(U^n)$, $\alpha_j > \alpha_{\omega_j}$, $j=\overline{1,n}$, according to the formula (3) generates a continuous linear functional on $A^{\vec{p}}_{\omega}(U^n)$, for which estimates (4) are valid.

The analogue of theorem 1 is true and in the case of $0<p_j\le 1,\ j=\overline{1,n}$.

Let $\vec{p}=(p_1,...,p_n),\ 0<p_j\le 1,\ j=\overline{1,n}$. We denote by $\lambda^{\vec{p}}_{\omega}(U^n)$ the class of analytic functions f in U^n, for which

$$\|f\|_{\lambda^{\vec{p}}_{\omega}(U^n)} = \sup_{z\in U^n}\frac{\left|D^{\alpha+1}f(z)\right|(1-|z|)^{\alpha+2-\frac{2}{p}}}{\omega^{\frac{1}{p}}(1-|z|)} < +\infty, \tag{5}$$

where $\omega\in\Omega^n$, $\alpha_j > \dfrac{\alpha_{\omega_j}+2}{p_j}$, $j=\overline{1,n}$.

Obviously, relative to the these norms $\lambda^{\vec{p}}_{\omega}(U^n)$ is a Banach space.

Theorem 2. *Let Φ be continuous linear functional on $A^{\vec{p}}_{\omega}(U^n)$, $\omega=(\omega_1,...,\omega_n)$, $\omega_j\in\Omega$, $\vec{p}=(p_1,...,p_n)$, $0<p_j\le 1$, $j=\overline{1,n}$, and $g(z)=\Phi(e_z)$, $e_z(\zeta)$ defined by the formula (2).*

Then $g\in\lambda^{\vec{p}}_{\omega}(U^n)$, the functional Φ has the form (3), and estimates

$$c_1(\vec{p},\alpha)\|\Phi\| \le \|g\|_{\lambda^{\vec{p}}_{\omega}(U^n)} \le c_2(\vec{p},\alpha)\|\Phi\|, \tag{6}$$

are valid.

The converse is also valid: any g such that $g\in\lambda^{\vec{p}}_{\omega}(U^n)$ according (3) generates a continuous linear functional on $A^{\vec{p}}_{\omega}(U^n)$, for which estimates (6) are valid.

Remark 1. For Hardy spaces in the unit disc H^p, $0<p<1$, description of the dual space was obtained in [1], and in the polydisc - in [4]. In [16], [17] the description of dual space $A^{\vec{p}}_{\omega}(U^n)$ differs substantially from the methods of [4], [8]. There the important role played the integral representation of $A^{\vec{p}}_{\omega}(U^n)$ classes, and the method of proof is similar to the methods of [13], [14].

Next, we give a complete description of continuous linear functionals in spaces $h^{\vec{p}}_{\omega}(U^n)$, $\tilde{h}^{\vec{p}}_{\omega}(U^n)$ for all $\omega,\vec{p}: 0<p_j<+\infty, j=\overline{1,n}$.

Note, in the work of A. Shields and D. Williams [18], similar results in L^1-metrics in spaces harmonic functions in the unit disc were given. Results F.A. Shamoyan, O.V. Yaroslavtseva generalize and improve the results of A. Shields and D. Williams in two directions: first, she considers the n-dimensional case; secondly, she studied the $L^{\vec{p}}_{\omega}$ -metric mixed norm for all $\vec{p}=(p_1,...,p_n)$.

First we give a complete description of continuous linear functionals in weighted spaces of n-harmonic functions in polydisc for $\vec{p}=(p_1,...,p_n)$, $1<p_j<+\infty$, $j=\overline{1,n}$.

Theorem 3. *Let Φ be continuous linear functional on $h^{\vec{p}}_{\omega}(U^n)$, $\overrightarrow{\omega}=(\omega_1,...,\omega_n)$, $\omega_j\in\Omega$, $\vec{p}=(p_1,...,p_n)$, $1<p_j<+\infty$, $j=\overline{1,n}$, and $g(z)=\Phi(e_z)$,*

$$e_z(\zeta)=\prod_{j=1}^{n}\left(\frac{1}{1-\zeta_j z_j}+\frac{1}{1-\bar{\zeta}_j\bar{z}_j}-1\right),\ z,\zeta\in U^n. \tag{7}$$

Then $g \in h_{\omega}^{\vec{p}}(U^n)$, $D^{\alpha+1}g \in h_{\omega_\alpha}^{\vec{q}}(U^n)$, $\alpha_j > \alpha_{\omega_j}$, $j = \overline{1,n}$, *the functional* Φ *has the form (3) and estimates*

$$c_1(\vec{p},\alpha)\left\|D^{\alpha+1}g\right\|_{h_{\omega_\alpha}^{\vec{q}}(U^n)} \le \|\Phi\| \le c_2(\vec{p},\alpha)\left\|D^{\alpha+1}g\right\|_{h_{\omega_\alpha}^{\vec{q}}(U^n)}, \tag{8}$$

are valid.

The converse is also valid: any ν *such that* $D^{\alpha+1}g \in h_{\omega_\alpha}^{\vec{q}}(U^n)$, $\alpha_j > \alpha_{\omega_j}$, $j = \overline{1,n}$, *according (3) generates a continuous linear functional on* $h_{\omega}^{\vec{p}}(U^n)$, *for which estimates (8) are valid.*

Let $\Lambda_{\omega}^{\vec{p}}(U^n)$ set of all n-harmonic function g in U^n, for which the norm is defined as in (5) and for the norm $\|f\|_{\Lambda_{\omega}^{\vec{p}}(U^n)}$ the condition (5) is valid. Clearly, relative to the these norms $\Lambda_{\omega}^{\vec{p}}$ is a Banach space. Then the following statement is valid:

Theorem 4. *Let* Φ *be continuous linear functional on* $h_{\omega}^{\vec{p}}(U^n)$, $\overrightarrow{\omega} = (\omega_1, ..., \omega_n)$, $\omega_j \in \Omega$, $\vec{p} = (p_1, ..., p_n)$, $0 < p_j \le 1$, $j = \overline{1,n}$, *and* $g(z) = \Phi(e_z)$, $e_z(\zeta)$ *defined by the formula (7).*

Then $g \in \Lambda_{\vec{\omega}}^{\vec{p}}(U^n)$, *the functional* Φ *has the form (3), and estimates*

$$c_1(\vec{p})\|\Phi\| \le \|g\|_{\Lambda_{\vec{\omega}}^{\vec{p}}(U^n)} \le c_2(\vec{p})\|\Phi\|, \tag{9}$$

are valid.

The converse is also valid: any $g \in \Lambda_{\vec{\omega}}^{\vec{p}}(U^n)$ *according (3) generates a continuous linear functional on* $h_{\omega}^{\vec{p}}(U^n)$, *for which estimates (9) are valid.*

In case of the spaces $\tilde{h}_{\omega}^{\vec{p}}(U^n)$ we define

$$\tilde{e}_z(\zeta) = \prod_{j=1}^{n}\frac{1}{1-\zeta_j z_j} + \prod_{j=1}^{n}\frac{1}{1-\bar{\zeta}_j \bar{z}_j} - 1,\ z,\zeta \in U^n,$$

as a result we obtain analogues of theorems 3 and 4 with similar formulations.

In [19] of F.A. Shamoyan, N.A. Chasova, and in [20] the above results are summarized in the following case: first, a complete description of linear continuous functional in the space $A_{\omega}^{\vec{p}}(U^n)$ for all the sets $\vec{p} = (p_1, \ldots, p_n)$, $0 < p_j < +\infty$, $j = \overline{1,n}$; secondly, for the first time introduced the generalized Hardy space in the polydisc with mixed norm and describes the continuous linear functionals in these spaces Hardy $H^{\vec{p}}(U^n)$, $\vec{p} = (p_1, \ldots, p_n)$, for $0 < p_j < 1$ and $1 < p_j < +\infty$, $j = \overline{1,n}$.

Let $\vec{p} = (p_1, \ldots, p_n)$, where $p_j > 1$, $j = \overline{1,k}$, $0 < p_j \le 1$, $j = \overline{k+1,n}$, $1 \le k \le n$, $\dfrac{1}{q_j} + \dfrac{1}{p_j} = 1$, $j = \overline{1,k}$, $\tilde{A} = A^{q_1,\ldots,q_k}(\omega_{\alpha_1}, \ldots, \omega_{\alpha_k})$, we denote by $\lambda_{\vec{\omega}}^{\vec{p}}$ be the set of analytic functions f in U^n, for which

$$\|f\|_{\lambda_{\vec{\omega}}^{\vec{p}}} = \sup_{(z_{k+1},\ldots,z_n)\in U^{n-k}} \prod_{j=k+1}^{n} \frac{(1-|z_j|)^{\alpha_j+2-\frac{2}{p_j}}}{(\omega_j(1-|z_j|))^{\frac{1}{p_j}}}\left\|D^{\alpha+1}f\right\|_{\tilde{A}} < +\infty,$$

where $\alpha_j > \dfrac{\alpha_{\omega_j}+2}{p_j} - 1$, $j = \overline{1,n}$.

If $\vec{p} = (p_1, \ldots, p_n)$, $0 < p_j \le 1$, $j = \overline{1,k}$, and $p_j > 1$, $j = \overline{k+1,n}$, $1 \le k \le n$; $\dfrac{1}{q_j} + \dfrac{1}{p_j} = 1$, $j = \overline{k+1,n}$, $\tilde{\tilde{A}} = A^{q_{k+1},\ldots,q_n}(\omega_{\alpha_{k+1}}, \ldots, \omega_{\alpha_n})$, we denote by $\tilde{\lambda}_{\vec{\omega}}^{\vec{p}}$ be the set of analytic functions f in U^n , for which

$$\|f\|_{\tilde{\lambda}^{\vec{p}}_{\vec{\omega}}} = \left\| \sup_{(z_1,\dots,z_k)\in U^k} \prod_{j=1}^{k} \frac{(1-|z_j|)^{\alpha_j+2-\frac{2}{p_j}}}{(\omega_j(1-|z_j|))^{\frac{1}{p_j}}} \left|D^{\alpha+1}f(z)\right| \right\|_{\tilde{\tilde{A}}} < +\infty,$$

where $\alpha_j > \dfrac{\alpha_{\omega_j}+2}{p_j} - 1$, $j=\overline{1,n}$.

Regarding these norms the sets $\lambda^{\vec{p}}_{\vec{\omega}}$ and $\tilde{\lambda}^{\vec{p}}_{\vec{\omega}}$ converted into quasi-Banach spaces.

Let $\Lambda^{\vec{p}}_{\vec{\omega}}$, where $\vec{p}=(p_1,\dots,p_n)$, $0<p_j<+\infty$, $j=\overline{1,n}$, coincides with the space $\lambda^{\vec{p}}_{\vec{\omega}}$, if $p_j>1$, $j=\overline{1,k}$, $0<p_j\le 1$, $j=\overline{k+1,n}$ and with the space $\tilde{\lambda}^{\vec{p}}_{\vec{\omega}}$, if $0<p_j\le 1$, $j=\overline{1,k}$, $p_j>1$, $j=\overline{k+1,n}$, $1\le k\le n$.

It is well - known, if that among the components of the vector $\vec{p}=(p_1,\dots,p_n)$, there is at least one p_{j_0}, such that $0<p_{j_0}<1$, then each continuous linear functional on $L^{\vec{p}}_{\omega}(U^n)$ is a trivial, this raises the question of the complete characterization of such functionals on $A^{\vec{p}}_{\omega}(U^n)$ for all $\vec{p}=(p_1,\dots,p_n)$, $0<p_j<+\infty$, $j=\overline{1,n}$.

The following theorem allows us to give a complete answer to this question.

Theorem 5. *Let Φ be continuous linear functional on $A^{\vec{p}}_{\omega}(U^n)$, $\omega=(\omega_1,...,\omega_n)$, $\omega_j\in\Omega$, $\vec{p}=(p_1,...,p_n)$, $0<p_j<+\infty$, $j=\overline{1,n}$, and $g(z)=\Phi(e_z)$, $e_z(\zeta)$ defined by the formula (2).*

Then $g\in\Lambda^{\vec{p}}_{\vec{\omega}}$, the functional Φ has the form (3), and estimates

$$c_1\|g\|_{\Lambda^{\vec{p}}_{\vec{\omega}}} \le \|\Phi\| \le c_2\|g\|_{\Lambda^{\vec{p}}_{\vec{\omega}}}, \tag{10}$$

are valid.

The converse is also valid: any g such that $g\in\Lambda^{\vec{p}}_{\vec{\omega}}$ according (3) generates a continuous linear functional on $A^{\vec{p}}_{\omega}(U^n)$, for which estimates (10) are valid.

Remark 2. In that case, when all the components of the vector $\vec{p}$ such that $p_j>1$, or $0<p_j\le 1$, $j=\overline{1,n}$, the description of continuous linear functionals is obtained in the works considered above F.A. Shamoyan and O.V. Yaroslavtseva [16].

In [19], [20] the authors found a complete description of continuous linear functionals in generalized Hardy spaces $H^{\vec{p}}(U^n)$, $\vec{p}=(p_1,\dots,p_n)$, for $1<p_j<+\infty$ and $0<p_j<1$, $j=\overline{1,n}$.

We denote by $L^{\vec{p}}(T^n)$, $\vec{p}=(p_1,\dots,p_n)$, $0<p_j<+\infty$, $j=\overline{1,n}$, be the set of measurable functions f in U^n , for quasinorm which is valid:

$$\|f\|_{L^{\vec{p}}(T^n)} = \left(\int_T \cdots \left(\int_T \left(\int_T |f(\zeta)|^{p_1}\, d\sigma(\zeta_1)\right)^{\frac{p_2}{p_1}} \dots \right)^{\frac{p_3}{p_2}} \cdots d\sigma(\zeta_n)\right)^{\frac{1}{p_n}} < +\infty.$$

Let $\vec{p}=(p_1,\dots,p_n)$, $0<p_j<+\infty$, $j=\overline{1,n}$, then we denote generalized Hardy spaces with mixed norm $H^{\vec{p}}(U^n)$ be the set of analytic functions f in U^n , for which

$$\sup_{0\le r<1} \left(\int_T \cdots \left(\int_T \left(\int_T |f(r\zeta)|^{p_1}\, d\sigma(\zeta_1)\right)^{\frac{p_2}{p_1}} \dots \right)^{\frac{p_3}{p_2}} \cdots d\sigma(\zeta_n)\right)^{\frac{1}{p_n}} < +\infty.$$

In case $p_1 = \ldots = p_n = p$ the space $H^{\vec{p}}(U^n)$ coincides with the usual Hardy space $H^p(U^n)$.

For $1 < p < +\infty$ the duality result for classical Hardy spaces in polydisk is a classical result and it is quite known in folklore, for other values of p, this result can be also seen in various papers. Our closest goal to extend both results to generalized Hardy spaces. For cases when all $1 < p_j < +\infty$ F.A. Shamoyan, N.A. Chasova found the following standard solution (via usual so-called Causchy duality representation) of the duality problem for such type mixed norm Hardy spaces in the polydisk: $(X)^* = Y$, where X is $H^{\vec{p}}(U^n)$, Y is $H^{\vec{q}}(U^n)$, $\vec{p} = (p_1, \ldots, p_n)$, $\vec{q} = (q_1, \ldots, q_n)$, $1 < p_j < +\infty$, $(p_j)^{-1} + (q_j)^{-1} = 1, j = \overline{1,n}$.

Let for all j $0 < p_j < 1$. We denote by $\lambda^{\vec{p}}$, $\vec{p} = (p_1, \ldots, p_n)$, $0 < p_j < 1$, $j = \overline{1,n}$, be the set of analytic functions f in U^n , for which

$$\left|D^{\alpha+1} f(z)\right| \leq c \prod_{j=1}^{n} \frac{1}{(1-|z_j|)^{\alpha_j+2-\frac{1}{p_j}}},$$

where $z = (z_1, \ldots, z_n) \in U^n$. We introduce in $\lambda^{\vec{p}}$ the norm:

$$\|f\|_{\lambda^{\vec{p}}} = \sup_{z\in U^n} \left\{ |D^{\alpha+1} f(z)| \prod_{j=1}^{n} (1-|z_j|)^{\alpha_j+2-\frac{1}{p_j}} \right\}.$$

Obviously, the class $\lambda^{\vec{p}}$ does not depend on $\alpha = (\alpha_1, \ldots, \alpha_n)$, satisfying $\alpha_j > \dfrac{1}{p_j} - 1$, $j = \overline{1,n}$. Regarding these norms the set $\lambda^{\vec{p}}$ converted into Banach spaces.

Theorem 6. *Let Φ be continuous linear functional on $H^{\vec{p}}(U^n)$, $\vec{p} = (p_1, ..., p_n)$, $0 < p_j < 1$, $j = \overline{1,n}$, and $g(z) = \Phi(e_z)$, $e_z(\zeta)$ defined by the formula (2).*

Then $g \in \lambda^{\vec{p}}$, the functional Φ has the form (3), and estimates

$$c_1 \|g\|_{\lambda^{\vec{p}}} \leq \|\Phi\| \leq c_2 \|g\|_{\lambda^{\vec{p}}}, \tag{11}$$

are valid.

The converse is also valid: any g such that $g \in \lambda^{\vec{p}}$ according (3) generates a continuous linear functional on $H^{\vec{p}}(U^n)$, for which estimates (11) are valid.

Remark 3. The results of [4] can be obtained from our theorems, if we put in formulation all $p_j = p$, $j = \overline{1,n}$. The case when all indexes are bigger than one in our last theorem can be also considered. In this case we have a standard duality theorem of two generalized Hardy mixed norm spaces with standard conjugate indexes and the same representation of functionals(see [19], [20] for details) .This also similarly provides another duality theorem in more general mixed norm Lebegues spaces again with usual conjugate indexes with the same representation of functionals.We omit details here.

Remark 4. For Hardy and Bergman spaces in bounded symmetric domains the dual spaces are known see, for example, [5], [21], [22], [23]. It is natural, to define mixed norm Hardy and Bergman spaces. As we did above, but on products of bounded symmetric domains and pseudoconvex domains with smooth boundary and to try to describe duals of such spaces. This will extend completely some results of this paper.

Remark 5. The study of direct analogues of same mixed norm Hardy spaces in products of tubular domains over symmetric cones and bounded strictly pseudoconvex domains is an open and interesting problem.Moreover, the same mixed norm

Hardy spaces in harmonic function classes and pluriharmonic classes deserves also a separate study.

We now consider the space $L^{p,q}_{\omega}(U^n)$, $\omega \in \Omega^n$, $0 < p, q < +\infty$, as the set of measurable functions f in U^n, for which

$$\|f\|_{L^{p,q}_{\omega}} = \left(\int\limits_{Q_n} \omega_{\Pi}\,(1-r) \left(\int\limits_{T^n} |f(rz)|^p \, d\sigma_n(z) \right)^{\frac{q}{p}} dr \right)^{\frac{1}{q}} < +\infty, \tag{12}$$

where $Q^n = (0,1]^n$.

The theory of function spaces with mixed norms of type $L^{p,q}_{\omega}(U^n)$ has its origins in the 60s of the last century from the work of A. Benedek, R. Panzone [24]. For these questions, it published a series of fundamental works. The results are highlighted in the well-known monographs S. Nikolsky [25], O.Besov, V. Ilin, S. Nikolsky [26], H. Triebel [27].

We remark that in the case of $p = q$ such results were obtained in [14].

We also introduce the notation $A^{p,q}_{\omega}(U^n) = H(U^n) \bigcap L^{p,q}_{\omega}(U^n)$ with the corresponding quasi-norm; $h^p(U^n)$, $0 < p < +\infty$, be the class Hardy n-harmonic functions in U^n, i.e. the set of n-harmonic functions in U^n, for which

$$\sup_{0<r<1} \int\limits_{T} |u(r\zeta)|^p \, d\sigma(\zeta) < +\infty.$$

In recent papers [28] E.V. Povprits, F.A. Shamoyan, and in [29], solve problems, concerning the description of continuous linear functionals in terms of Cauchy transform in these spaces of analytic functions.

Recall that if Φ *in* $(A^{p,q}_{\omega}(U^n))^*$, then the Cauchy transform of this functional is a function $g(z) = \Phi(e_z)$, where $e_z(\zeta)$ defined by the formula (2). Clearly, the function g is analytic in U^n. Note that when p, q belong to $(1, +\infty)$ and $(0, 1]$, and in the case, when one of the parameters in the interval $(0, 1]$ and other is in the interval $(1, +\infty)$ the characterization of the Cauchy transform has a completely different description.

We denote by $\omega_{\alpha}(t) = \omega(t) \left(\frac{t^{\alpha}}{\omega(t)} \right)^q$, $t \in Q_n$.

Theorem 7. *Let Φ be continuous linear functional on $A^{p,q}_{\omega}(U^n)$, $\omega = (\omega_1, ..., \omega_n)$, $\omega_j \in \Omega$, $j = \overline{1,n}$, $1 < p, q < +\infty$, $g(z) = \Phi(e_z)$, $e_z(\zeta)$ defined by the formula (2).*

Then $g \in H(U^n)$, $D^{\alpha+1} g \in A^{p',q'}_{\omega_{\alpha}}(U^n)$, $p' = \dfrac{p}{p-1}$, $q' = \dfrac{q}{q-1}$, $\alpha > \alpha_{\omega}$, the functional Φ has the form (3), and estimates

$$c_1(p,\alpha) \left\| D^{\alpha+1} g \right\|_{A^{p',q'}_{\omega_{\alpha}}(U^n)} \le \|\Phi\| \le c_2(p,\alpha) \left\| D^{\alpha+1} g \right\|_{A^{p',q'}_{\omega_{\alpha}}(U^n)}. \tag{13}$$

are valid.

The converse is also valid: any g such that $D^{\alpha+1} g \in A^{p',q'}_{\omega_{\alpha}}(U^n)$ according (3) generates a continuous linear functional on $A^{p',q'}_{\omega_{\alpha}}(U^n)$, for which estimates (13) are valid.

Let $0 < p, q \leq 1$, we denote by $\lambda_\omega^{p,q}(U^n)$ the class of analytic functions f in U^n, for which

$$\|f\|_{\lambda_\omega^{p,q}(U^n)} = \sup_{z\in U^n} \frac{|D^{\alpha+1} f(z)|\,(1-|z|)^{\alpha-\frac{1}{p}-\frac{1}{q}+2}}{\omega_\Pi^{\frac{1}{q}}(1-|z|)} < +\infty,$$

where $\alpha_j > \dfrac{\alpha_{\omega_j}+1}{q} + \dfrac{1}{p} - 2, j = \overline{1,n}$.

If $0 < p \leq 1, \quad 1 < q < +\infty$, we denote by $\widetilde{\lambda}_\omega^{p,q}(U^n)$ the class of analytic f functions in U^n, for which

$$\|f\|_{\widetilde{\lambda}_\omega^{p,q}(U^n)} = \left(\int_{Q_n} \frac{(1-r)^{\alpha q' - \frac{q'}{p} + q'}}{\omega_\Pi^{\frac{q'}{q}}(1-r)} \left(\sup_{z\in T^n} |D^{\alpha+1} f(rz)|\right)^{q'} r dr\right)^{\frac{1}{q'}} < +\infty,$$

where $\alpha_j > \dfrac{\alpha_{\omega_j}}{q} + \dfrac{1}{p} - \dfrac{1}{q'} - 1, j = \overline{1,n}$.

Lastly, if $1 < p < +\infty, 0 < q \leq 1$, we denote by $\overset{\approx}{\lambda}{}_\omega^{p,q}(U^n)$ the class of analytic f functions in U^n, for which

$$\|f\|_{\overset{\approx}{\lambda}{}_\omega^{p,q}(U^n)} = \sup_{r\in Q_n} \frac{(1-r)^{\alpha-\frac{1}{q}+1}}{\omega_\Pi^{\frac{1}{q}}(1-r)} \left(\int_{T^n} \left|D^{\alpha+1} g(rz)\right|^{p'} d\sigma_n(z)\right)^{\frac{1}{p'}} < +\infty,$$

where $\alpha_j > \dfrac{\alpha_{\omega_j}+1}{q} - 2,\ j = \overline{1,n}$.

We also introduce the following notation

$$\Lambda_\omega^{p,q}(U^n) = \begin{cases} \lambda_\omega^{p,q}(U^n), & 0 < p, q \leq 1; \\ \widetilde{\lambda}_\omega^{p,q}(U^n), & 0 < p \leq 1,\ 1 < q < +\infty; \\ \overset{\approx}{\lambda}{}_\omega^{p,q}(U^n), & 1 < p < +\infty,\ 0 < q \leq 1. \end{cases}$$

It is well known that if $\min(p,q) < 1$, then any continuous functional on $L_\omega^{p,q}(U^n)$ is identically zero. In the case of analytic functions, this assertion is not true. A continuous linear functional in these spaces is the value of the function $f \in A_\omega^{p,q}(U^n)$ at $\Phi_{z_0}(f) = f(z_0)$, $z_0 \in U^n$, and the moreover following is true:

Theorem 8. *Let Φ be continuous linear functional on $A_\omega^{p,q}(U^n)$, p, q belong $(0,1]$ or one of the parameters in the interval $(0,1]$ and the other - the interval $(1;+\infty)$; $\omega = (\omega_1, ..., \omega_n)$, $\omega_j \in \Omega, j = \overline{1,n}$, and $g(z) = \Phi(e_z)$, $e_z(\zeta)$ defined by the formula (2).*

Then $g \in \Lambda_\omega^{p,q}(U^n)$, the functional Φ has the form (3), and the estimates

$$c_1(p,\alpha)\,\|g\|_{\Lambda_\omega^{p,q}(U^n)} \leq \|\Phi\| \leq c_2(p,\alpha)\,\|g\|_{\Lambda_\omega^{p,q}(U^n)}, \tag{14}$$

are valid.

The converse is also valid: any $g \in \Lambda_\omega^{p,q}(U^n)$ according (3) generates a continuous linear functional on $A_\omega^{p,q}(U^n)$, for which the estimates (14) are valid.

We next consider the mixed norm classes of Hardy-Sobolev type $F^{p,q}_{\alpha}(U^n)$, for $0 < p, q < +\infty$, $-1 < \alpha < +\infty$, analytic in the polydisc U^n functions f , such that

$$\|f\|^p_{F^{p,q}_{\alpha}(U^n)} = \int\limits_{T^n} \left(\int\limits_{I^n} |f(r\zeta)|^q (1-r)^{\alpha} r dr \right)^{\frac{p}{q}} d\sigma_n(\zeta) < +\infty,$$

where $I^n = [0,1]^n$.

Note that $F^{p,2}_{2\alpha-1}(U) = H^p_{\alpha}(U)$, $U = U^1$, $0 < p < +\infty$, and $F^{p,p}_{\alpha}(U^n) = A^p_{\alpha}(U^n)$, $0 < p < +\infty$, (see [26]), where $H^p_{\alpha}(U)$, $A^p_{\alpha}(U^n)$ there are well-known analytic classes of Hardy-Sobolev and Bergman-Djrbashian in the unit disc U and polydisc U^n respectively (see [30]). Furthermore, before R^n various authors studied in detail some analogues of $F^{p,q}_{\alpha}(U^n)$ classes is the $F^{p,q}_{\alpha}(R^n)$ classes (see [27]).

Note of [31]-[36], which, in fact, the beginning of research properties of analytic functions of the $F^{p,q}_{\alpha}$ class in the unit disc U, in the ball B^n, polydisk U^n and smoothness analogues.

Let also $F^{p,\infty}_{\alpha}(U^n)$, $0 < p \leq 1$, $-1 < \alpha < +\infty$, the class of analytic in U^n functions f, for which

$$\left(\int\limits_{T^n} \left(\sup_{r \in I^n} \left| D^{\alpha+1} \left(D^1_{\alpha} f(r\zeta) \right) \right| (1-r) \right)^p d\sigma_n(\zeta) \right)^{\frac{1}{p}} < +\infty,$$

where for $f \in H(U^n):\ f(z) = \sum\limits_{k=0}^{+\infty} a_k z^k$, $D^1_{\alpha} f(z) = \sum\limits_{k=0}^{+\infty} (\alpha + k + 2) a_k z^k$.

Finally, let $S^{p,q}_{\alpha,\beta}(U^n)$, $0 < p \leq 1$, $0 < q < +\infty$, $-1 < \alpha < +\infty$, the class of analytic functions f in U^n, for which

$$\|f\|_{S^{p,q}_{\alpha,\beta}(U^n)} = \sup_{z \in U^n} \left(\left| D^{\beta+1} f(z) \right| (1-|z|)^{\beta+2-\frac{\alpha+1}{q}-\frac{1}{q}} \right) < +\infty,$$

where $\beta > \dfrac{\alpha+1}{q} + \dfrac{1}{p}$.

It is easy to see that relative to these quasinorms of the space $S^{p,q}_{\alpha,\beta}(U^n)$ is a extending Banach space.

In [37] the author describes the conjugate spaces of the $F^{p,q}_{\alpha}(U^n)$ spaces under some restrictions on p, q, known results about the representation functionals in classes $F^{p,p}_{\alpha}(U^n)$. The description of duals spaces $F^{p,q}_{\alpha}(U^n)$ for $p = q$, $0 < q < +\infty$, and their numerous applications are well known (see, eg, [30] and [38]). However, the question of description of continuous linear functionals even in the case of the unit disc for these mixed norm classes remained open for other p and q.

In [37] the following result was, we will recall the main steps of the proof.

Theorem 9. *Let Φ be continuous linear functional on $F^{p,q}_{\alpha}(U^n)$, $0 < p < 1 < q < +\infty$, or $0 < p, q \leq 1$, $g(z) = \Phi(e_z)$, where $e_z(\zeta)$ defined by the formula (2).*

Then $g \in S^{p,q}_{\alpha,\beta}(U^n)$, and the functional Φ has the form (3).

The converse is also valid: any $g \in S^{p,q}_{\alpha,\beta}(U^n)$ according (3) generates a continuous linear functional on $F^{p,q}_{\alpha}(U^n)$, more addition, the following estimates

$$c_1(\alpha, p, q) \|\Phi\| \leq \|g\|_{S^{p,q}_{\alpha,\beta}(U^n)} \leq c_2(\alpha, p, q) \|\Phi\| . \tag{15}$$

are valid.

Proof of theorem 9. We provide the complete sketch of the proof below.

Let Φ be continuous linear functional on $F^{p,q}_{\alpha}(U^n)$, $0<p<1<q<+\infty$.

Note that from Holder's inequality we have

$$\|g\|_{F^{p,q}_{\alpha}(U^n)} \le c\,\|g\|_{F^{q,q}_{\alpha}(U^n)}\,,\ 0<p<1<q<\infty. \tag{16}$$

Therefore, taking into account equality $e_z(\zeta)=\sum\limits_{|k|\ge 0}^{+\infty}(z\zeta)^k$ and that this series converges to the function $e_z(\zeta)$ in the space $F^{q,q}_{\alpha}(U^n)$, and for all $z\in U^n$ get $g(z)=\Phi(e_z)=\sum\limits_{|k|\ge 0}^{+\infty}\Phi(\delta_k)z^k$, where $\delta_k=z_1^{k_1}...z_n^{k_n}$, $(k_1,...,k_n)\in \mathrm{Z}^n_+$. So $g\in H(U^n)$.

Next, using the evaluation [37, p. 676], we obtain

$$\left|D^{\beta+1}g(z)\right| = \left|\Phi(D^{\beta+1}e_z(\zeta))\right| \le \|\Phi\|\,\left\|D^{\beta+1}e_z\right\|_{F^{p,q}_{\alpha}(U^n)} \le \frac{c\,\|\Phi\|}{(1-|z|)^{\beta+2-\frac{\alpha+1}{q}-\frac{1}{p}}},$$

for $\beta>\dfrac{\alpha+1}{q}+\dfrac{1}{p}-2$.

Therefore, $g\in S^{p,q}_{\alpha,\beta}(U^n)$, and $\|g\|_{S^{p,q}_{\alpha,\beta}}\le c\,\|\Phi\|$.

To prove the representation of the functional (2) it is sufficient to take into account that (see [34])

$$\|f_\rho - f\|_{F^{p,q}_{\alpha}(U^n)} \to 0, \rho\to 1-0, \tag{17}$$

to bring equality $\Phi(f_\rho)=\sum\limits_{|k|\ge 0}^{\infty}a_k\rho^k\Phi(\delta_k)$, $\rho^k=\rho_1^{k_1}...\rho_n^{k_n}$, $f(z)=\sum\limits_{|k|\ge 0}^{\infty}a_kz^k$, and use standard arguments (see, for example, theorem 6.9 в [16]).

Let us prove the second part of theorem 9 and the right estimate contained in it. We assume that (see, in detail, [37, c. 681])

$$|\Phi(f)| = \left|\lim_{\rho\to 1-0}\int\limits_{T^n} f(\rho\zeta)g(\rho\bar{\zeta})d\sigma_n(\zeta)\right| \le$$

$$\le c\left|\lim_{\rho\to 1-0}\int\limits_{T^n}\int\limits_{I^n}\int\limits_{I^n} f_\rho\left(D^{\beta+\alpha+1}g_\rho\right)(1-r^2)^{\alpha}(1-R^2)^{\beta-1}rR^{2\alpha+3}drdRd\sigma_n(\xi)\right|,$$

where $f_\rho := f_\rho(Rr\bar{\xi})$, $g_\rho := g_\rho(Rr\bar{\xi})$, $f_\rho(z)=f(\rho z)$, $\rho\in I$.

Next, using Holder's inequality, we have

$$|\Phi(f)| = \left|\lim_{\rho\to 1-0}\int\limits_{T^n} f(\rho\zeta)g(\rho\bar{\zeta})d\sigma_n(\zeta)\right| \le$$

$$\le c_1(\alpha,\beta)\sup_{R\in I^n}\left(\int\limits_{I^n}\left(M_\infty\left(D^{\beta+\alpha+1}g_R,r\right)\right)^{q'}(1-r)^{\alpha}dr\right)^{\frac{1}{q'}}(1-R)^{\beta+\alpha-\frac{1}{p}}$$

$$\times\int\limits_{T^n}\int\limits_{I^n}\left(\int\limits_{I^n}|f(Rr\xi)|^q(1-r)^l dr\right)^{\frac{1}{q}}(1-R)^{\frac{1}{p}-2}RdRd\sigma_n(\xi),$$

where $M_\infty(f,r)=\sup\limits_{|z_j|=r_j}|f(z)|,\quad r\in I^n, j=\overline{1,n}$.

In the paper [37] the author established estimate

$$\left(M_\infty\left(D^{\beta+\alpha+1}g_R,r\right)\right)^{q'}\le\left(M_\infty\left(D^{\beta+\alpha+1}D^{-\gamma}g_R,r\right)\right)^{q'}(1-rR)^{-\gamma q'},$$

where $\frac{\alpha+1}{q'}<\gamma<\alpha+2,\beta>\frac{\alpha+1}{q}+\frac{1}{p},\alpha>-1$. Therefore,

$$|\Phi(f)|\le c_2(\alpha,\beta)\tilde{S}\times$$

$$\times\sup_{R\in I^n}\left(\int\limits_{I^n}\left(M_\infty\left(D^{\beta+l+1}D^{-\gamma}g_R,r\right)\right)^{q'}(1-Rr)^{-\gamma q'}(1-r)^\alpha dr\right)^{\frac{1}{q'}}(1-R)^{\beta+1-\frac{1}{p}}.$$

Further, given that $M_\infty(G,Rr)\le M_\infty(G,R)$, $G\in H(U^n)$, $r,R\in I^n$, let $\tilde{M}_\infty:=M_\infty(D^{\beta+\alpha+1}D^{-\gamma}g,R)$, we write

$$|\Phi(f)|\le c\tilde{S}\sup_{R\in I^n}\left(\tilde{M}_\infty\left(\int\limits_{I^n}(1-Rr)^{-\gamma q'}(1-r)^\alpha dr\right)^{\frac{1}{q'}}(1-R)^{\beta+1-\frac{1}{p}}\right).$$

Using the elementary estimate

$$\int\limits_0^1(1-Rr)^{-l}(1-r)^t dr\le c(l,t)(1-R)^{-l+t+1},\ t>-1,\ l>t+1,\ R\in(0,1),$$

we get for $\frac{\alpha+1}{q'}<\gamma<\alpha+2:$

$$|\Phi(f)|\le c(\alpha,\beta)\tilde{S}\sup_{R\in I^n}M_\infty\left(D^{\beta+\alpha+1}D^{-\gamma}g,R\right)(1-R)^{\beta+1-\frac{1}{p}-\gamma+\frac{\alpha+1}{q'}}.$$

Given that

$$\sup_{R\in I^n}M_\infty\left(D^{\beta+\alpha+1}D^{-\gamma}g,R\right)(1-R)^{\beta+1-\frac{1}{p}-\gamma+\frac{\alpha+1}{q'}}\cong$$

$$\cong\sup_{R\in I^n}M_\infty\left(D^{\beta+1}g,R\right)(1-R)^{\beta+2-\frac{1}{p}-\frac{\alpha+1}{q'}},$$

for $\beta>\frac{1}{p}+\frac{l+1}{q}$, $\gamma\in\left(\frac{l+1}{q'},l+2\right)$ (see, in detail, [37]), we obtain

$$|\Phi(f)|\le c(\alpha,\beta)\left\|g\right\|_{S^{p,q}_{\alpha,\beta}(U^n)}\tilde{S}. \tag{18}$$

We now show that the inequality

$$\tilde{S}\le c\left\|f\right\|_{F^{p,q}_\alpha(U^n)},\ 0<p<1<q<\infty, \tag{19}$$

are valid. For this we use the proposition set forth in [39]:

Let X be the set measurable functions G in $[0,1]$, for which

$$\left(\int\limits_0^1(G(r))^q(1-r)^\alpha dr\right)^{\frac{1}{q}}<+\infty,\ 1<q<+\infty,\ -1<\alpha<+\infty.$$

The function $f(Rr\xi)=f(zr)=f_z(r)$ is a X -valued measure,

$$\mu(z)=(1-|z|)^{\frac{1}{p}-2}dm(z),\quad 0<p<1,$$

and satisfies the condition of proposition [39], we have

$$\int_0^1\int_0^{2\pi}\left(\int_0^1 |f(Rr\xi)|^q (1-r)^\alpha dr\right)^{\frac{1}{q}} (1-R)^{\frac{1}{p}-2} Rd\theta dR$$

$$\leq c(\alpha,p,q)\left(\int_T\left(\int_0^1 |f(r\xi)|^q (1-r)^\alpha dr\right)^{\frac{p}{q}} d\sigma(\xi)\right).$$

Thus, inequality (17) for $n=1$ is established.

Let X be the set measurable functions G in I^n, for which

$$\left(\int_{I^n} (G(r))^q (1-r)^\alpha d\rho\right)^{\frac{1}{q}} < +\infty,\ 1<q<+\infty,\ -1<\alpha<+\infty.$$

Applying induction and reasoning as above, we obtain for any n (see, in detail, [37]):

$$\int_{U^n}\left(\int_{I^n} |f(Rr\xi)|^q (1-r)^\alpha dr\right)^{\frac{1}{q}} (1-R)^{\frac{1}{p}-2} RdRd\sigma_n(\zeta) \leq c\,\|f\|_{F^{p,q}_\alpha(U^n)}.$$

From (17) and (19) we have the estimate $\|\Phi\| \leq c\,\|g\|_{S^{p,q}_{\alpha,\beta}(U^n)}$, i.e. the validity of theorem 9 when $0<p<1<q<+\infty$.

Consider the case $0<p,q\leq 1$.

The proof of the first part of the statement and the left estimate easily obtained by repeating the relevant argument provided in the proof of $0<p<1<q<+\infty$ case. Next, using analogs of Littlewood - Paley equality proved in [37, s. 674], we obtain for $\beta > \dfrac{\alpha+1}{q}+\dfrac{1}{p}$:

$$\left|\int_{T^n} f(\rho\zeta)g(\rho\bar{\zeta})d\sigma_n(\zeta)\right| \leq \frac{c(\beta,p)}{\rho^{2(\beta+1)}}\int_{U^n}\left|D^{\beta+1}g(\bar{\xi})\right|(1-|\xi|)^\beta |f(\xi)|\,dm_{2n}(\xi)$$

$$\leq \frac{c(\beta,p)}{\rho^{2(\beta+1)}}\,\|g\|_{S^{p,q}_{\alpha,\beta}(U^n)}\,\|f\|_{F^{p,q}_\alpha(U^n)}.$$

Hence it is easy to deduce that any $g \in S^{p,q}_{\alpha,\beta}(U^n)$ function by the formula (3) generates a continuous linear functional on $F^{p,q}_\alpha(U^n)$ and the right estimate in (15) is valid. Theorem 9 is proved completely.

Theorem 10. *Let Φ be continuous linear functional on $F^{p,1}_\alpha(U^n)$, $1<p<+\infty$, $-1<\alpha<+\infty$, $g(z)=\Phi(e_z)$,$g(z)=\Phi(e_z)$, $e_z(\zeta)$ defined by the formula (2).*

Then $g \in F^{p',\infty}_\alpha(U^n)$, and the functional Φ has the form (3).

The converse is also valid: any $g \in F^{p',\infty}_\alpha(U^n)$ according (3) generates a continuous linear functional on $F^{p,1}_\alpha(U^n)$, more addition, the following estimates

$$c_1(\alpha,p)\,\|g\|_{F^{p',\infty}_\alpha(U^n)} \leq \|\Phi\| \leq c_2(\alpha,p)\,\|g\|_{F^{p',\infty}_\alpha(U^n)}, \tag{20}$$

are valid.

Proof of theorem 10. We provide the complete sketch of the proof below.

We first prove the second assertion of the theorem and the right estimate in (20). Given the inequality (see, for details, [37])

$$|\Phi(f)| = \left| \lim_{\rho\to 1-0} \int\limits_{T^n} f(\rho\zeta) g(\rho\bar{\zeta}) d\sigma_n(\zeta) \right|$$

$$\leq c(\alpha) \int\limits_{U^n} \left| D^{\alpha+1} D^1_\alpha g(\xi) \right| |f(\xi)| (1-|\xi|)^{\alpha+1} dm_{2n}(\xi), \alpha > -1,$$

and applying to the right side of the Holder inequality, we obtain

$$|\Phi(f)| \leq c(\alpha) \left(\int\limits_{T^n} \left(\sup_{|\xi|\in I^n} \left| D^{\alpha+1} D^1_\alpha g(\xi) \right| (1-|\xi|) \right)^{p'} d\sigma_n(\zeta) \right)^{\frac{1}{p'}}$$

$$\times \left(\int\limits_{T^n} \left(\int\limits_{I^n} |f(\xi)| (1-|\xi|)^\alpha d|\xi| \right)^p d\sigma_n(\zeta) \right)^{\frac{1}{p}} \leq c(\alpha,p) \|g\|_{F^{p',\infty}_\alpha(U^n)} \|f\|_{F^{p,1}_\alpha(U^n)}.$$

This implies the second assertion of the theorem and the right estimate in (20).

We now prove the converse theorem and left in the estimate (20). Note that

$$\left\| D^{-\alpha-1} \right\|_{H^p(U^n)} \leq c(\alpha,p) \int_{T^n} \left(\int_{I^n} |g(R\xi)| (1-R)^\alpha dR \right)^p d\sigma_n(\xi),$$

for $0 < p < +\infty$, $-1 < \alpha < +\infty$.

Consequently, the functional Φ can be extended to space

$$H^p_{-\alpha-1}(U^n) = \left\{ g \in H(U^n) : \left\| D^{-\alpha-1} g \right\|_{H^p(U^n)} < +\infty \right\}$$

continuous manner. Further, considering the form of a continuous linear functional classes $H^p(U^n)$, we can derive the representation

$$\Phi(f) = \lim_{\rho\to 1-0} \int\limits_{T^n} f(\rho\zeta) g(\rho\bar{\zeta}) d\sigma_n(\zeta),\ f \in F^{p,1}_\alpha(U^n),\ \left\| D^{\alpha+1} g \right\|_{H^{p'}(U^n)} < +\infty.$$

Note that

$$\|g\|_{F^{p,1}_\alpha(U^n)} \leq c \|g\|_{F^{p,p}_{\alpha p}(U^n)},\ p \in (1,+\infty). \tag{21}$$

Therefore, given that $e_z(\zeta) = \sum\limits_{|k|\geq 0} (z\zeta)^k$, and the series converges to the function $e_z(\zeta)$ in the space $F^{p,p}_{\alpha p}(U^n)$, $1 < p < +\infty$, and therefore, by (21), and in $F^{p,1}_{\alpha p}(U^n)$, for all $z \in U^n$, we obtain $g(z) = \Phi(e_z) = \sum\limits_{|k|\geq 0} \Phi(\delta_k) z^k$.

Consequently, $g(z) \in H(U^n)$.

Further, using the estimates

$$\int\limits_{T^n} \left(\sup_{r\in I^n} \left| D^1_{\alpha-1} f(r\xi) \right| (1-r) \right)^q d\sigma_n(\xi) \leq c(\alpha,q) \|f\|^q_{H^q(U^n)},$$

$f \in H^q(U^n)$, $0 < q \leq +\infty$, and

$$\int\limits_{T^n} \left| \int\limits_{T^n} \frac{G(\xi) d\sigma_n(\xi)}{1-\bar{\xi}z} \right|^p d\sigma_n(z) \leq c(p) \int\limits_{T^n} |G(t)|^p d\sigma_n(t),$$

where $\|G\|_{L^p(T^n)} < +\infty$, $1 < p < +\infty$, and where the inner integral is understood in the sense of the principal value, we finally obtain ($z = Rt$):

$$\left|D^{\alpha+1}g(z)\right| = \left|\Phi\left(D^{\alpha+1}e_z(\zeta)\right)\right| = \left|\lim_{r\to 1}\int_{T^n}\frac{D^{\alpha+1}G(r\xi)}{(1-Rrt\bar{\xi})}d\sigma_n(\zeta)\right|,$$

$$\left(\int_{T^n}\left(\sup_{|z|\in I^n}\left|D^{\alpha+1}D^1_\alpha g(z)\right|(1-|z|)\right)^{p'}d\sigma_n(\xi)\right)^{\frac{1}{p'}} \le c(\alpha)\left\|D^{\alpha+1}g\right\|_{H^{p'}(U^n)}$$

$$\le c(\alpha)\sup_{R\in I}\lim_{r\to 1}\left(\int_{T^n}\left|D^{\alpha+1}G(Rr\xi)\right|^{p'}d\sigma_n(\xi)\right)^{\frac{1}{p'}} \le c(\alpha)\left\|D^{\alpha+1}G\right\|_{H^{p'}(U^n)} < +\infty.$$

To get an representation (2) for the functional Φ it is enough to note

$$\|g\|_{F^{p,1}_\alpha(U^n)} \le c\,\|g\|_{F^{p,p}_{p\alpha}(U^n)}, \quad 1 < p < +\infty,$$

the (16) hold by standard arguments (see, eg, [35], theorem 6.9). Theorem 10 proved.
Remark 6. Note, duality theorems for analytic Lizorkin-Triebel type spaces in unit ball, where given in papers of J. Ortega and J. Fabrega, our theorems can be considered as direct analogues of these results in case of polydisk. We refer the reader to [32]-[34] for these interesting results and other results on $F^{p,q}_\alpha$ analytic Lizorkin-Triebel type spaces in higher dimension.
Remark 7. Approaches used in [37] can be used also to describe bounded linear functionals in analytic spaces with quazinorms like

$$\int_0^1\cdots\int_0^1\left(\int_{|z_1|<r_1}\cdots\int_{|z_n|<r_n}|f(z_1,\dots,z_n)|^q(1-|z|)^t dm_{2n}(z)\right)^{p/q}(1-r)^\beta r dr$$

for various p and q, or

$$\int_{U^n}\left(\int_0^{r_1}\cdots\int_0^{r_n}|f(z)|^q(1-|z|)^t d|z|\right)^{p/q}(1-r)^\beta dm_{2n}(z)$$

for various p and q.

For unit disk case see [5]. These results were gived in [40].

3. On representation of continuous linear functionals in weighted spaces in a unit ball and unit polyball

In this section we provide a complete description of continuous linear functionals in weight spaces with mixed norm of analytic functions in the unit ball and polyball.

Let

$$B_n = \{z \in \mathbb{C}^n : |z| < 1\} -$$

be a unit ball в n-dimension complex plane $\mathbb{C}^n$, S_n be the boundary of B_n.

Let also $L^{p,q}_{\omega}(B_n)$, $\omega\in\Omega^n$, $0<p,q<+\infty,$ be the set of measurable functions f in B_n, for which

$$\|f\|_{L^{p,q}_{\omega}(B_n)}=\left(\int\limits_0^1\omega\,(1-r)\left(\int\limits_{S_n}|f(rz)|^p\,d\sigma_n(z)\right)^{\frac{q}{p}}r^{2n-1}dr\right)^{\frac{1}{q}}<+\infty.$$

We denote by $H(B_n)$ the class of analytic functions f in B^n; $A^{p,q}_{\omega}(B_n)=H(B_n)\bigcap L^{p,q}_{\omega}(B_n)$.

In the paper O. E. Antonenkova, F. A. Shamoyan [41] in terms of Cauchy transform a complete description of continuous linear functionals in the space $A^{p,q}_{\omega}(B_n)$ for all $0<p,q<+\infty$ was obtained.

Note that in [42] a different representation of linear continuous functional in the space $A^{p,q}_{\omega}(B_n)$ for $\omega(t)=t^{\beta}$, $\beta>-1$, if $1<p<+\infty,$ $\max(1,1+\beta)<q<+\infty$ was given. For the remaining p,q proposed in this paper method does not work.

Suppose, as above, $\dfrac{1}{q}+\dfrac{1}{q'}=1$, $\dfrac{1}{p}+\dfrac{1}{p'}=1$.

If $0<p,q\le 1$, we denote by $\lambda^{p,q}_{\omega}(B_n)$ the class of analytic functions f in B_n, for which

$$\|f\|_{\lambda^{p,q}_{\omega}(U^n)}=\sup_{z\in B_n}\frac{|D^{\alpha+1}f(z)|(1-|z|)^{\alpha-n(\frac{1}{p}-1)-\frac{1}{q}+1}}{\omega^{\frac{1}{q}}(1-|z|)}<+\infty,$$

where $\alpha>\dfrac{\alpha_{\omega}+1}{q}+n(\dfrac{1}{p}-1)-1$.

If $0<p\le 1$, $1<q<+\infty$, we denote $\widetilde{\lambda}^{p,q}_{\omega}(B_n)$ the class of analytic functions f in B_n, for which

$$\|f\|_{\widetilde{\lambda}^{p,q}_{\omega}(B_n)}=\left(\int\limits_0^1\frac{(1-r)^{\alpha q'-nq'(\frac{1}{p}-1)}}{\omega^{\frac{q'}{q}}(1-r)}\left(\sup_{z\in S_n}|D^{\alpha+1}f(rz)|\right)^{q'}r^{2n-1}dr\right)^{\frac{1}{q'}}<+\infty,$$

where $\alpha>\dfrac{\alpha_{\omega}}{q}+n\left(\dfrac{1}{p}-1\right)-\dfrac{1}{q'}$.

If $1<p<+\infty$, $0<q\le 1$, we denote $\tilde{\tilde{\lambda}}^{p,q}_{\omega}(B_n)$ the class of analytic functions f in B_n, for which

$$\|f\|_{\tilde{\tilde{\lambda}}^{p,q}_{\omega}(B_n)}=\sup_{r\in(0;1]}\frac{(1-r)^{\alpha-\frac{1}{q}+1}}{\omega^{\frac{1}{q}}(1-r)}\left(\int\limits_{S^n}\left|D^{\alpha+1}f(rz)\right|^{p'}d\sigma_n(z)\right)^{\frac{1}{p'}}<+\infty,$$

where $\alpha>\dfrac{\alpha_{\omega}+1}{q}-2$.

It is easy to note that the definitions of these classes do not depend on α, while in relation to these norms above this space becomes a Banach space.

If $1<p,q<+\infty$, we denote $\widetilde{\lambda}^{p,q}_{\omega}(B_n)$ the class of analytic functions f in B_n, for which

$$\|f\|_{\tilde{\tilde{\lambda}}^{p,q}_{\omega}(B_n)}=\left\|D^{\alpha+1}f\right\|_{A^{p',q'}_{\omega_\alpha}(B_n)}.$$

Let $I=(0,1]$, $J=(1,+\infty)$, and introduce the following notation

$$\Lambda_\omega^{p,q}(B_n)=\begin{cases}\lambda_\omega^{p,q}(B_n), & p,\, g\in I;\\ \widetilde{\lambda}_\omega^{p,q}(B_n), & p\in I,\, g\in J;\\ \tilde{\tilde{\lambda}}_\omega^{p,q}(B_n), & p\in J,\, g\in I;\\ \tilde{\tilde{\tilde{\lambda}}}_\omega^{p,q}(B_n), & p,q\in J.\end{cases}$$

The following theorem holds:

Theorem 11. *Let $\omega\in\Omega$, $p,q\in I\bigcup J$, Φ be continuous linear functional on $A_\omega^{p,q}(B_n)$, $g\left(z\right)=\Phi\left(e_z\right)$, where $e_z(\zeta)=\frac{1}{(1-\langle z,\zeta\rangle)^n}$, $z,\zeta\in B_n$.*

Then $g\in\Lambda_\omega^{p,q}(B_n)$, and the functional Φ has the form

$$\Phi(f)=\lim_{\rho\to1-0}\frac{1}{2\pi}\int\limits_{S_n}f(\rho\zeta)\overline{g(\rho\zeta)}d\sigma_n(\zeta), \tag{22}$$

and estimates

$$c_1\left\|g\right\|_{\Lambda_\omega^{p,q}(B_n)}\le\left\|\Phi\right\|\le c_2\left\|g\right\|_{\Lambda_\omega^{p,q}(B_n)}, \tag{23}$$

are valid.

The converse is also valid: any $g\in\Lambda_\omega^{p,q}(B_n)$ according (22) generates a continuous linear functional on $A_\omega^{p,q}(B_n)$, for which estimates (23) are valid.

Remark 8. Various weighted spaces in the unit ball were studied in recent decades. We refer the reader for various other duality theorems for similar weighted mixed norm spaces in the unit ball and in the unit disk to the [43], where other types of weights were considered in mixed norm spaces.

Let B_n^m denote the polyball $B_n^m=B_n\times B_n\times...\times B_n$. Let also $S_n^m=S_n\times S_n\times...\times S_n$. As usual, we denote by $H(B_n^m)$ the space of all analytic functions in B_n^m by each variable separately.

In [44] the authors gives the results related to the description of bounded linear functionals in polyball. The results for n = 1 coincide with the results taken from [16].

We now introduce the mixed norm classes in polyballs

$$A_{\overrightarrow{\alpha}}^{\overrightarrow{p}}(B_n^m)=\{f\in H(B_n^m):\ \|f\|_{A_{\overrightarrow{\alpha}}^{\overrightarrow{p}}}:=(\int\limits_{B_n}(1-|z_m|)^{\alpha_m}(\int\limits_{B_n}(1-|z_{m-1}|)^{\alpha_{m-1}}$$

$$...\int\limits_{B_n}|f(z_1,...,z_m)|^{p_1}\,(1-|z_1|)^{\alpha_1}d\nu(z_1))^{\frac{p_2}{p_1}}...d\nu(z_{m-1}))^{\frac{p_m}{p_{m-1}}}d\nu(z_m))^{\frac{1}{p_m}}<+\infty\},$$

where $0<p_j<\infty$, $\alpha_j>-1$, $j=\overline{1,m}$.

Note that for $n=1$ these classes studied in [16]. For $m=1$ we have the classical Bergman spaces on the unit ball. Formally replacing B_n by R_n we arrive at well studied function classes in R^n (see [25], [26]).

It is not difficult to show that $A_{\overrightarrow{\alpha}}^{\overrightarrow{p}}$ is a Banach space for $1\le p_j<\infty$, $j=\overline{1,m}$. Moreover, it can be shown that $A_{\overrightarrow{\alpha}}^{\overrightarrow{p}}$ is a complete metric space for $0<p_j<1$, $j=\overline{1,m}$.

Theorem 12. *Let Φ be bounded linear functional on $A_{\overrightarrow{\alpha}}^{\overrightarrow{p}}$, $1<p_j<\infty$, $\alpha_j>-1$, $j=\overline{1,m}$, and $g(z)=\Phi(e_z)$,*

$$e_z = \prod_{j=1}^{m} \frac{1}{1 - z_j\overline{\zeta_j}}, z \in B_n^m. \tag{24}$$

Then $g \in H(B_n^m)$, $D^{\alpha+1}_{z_1,\dots,z_m} g \in A^{\vec{q}}_{\vec{\alpha}}$, $\dfrac{1}{p_j} + \dfrac{1}{q_j} = 1$, $j = \overline{1,m}$, *and*

$$\Phi(f) = \lim_{\rho\to 1-0} \int\limits_{S_n^m} f(\rho\zeta) g(\rho\zeta) d\zeta, \tag{25}$$

and estimates

$$c_1 \left\| D^{\alpha+1}_{z_1\dots z_m} g \right\|_{A^{\vec{q}}_{\vec{\alpha}}} \le \|\Phi\| \le c_2 \left\| D^{\alpha+1}_{z_1\dots z_m} g \right\|_{A^{\vec{q}}_{\vec{\alpha}}}, \tag{26}$$

are valid.

The reverse is also true: each g function so that $D^{\alpha+1} g \in A^{\vec{q}}_{\vec{\alpha}}$, by (25) produce a bounded linear functional on $A^{\vec{p}}_{\vec{\alpha}}$, $\alpha_j > -1, 1 < p_j < +\infty, j = \overline{1,m}$, for which estimate (26) holds.

Let $0 < p_j \le 1$, $j = \overline{1,m}$. We denote $\lambda^{\vec{p}}_{\vec{\alpha}}(B_n^m)$ the class of analytic functions f in B_n^m, for which

$$\left| D^{\tilde{\alpha}+1}_{z_1,\dots,z_n} g(z) \right| \le c \prod_{j=1}^{m} (1 - |z_j|)^{\frac{\alpha_j+n+1}{p_j} - \tilde{\alpha} - (n+1)}, z_j \in B_n,$$

where $\tilde{\alpha} > \dfrac{\alpha_j + n + 1}{p_j} - (n+1)$, $j = \overline{1,m}$.

It can be shown as in case of polydisc (see [16]) these spaces are independent from $\tilde{\alpha}$.

Theorem 13. *Let Φ be bounded linear functional on $A^{\vec{p}}_{\vec{\alpha}}$, $0 < p_j \le 1$, $\alpha_j > -1$, $j = \overline{1,m}$, and $g(z) = \Phi(e_z)$, e_z defined by the formula (24).*

Then $g \in \lambda^{\vec{q}}_{\vec{\alpha}}$, and the functional Φ has the form (25)and estimates

$$c_1 \|g\|_{\lambda^{\vec{q}}_{\vec{\alpha}}} \le \|\Phi\| \le c_2 \|g\|_{\lambda^{\vec{q}}_{\vec{\alpha}}}, \tag{27}$$

are valid.

The reverse is also true: each g function so that $g \in \lambda^{\vec{p}}_{\vec{\alpha}}$, by (25) produce a bounded linear functional on $A^{\vec{p}}_{\vec{\alpha}}$, $\alpha_j > -1, 0 < p_j < 1, j = \overline{1,m}$, for which estimate (27) holds.

4. On representation of continuous linear functionals in weighted analytic spaces in simply connected domain

In this section we give a proof of a recent result of the second author, reflecting the possibility of using integral representations of functions and action of Bergman-Djrbashian type integral operators in appropriate classes for solutions of the problems related with duality.

Questions description of continuous linear functionals on the weighted spaces of analytic functions are closely related to the construction of an bounded integral operator of the space under consideration of measurable functions in the corresponding space of analytic functions.

In the case of domains other than the unit disc, these problems solved, for example, in the articles P.H. Tatoyan [45], A.M. Shihvatov [46], [47], A.A. Solovyov [48], [49], J. Burbea [50], H. Hedenmalm [51].

For instance, A.A. Solovyov analyzed the issue of the existence of a bounded projection in the case of a simply connected domain whose boundary is piecewise smooth curve; A.M. Shihvatov is in areas with angles. The problem was solved with significant limitations for the angle.

In the paper H. Hedenmalm [51] found that the operator with Bergman kernel is bounded from the space of measurable functions without weight in the corresponding space of analytic functions for any simply connected domain with rectifiable boundary, when $p_0 < p < \dfrac{p_0}{p_0 - 1}$, where $p_0 \in \left[\dfrac{4}{3}; 2\right)$.

Let G – some simply connected bounded domain in the $\mathbb{C}$; $d(w, \partial G)$ be a distance from the point w to ∂G; suppose $\varphi : U \to G$ maps conformally, ψ reverse function for the φ.

Let $L^p(\alpha, G)$, $0 < p < +\infty, \alpha > -1$, be the class of measurable functions f in G, for which

$$\|f\|_{L^p(\alpha,G)} = \int\limits_G |f(z)|^p \, (1 - |\psi(z)|^2)^\alpha \, |\psi'(z)|^2 \, dm_2(z) < +\infty, \tag{28}$$

$A^p(\alpha, G)$ the set of analytic function f in G, for which (28) are valid.

In this section we obtain a description of the dual space of weighted spaces $A^p(\alpha, G)$, $1 < p < +\infty$, $\alpha \in Z_+$, $\alpha > -1$.

Theorem 14. *Let G simply connected bounded domain in the complex plane $\mathbb{C}$, whose boundary contains more than one point; $\varphi : U \to G$ conformally, $\varphi(0) = z_0$, $z_0 \in G$, $\varphi'(0) > 0$, ψ is a reverse function for the φ. Let also $1 < p < +\infty$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$; and $e_z(\zeta) = \dfrac{\alpha + 1}{\pi} \dfrac{1}{(1 - \psi(\zeta)\overline{\psi(z)})^{\alpha+2}}, \alpha > -1$.*

If $\Phi \in (A^p(\alpha, G))^*$, $g(z) = \overline{\Phi(e_z)}$, then $g \in A^q(\alpha, G)$, and the functional Φ has the form

$$\Phi(f) = \int\limits_G f(z)\overline{g(z)}(1 - |\psi(z)|^2)^\alpha \, |\psi'(z)|^2 \, dm_2(z), \tag{29}$$

and the estimates

$$c_1 \, \|g\|_{A^q(\alpha,G)} \le \|\Phi\| \le c_2 \, \|g\|_{A^q(\alpha,G)} \, . \tag{30}$$

are valid.

The converse is also valid: for any $g \in A^q(\alpha, G)$ according (29) generates a continuous linear functional Φ on $A^p(\alpha, G)$, for which the estimates (30) are valid.

Proof of theorem 14. We provide the complete sketch of the proof below.

Let Φ be continuous linear functional on $A^p(\alpha, G)$. We show that there exists a function $g \in A^q(\alpha, G)$, for which estimates (30) are valid.

By the Hahn-Banach theorem there exists Φ_1 the linear continuous functional on $L^p(\alpha, G)$, such that $\|\Phi\| = \|\Phi_1\|$, and $\Phi_1(f) = \Phi(f)$, if $f \in A^p(\alpha, G)$.

Further, by Riesz's theorem there exists a function $h \in L^q(\alpha, G)$, such that $\|h\|_{L^q(\alpha,G)} = \|\Phi_1\|$ and

$$\Phi_1(f) = \int\limits_G f(z)\overline{h(z)}(1 - |\psi(z)|^2)^\alpha \, |\psi'(z)|^2 \, dm_2(z).$$

Clearly, $e_z \in A^p(\alpha, G), \ \forall z \in G$. We denote for $h \in L^q(\alpha, G)$

$$g(z) = \overline{\Phi(e_z)} = \frac{\alpha+1}{\pi} \int\limits_G \frac{(1-|\psi(\zeta)|^2)^\alpha}{(1-\overline{\psi(\zeta)}\psi(z))^{\alpha+2}} h(\zeta)\, |\psi'(\zeta)|^2 \, dm_2(\zeta).$$

In [52] was shown that the following Bergman-Djrbashian type integral operator

$$P_\beta(f)(z) = F(z) = \frac{\beta+1}{\pi} \int\limits_G \frac{(1-|\psi(\zeta)|^2)^\beta}{(1-\overline{\psi(\zeta)}\psi(z))^{\beta+2}} f(\zeta)\, |\psi'(\zeta)|^2 \, dm_2(\zeta) \tag{31}$$

continuously maps the space $L^p(\alpha, G)$ into $A^p(\alpha, G)$, space for $1 \le p < +\infty$, $\alpha > -1$, $\beta \ge \alpha$, for $1 < p < +\infty$, and there exists a constant $c(\alpha, p)$ such that the estimate

$$\|F\|_{A^p(\alpha,G)} \le c(\alpha, p) \, \|f\|_{L^p(\alpha,G)} \, ,$$

is valid.

Also note that if $f \in A^p(\alpha, G)$, $1 \le p < +\infty$, $\alpha > -1$, then $F(z) = f(z), \ z \in G$, (see, for example, [52], [523]).

Then $g \in A^q(\alpha, G)$, and $c_1 \|g\|_{A^q(\alpha,G)} \le \|h\|_{L^q(\alpha,G)}$, $\alpha > -1$.

But if $f \in A^p(\alpha, G)$, then

$$\int\limits_G f(z)\overline{g(z)}(1-|\psi(z)|^2)^\alpha \, |\psi'(z)|^2 \, dm_2(z) = \frac{\alpha+1}{\pi}$$

$$\times \int\limits_G \overline{h(\zeta)}(1-|\psi(\zeta)|^2)^\alpha \, |\psi'(\zeta)|^2 \int\limits_G \frac{(1-|\psi(z)|^2)^\alpha}{(1-\overline{\psi(z)}\psi(\zeta))^{\alpha+2}} f(z)\, |\psi'(z)|^2 \, dm_2(z) dm_2(\zeta)$$

$$= \int\limits_G f(\zeta)\overline{h(\zeta)}(1-|\psi(\zeta)|^2)^\alpha \, |\psi'(\zeta)|^2 \, dm_2(\zeta) = \Phi_1(f) = \Phi(f).$$

We use the integral representation of the for a type (31) function from the $A^p(\alpha, G)$ class.

Hence, based on the above considerations, namely $\|\Phi\| = \|\Phi_1\|$, $\|h\|_{L^q(\alpha,G)} = \|\Phi_1\|$, and $c_1 \|g\|_{A^q(\alpha,G)} \le \|h\|_{L^q(\alpha,G)}$, we obtain $c_1 \|g\|_{A^q(\alpha,G)} \le \|h\| = \|\Phi\|$.

To prove the right inequality (30) we first show the converse proposition of the theorem.

Let $g(z)$ be an arbitrary function of $A^q(\alpha, G)$ and (29) are valid.

We show that Φ is a continuous linear functional on $A^p(\alpha, G)$, for which the estimates (30) are valid.

We apply Holder's inequality, we have

$$|\Phi(f)| \le \left(\int\limits_G |f(z)|^p \, (1-|\psi(z)|^2)^\alpha \, |\psi'(z)|^2 \, dm_2(z) \right)^{\frac{1}{p}}$$

$$\times \left(\int\limits_G |g(z)|^q \, (1-|\psi(z)|^2)^\alpha \, |\psi'(z)|^2 \, dm_2(z) \right)^{\frac{1}{q}} .$$

From where we have $\|\Phi\| \le c_2 \, \|g\|_{A^q(\alpha,G)}$, i.e. Φ is a continuous linear functional on $A^p(\alpha, G)$.

In addition, again using the integral representation (31) for functions of the $A^p(\alpha, G)$ class, we obtain

$$\Phi(e_z) = \frac{\alpha+1}{\pi} \int\limits_G \frac{(1-|\psi(\zeta)|^2)^\alpha}{(1-\overline{\psi(\zeta)}\psi(z))^{\alpha+2}} h(\zeta) \left|\psi'(\zeta)\right|^2 dm_2(\zeta) = \overline{g(z)}.$$

Considering now the first part of the proof of the theorem, we get $c_1 \left\|g\right\|_{A^q(\alpha,G)} \leq \left\|\Phi\right\|$. The uniqueness of the representation (29) is clear. Theorem 14 is proved.
Remark 9. It is an interesting, open problem to extend these results to the case of same function spaces, but on product of domains, the machinery we used above, can be also used in case of such type mixed norm spaces of Bergman type spaces in simple connected domains.

5. Final remarks and some lines of development

New results on Bergman type projections often give as application various theorems on duality in analytic function spaces (see, for example, [5] and references there also). In theorems of this review they also play an important role. We add in this section some new theorems on projections for mixed norm spaces in tubular domains and pseudoconvex domains, hoping they will find applications also for new duality results.

For formulation of our results will need a series of standard notations and definitions for analytic spaces in tubular domains over symmetric cones and strongly pseudoconvex domains with smooth boundary in C^n.

Let $T_\Omega = V + i\Omega$ be the tube domain over an irreducible symmetric cone Ω in the complexification $V^{\mathbb{C}}$ of an n-dimensional Euclidean space $\widetilde{V}$. We denote the rank of the cone Ω by r and by Δ the determinant function on $\widetilde{V}$. Letting $\widetilde{V} = \mathbb{R}^n$, we have as an example of a symmetric cone on $\mathbb{R}^n$ the forward light cone Λ_n defined for $n \geq 3$ by

$$\Lambda_n = \{y \in \mathbb{R}^n : y_1^2 - \cdots - y_n^2 > 0, y_1 > 0\}.$$

Light cones have rank 2. The determinant function in this case is given by the Lorentz form

$$\Delta(y) = y_1^2 - \cdots - y_n^2.$$

$H(T_\Omega)$ denotes the space of all holomorphic functions on T_Ω. We denote m cartesian products of tubes by T_Ω^m, $T_\Omega^m = T_\Omega \times ... \times T_\Omega$. The space of all analytic functions on this new product domain which are analytic by each variable separately will be denoted by $H(T_\Omega^m)$. In this paper we will be interested on properties of certain analytic subspaces of $H(T_\Omega^m)$. By m we denote below a natural number, $m > 1$.

For $\tau \in \mathbb{R}_+$ and the associated determinant function $\Delta(x)$ we set

$$A_\tau^\infty(T_\Omega) = \left\{ F \in H(T_\Omega) : \|F\|_{A_\tau^\infty} = \sup_{x+iy \in T_\Omega} |F(x+iy)| \Delta^\tau(y) < +\infty \right\}. \tag{32}$$

It can be checked that this is a Banach space.

For $1 \le p, q < +\infty$ and $\nu \in \mathbb{R}$, $\nu > \dfrac{n}{r} - 1$, we denote by $A^{p,q}_{\nu}(T_\Omega)$ the mixed-norm weighted Bergman space consisting of analytic functions f in T_Ω that

$$\|F\|_{A^{p,q}_{\nu}} = \left(\int\limits_{\Omega} \left(\int\limits_{\widetilde{V}} |F(x+iy)|^p dx \right)^{q/p} \Delta^{\nu}(y) \frac{dy}{\Delta(y)^{n/r}} \right)^{1/q} < +\infty.$$

We put $A^{p,p}_{\nu} = A^{p}_{\nu}(T_\Omega),\ 1 \le p \le +\infty$.

This is a Banach space. Replacing above simply A by L we will get as usual the corresponding larger space of all measurable functions in tube over symmetric cone with the same quazinorm (see, for example, [54]).

To define related two Bergman-type spaces $A^{p}_{\nu}(T^{m}_{\Omega})$ and $A^{\infty}_{\tau}(T^{m}_{\Omega})$ (ν and τ can be also vectors) in m-products of tube domains T^{m}_{Ω} we follow standard procedure which is well-known in case of unit disk and unit ball (see, i.a., [55]). Namely we consider analytic F functions $F = F(z_1, \dots z_m)$ which are analytic by each $z_j, j = \overline{1,m}$, variable, and where each such variable belongs to T_Ω tube. For example we set, for all $z_j = x_j + y_j$, $F(z) = F(z_1, \dots, z_m)$, $\tau = (\tau_1, \dots, \tau_m)$, $\tau_j \in R$, $j = \overline{1,m}$.

$$A^{\infty}_{\tau}(T^{m}_{\Omega}) = \left\{ F \in H(T^{m}_{\Omega}) : \|F\|_{A^{\infty}_{\tau}} = \sup_{x+iy \in T^{m}_{\Omega}} |F(x+iy)| \Delta^{\tau}(y) < \infty \right\}, \tag{33}$$

$$|F(x+iy)| = |F(x_1 + iy_1, ..., x_m + iy_m)|,$$

where $\Delta^{\tau}(y)$ is a product of m-onedimensional $\Delta^{\tau_j}(y_j)$ functions,$= \overline{1,m}$..

Similarly the Bergman space A^{p}_{τ} can be defined on products of tubes T^{m}_{Ω} for all $\tau = (\tau_1, \dots, \tau_m)$, $\tau_j > \dfrac{n}{r} - 1$, $j = \overline{1,m}$.. It can be shown that all spaces are Banach spaces. Replacing simply A by L we will get as usual the corresponding larger space of all measurable functions in products of tubes over symmetric cone with the same quazinorm.

Further we will use the standard notation (see, i.a., [56])

$$dV_{\nu}(w) = \Delta^{\nu - \frac{n}{r}}(v) du dv, w = u + iv \in T_\Omega, z = x + iy \in T_\Omega.$$

We add some basic definitions for pseudoconvex domains.

Let k_Δ denote the Poincare distance on the unit disk $U \subset \mathbb{C}^n$. If X is a complex manifold, the Lempert function δ_X: $X \times X \to \mathbb{R}^+$ of X is defined by

$$\delta_X(z, w) = \inf\{k_\Delta(\zeta, \eta) | \exists \phi \in H(U) :\ U \ \to X;\ \phi(\zeta) = z,\ \phi(\eta) = w\},$$

for all $z, w \in X$.

We denote by D a bounded strictly pseudoconvex domain with smooth boundary $\delta(z) = dist(z, \partial D), dv(z)$ is a Lebeques measure on D, $dv_\beta(z)$ is a weighted such measure $dv_\beta = \delta^\beta dv$.

Let $K : D \times D \to \mathbb{C}$ will be the Bergman kernel of D, the K_t is a weighted kernel of type t. Note $K = K_{n+1}$.

We define new Banach mixed norm Bergman-type spaces in products of pseudoconvex domains and mixed norm Bergman-type spaces in products of tubular domains over symmetric cones as follows.

Let $m \ge 1, p_j \in (1;\infty)$, $\nu_j > \frac{n}{r} - 1$, $\quad \alpha_j > -1$, $j = \overline{1,m}$.

$$A^{\vec{p}}_{\vec{\nu}}(T^m_\Omega)=\Bigg\{f\in H(T^m_\Omega):\Bigg(\int\limits_{T_\Omega}...\Bigg(\int\limits_{T_\Omega}|f(z_1,...,z_m)|^{p_1}\Big[\Delta^{\nu_1-n/r}(y_1)\Big]dx_1dy_1\Bigg)^{\frac{p_2}{p_1}}$$

$$\dots\Big[\Delta^{\nu_m-n/r}(y_m)\Big]dx_mdy_m\Bigg)^{\frac{1}{p_m}}<+\infty\Bigg\}.$$

$$A^{\vec{p}}_{\vec{\alpha}}(D^m)=\Bigg\{f\in H(D^m):\Bigg(\int\limits_{D}...\Bigg(\int\limits_{D}|f(z_1,...,z_m)|^{p_1}\delta^{\alpha_1}(z_1)dv(z_1)\Bigg)^{\frac{p_2}{p_1}}$$

$$\dots\delta^{\alpha_m}(z_m)dv(z_m)\Bigg)^{\frac{1}{p_m}}<+\infty\Bigg\}.$$

Replacing A by L we get larger spaces of measurable functions with the same norms.

Theorem 15. *1) Let*

$$T_{\vec{\beta}}f(\vec{z})=\int\limits_{T^m_\Omega}\frac{f(w_1,...,w_m)dV_{\beta_1}(w_1)...dV_{\beta_m}(w_m)}{\Delta^{\beta_1+\frac{n}{r}}(\frac{z_1-\overline{w}_1}{i})...\Delta^{\beta_m+\frac{n}{r}}(\frac{z_m-\overline{w}_m}{i})},$$

$\vec{z}=(z_1,...,z_m)\in T^m_\Omega$. *Let* $\beta_j>\beta_0$, $j=\overline{1,m}$, *for some fixed large enough* β_0.

Then T_β *maps* $L^{\vec{p}}_{\vec{\nu}}(T^m_\Omega)$ *into* $A^{\vec{p}}_{\vec{\nu}}(T^m_\Omega), p_j>1$, $\nu_j>\dfrac{n}{r}-1$, $j=\overline{1,m}$.

2)Let

$$\widetilde{T}_\beta f(\vec{z})=\int\limits_{T_\Omega}\frac{f(w)dV_\beta(w)}{\prod\limits_{j=1}^m\Delta^{\frac{\beta+\frac{n}{r}}{m}}(\frac{z_j-w}{i})},$$

$\vec{z}=(z_1,...,z_m), z_j\in T_\Omega$, $j=\overline{1,m}$. *Let* $\beta_j>\beta_0, j=\overline{1,m}$, *for some fixed large enough* β_0.

Then $\widetilde{T}_\beta$ *maps* $A^{p_m}_\nu(T_\Omega)$ *to* $A^{\vec{p}}_{\vec{\nu}}(T^m_\Omega)$, $\nu=\sum\limits_{j=1}^m\Big[\nu_j-\dfrac{n}{r}+\dfrac{2n}{r}(m-1)\Big]\dfrac{p_m}{p_j}$.

For unit ball or unit disk case this theorem 1 can be seen in [16], [17], [57]. For $p_j=p, j=\overline{1,m}$, part 2) can be seen in [578].

We provide a complete analogue of theorem 1 for bounded strictly pseudoconvex domains with smooth boundary.

Theorem 16. *1) Let*

$$S_{\vec{\beta}}f(\vec{z})=\int\limits_{D^m}f(w_1,...,w_m)K_{\beta_1}(z_1,w_1)...K_{\beta_m}(z_m,w_m)dv_{\beta_1}(w_1)...dv_{\beta_m}(w_m),$$

$\vec{z}=(z_1,...,z_m)\in D^m$. *Let* $\beta_j>\beta_0$, $j=\overline{1,m}$, *for some fixed large enough* β_0.

Then S_β *maps* $L^{\vec{p}}_{\vec{\nu}}(D^m)$ *into* $A^{\vec{p}}_{\vec{\nu}}(D^m), p_j>1, \nu_j>-1, j=\overline{1,m}$.

2)Let

$$\widetilde{S}_\beta f(\vec{z})=\int\limits_D f(w)\prod_{j=1}^m K_\beta(z_j,w)dv_\beta(w),$$

$\overrightarrow{z} = (z_1, ..., z_m), z_j \in D,\ j = \overline{1,m}$. *Let* $\beta_j > \beta_0,\ j = \overline{1,m}$, *for some fixed large enough* β_0.

Then $\widetilde{S}_\beta$ *maps* $A^{p_m}_{\nu}(D)$ *to* $A^{\overrightarrow{p}}_{\overrightarrow{\nu}}(D^m), \nu = \sum\limits_{j=1}^{m} \nu_j + (m-1)(n-1)$.

For $p_j = p, j = \overline{1,m}$, this theorem (the second part) can be seen in [38], [55], [57] in unit disk and ball. For unit disk case theorems (first parts) can be seen in [16], [17]. Also the proof in all cases is almost parallel to unit disk case for $p_j = p, j = \overline{1,m}$ (see [38]).

Note again, these projections theorems can be used for new duality theorems, which will extend some results we provided in previous section to the case of more complicated domains such as tubular domains over symmetric cones and bounded strictly pseudoconvex domains with smooth boundary.

New results on duality in tubular domains can be seen in [54], it will be nice to extend them to the case of analytic spaces with mixed norm on product domains.

Finnaly, we also would like to mention [59], [60], [61] for duality theorems in Hardy and Bergman spaces in bounded symmetric domains. It will be nice to define their complete extentions as some new mixed norm spaces on product of bounded symmetric domains and to study duality problem in such new spaces similarly as we did in this paper in less general case of the unit disk or the unit ball.

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