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ON ASYMPTOTIC CURVES AND VALUES IN THE THEORY OF MAPPINGS WITH WEIGHTED BOUNDED DISTORTION

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Abstract. We show that a mapping with weighted bounded (p, q)-distortion can be extended to the set whose family of asymptotic curves has weighted modulus zero. We also state some results about asymptotic values, in particular, the counterpart to Iversen's theorem for mappings with weighted bounded (n, n)-distortion.

Keywords: mapping with weighted bounded (p, q)-distortion, asymptotic curve, asymptotic value, singularity, capacity, modulus.

1. Introduction

In the 1960s and 1970s, Yu. G. Reshetnyak published a series of papers that laid the foundations of the theory of mappings with bounded distortion (see the monograph [1]). Let Ω be a domain in the Euclidean space $\mathbb{R}^n$, $n \geq 2$. A mapping $f = (f_1, \dots, f_n)\colon \Omega \to \mathbb{R}^n$ of class $W^1_{n,\mathrm{loc}}(\Omega)$ is called *a mapping with bounded distortion* if there exists a constant $K \in [1, \infty)$ such that the inequality $|Df(x)|^n \leq KJ(x, f)$ holds almost everywhere in Ω. The symbol $|Df(x)|$ denotes the operator norm of the Jacobi matrix $Df(x) = (\frac{\partial f_i}{\partial x_j}(x))_{i,j=1,\dots,n}$, and $J(x, f) = \det Df(x)$.

A number of investigations in quasiconformal analysis deal with the problem of removable singularities. Let $F \subset \Omega$ be a closed set and $f\colon \Omega \setminus F \to \mathbb{R}^n$ a mapping with bounded distortion. It is known [2, Theorem 4.1] that if $\operatorname{cap}(F; W^1_n(\mathbb{R}^n)) = 0$ and $\operatorname{cap}(\mathbb{R}^n \setminus f(\Omega \setminus F); W^1_n(\mathbb{R}^n)) > 0$, then f admits a continuous extension $\tilde{f}\colon \Omega \to \overline{\mathbb{R}^n}$, where $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$. This result can be strengthened. Recall that a

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curve $\beta\colon [0,1) \to \mathbb{R}^n$ is said to be *asymptotic* for a point $x \in F$ if there is a curve $\alpha\colon [0,1) \to \Omega \setminus F$ such that $f \circ \alpha = \beta$ and $\lim\limits_{t \to 1-0} \alpha(t) = x$. E. A. Poletskiĭ proved the following

Proposition 1 ([3, Theorem 3]). *Suppose $f\colon \Omega \setminus F \to \mathbb{R}^n$ is a mapping with bounded distortion, $F \subset \Omega$ is a closed set, $\dim F \le n-2$, and $\widehat{\Gamma}$ is a family of asymptotic curves for the set F. If $\operatorname{mod}_n \widehat{\Gamma} = 0$ and $\operatorname{cap}(\mathbb{R}^n \setminus f(\Omega \setminus F); W_n^1(\mathbb{R}^n)) > 0$, then f can be extended to a continuous mapping $\tilde{f}\colon \Omega \to \overline{\mathbb{R}^n}$.*

This assertion is stronger than [2, Theorem 4.1], since, as shown in [3, Example 1], the following situation is possible: $\operatorname{cap}(F; W_n^1(\mathbb{R}^n)) > 0$ but $\operatorname{mod}_n \widehat{\Gamma} = 0$.

Along with removal of singularities we are also interested in questions relating to the notion of asymptotic value. A mapping $f\colon \Omega \to \mathbb{R}^n$ has *an asymptotic value* c at a point $b \in \partial\Omega$ if $c = \lim\limits_{t \to 1-0} f(\gamma(t))$ for some curve $\gamma\colon [0,1) \to \Omega$ with $\gamma(t) \to b$ as $t \to 1-0$. We mention a couple of results concerning this notion.

Proposition 2 ([4, Theorem 2.4]). *Let $f\colon B(0,1) \to \mathbb{R}^n$ be a mapping with bounded distortion and $\operatorname{cap}(\mathbb{R}^n \setminus f(B(0,1)); W_n^1(\mathbb{R}^n)) > 0$. If $E \subset S(0,1)$ is the set of points at which f has some asymptotic value, then $\operatorname{cap}(E \cap B(y,\varepsilon); W_n^1(\mathbb{R}^n)) > 0$ for all $y \in S(0,1)$ and $\varepsilon > 0$.*

Proposition 3 (the counterpart to Iversen's theorem, [4, Theorem 2.6]). *Let $f\colon \Omega \to \mathbb{R}^n$ be a mapping with bounded distortion and $b \in \partial\Omega$ an isolated boundary point. If b is an essential singularity of f, then every point in $\mathbb{R}^n \setminus f(\Omega)$ is an asymptotic value of f.*

The aim of this paper is to establish the analogs of the above-stated propositions for the class of mappings which have recently been introduced by S. K. Vodop'yanov. This class serves as a natural generalization of the class of mappings in the Reshetnyak sense.

Definition 1 ([5]). Let $\theta, \sigma\colon \mathbb{R}^n \to [0,\infty]$ be locally integrable functions (called *weighted*) such that $\theta > 0$, $\sigma > 0$ almost everywhere. A mapping $f\colon \Omega \to \mathbb{R}^n$ is called *a mapping with (θ,σ)-weighted bounded (p,q)-distortion*, $n-1 < q \le p < \infty$, if

1) f is continuous, open, and descrete;
2) f belongs to the Sobolev class $W^1_{q,\mathrm{loc}}(\Omega)$;
3) $J(x,f) \ge 0$ for almost all $x \in \Omega$;
4) The mapping f has *finite distortion*, which means that for almost all $x \in \Omega$ the equality $J(x,f) = 0$ implies $Df(x) = 0$;
5) The function of local (θ,σ)-weighted q-distortion

$$\Omega \ni x \mapsto K_q^{\theta,\sigma}(x,f) = \begin{cases} \frac{\theta^{\frac{1}{q}}(x)|Df(x)|}{\sigma^{\frac{1}{p}}(f(x))J(x,f)^{\frac{1}{p}}}, & \text{if } J(x,f) \ne 0, \\ 0, & \text{if } J(x,f) = 0, \end{cases}$$

belongs to the class $L_\varkappa(\Omega)$, where $\varkappa$ is determined from the condition $\frac{1}{\varkappa} = \frac{1}{q} - \frac{1}{p}$ ($\varkappa = \infty$ for $q = p$).

Denote by $K_{q,p}^{\theta,\sigma}(f;\Omega)$ the quantity $\|K_q^{\theta,\sigma}(\cdot,f) \mid L_\varkappa(\Omega)\|$.

In Section 3 we state and prove the theorems which generalize Propositions 1, 2, and 3 to the class of mappings satisfying Definition 1.

2. Preliminaries

2.1. Sobolev classes. Throughout the text the symbol Ω denotes a domain (i. e., an open connected set) in $\mathbb{R}^n$. For a domain $U \subset \mathbb{R}^n$, we use the notation $U \Subset \Omega$ in order to indicate that U is bounded and $\overline{U} \subset \Omega$. Given $x_0 \in \mathbb{R}^n$ and $r > 0$, let $B(x_0, r) = \{y \in \mathbb{R}^n \mid |y - x_0| < r\}$, $S(x_0, r) = \partial B(x_0, r)$.

Suppose $u\colon \Omega \to \mathbb{R}$ is a function of class $L_{1,\mathrm{loc}}(\Omega)$. If there exists a function $v_i \in L_{1,\mathrm{loc}}(\Omega)$, $i = 1, \dots, n$, such that for every test function $\varphi \in C_0^\infty(\Omega)$ the equality

$$\int\limits_\Omega u(x) \frac{\partial \varphi}{\partial x_i}(x)\, dx = -\int\limits_\Omega v_i(x) \varphi(x)\, dx$$

holds, then v_i is called *the generalized partial derivative of u with respect to x_i* and written as $\frac{\partial u}{\partial x_i}$. Denote by ∇u the vector-function $(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n})$.

Let $p \geq 1$. A mapping $u\colon \Omega \to \mathbb{R}$ that has in Ω the generalized partial derivatives with respect to all variables belongs to *the Sobolev space $W_p^1(\Omega)$* whenever $u \in L_p(\Omega)$ and $\frac{\partial u}{\partial x_i} \in L_p(\Omega)$ for all $i = 1, \dots, n$.

We say that a mapping $f = (f_1, \dots, f_n)\colon \Omega \to \mathbb{R}^n$ lies in *the Sobolev class $W_p^1(\Omega)$* ($W^1_{p,\mathrm{loc}}(\Omega)$) if all $f_i \in W_p^1(\Omega)$ (all $f_i \in W_p^1(U)$ for any domain $U \Subset \Omega$).

2.2. Dimension. In this section we recall the notion of topological dimension and some of its properties.

Definition 2 ([6, Definition III 1])**.** Let F be a subset of $\mathbb{R}^n$. The empty set and only the empty set has *dimension* -1. The set F has dimension $\leq k$, $0 \leq k \leq n$, at a point $x \in F$ if x has arbitrarily small neighborhoods whose boundaries have intersections with F of dimension $\leq n-1$. The set F has *dimension* $\leq k$, $\dim F \leq k$, if F has dimension $\leq k$ at each of its points. The set F has *dimension k at a point* $x \in F$ if it is true that F has dimension $\leq k$ at x and it is false that F has dimension $\leq k - 1$ at x. Finally, the set F has *dimension k*, $\dim F = k$, if it is true that $\dim F \leq k$ and it is false that $\dim F \leq k - 1$.

Proposition 4 ([6, Theorem IV 3])**.** *Suppose U is a subset of $\mathbb{R}^n$. Then $\dim U = n$ if and only if U contains a non-empty set which is open in $\mathbb{R}^n$.*

Proposition 5 ([6, Ch. IV, Sec. 5, Corollary 1])**.** *Let Ω be a domain in $\mathbb{R}^n$ and let $F \subset \Omega$. If $\dim F \leq n - 2$, then the set $\Omega \setminus F$ is connected.*

Remark 1. In $\mathbb{R}^n$ the concept of Hausdorff dimension is widely used as well. It follows from [6, Theorem VII 2] that the topological dimension of a set does not exceed its Hausdorff dimension.

2.3. Capacity. Here we present several kinds of capacity. A more thorough exposition can be found in [7, 8].

In what follows we shall assume that a function $\omega\colon \mathbb{R}^n \to [0, \infty]$ (called *weighted*) is different from zero almost everywhere and locally integrable.

Definition 3. By *a condenser* we understand a pair $E = (U, K)$, where $U \subset \mathbb{R}^n$ is open and $K \subset U$ is compact. The number

$$\operatorname{cap}_p^\omega E = \inf_{u \in W_0(U,K)} \int\limits_U |\nabla u(x)|^p \omega(x)\, dx,$$

where $W_0(K, U) = \{u \in C_0^\infty(U) \mid 0 \leq u \leq 1,\ u|_K = 1\}$, $p \in [1, \infty)$, is said to be *the ω-weighted p-capacity of the condenser E*. For $\omega \equiv 1$, we simply write $\operatorname{cap}_p E$.

Definition 4. Let U be a domain in $\mathbb{R}^n$ and let $K \subset U$ be compact. *The ω-weighted p-capacity of the compact set K in the space $W_p^1(U,\omega)$* is the quantity

$$\operatorname{cap}(K; W_p^1(U,\omega)) = \inf_{u \in W_0(K,U)} \left(\int\limits_U u^p(x)\omega(x)\,dx + \int\limits_U |\nabla u(x)|^p \omega(x)\,dx \right).$$

If $\omega \equiv 1$, we write $\operatorname{cap}(K; W_p^1(U))$. The notion of capacity (for $\omega \equiv 1$) can be extended to arbitrary sets in the well-known way (see, e. g., [7, Sec. 7.2.1]).

We shall consider separately the case when K is a one-point set. Let Ω be a domain in $\mathbb{R}^n$ and $x_0 \in \Omega$. The point x_0 is said to *have ω-weighted p-capacity zero* in Ω if, for some open ball $B(x_0,R) \subset \Omega$, we have $\operatorname{cap}(\{x_0\}; W_p^1(B(x_0,R),\omega)) = 0$. The latter relation, in view of Definition 3, 4, and [8, Theorem 2.2 (iv)], implies the equalities

$$\operatorname{cap}_p^\omega(\{x_0\}, B(x_0,R)) = \lim_{r\to 0} \operatorname{cap}_p^\omega(B(x_0,R), \overline{B(x_0,r)}) = 0.$$

The following statements will be needed below.

Proposition 6 ([4, Lemma III.2.6])**.** *Let E be a compact set in $\mathbb{R}^n$, $\operatorname{cap}(E; W_n^1(\mathbb{R}^n)) > 0$, and C a continuum in $\mathbb{R}^n \setminus E$. Then, for every $a > 0$, there exists $\delta > 0$ such that $\operatorname{cap}_n(\mathbb{R}^n \setminus E, C) \geq \delta$, provided that $\operatorname{diam} C \geq a$.*

Proposition 7 ([7, Sec. 9.1.2, Proposition 1])**.** *Let C be a continuum in $\mathbb{R}^n$ and $p \in (n-1,n)$. Then $\operatorname{cap}_p(\mathbb{R}^n, C) \geq \lambda(\operatorname{diam} C)^{n-p}$, where a constant $\lambda > 0$ depends only on p and n.*

Next we state two assertions about the relationship between Hausdorff measures and capacities. The symbol $\mathcal{H}^s$ stands for the usual s-dimensional Hausdorff measure in $\mathbb{R}^n$. In the unweighted case we have the following

Proposition 8 ([8, Theorem 2.26])**.** *Suppose that E is a subset of $\mathbb{R}^n$ such that $\operatorname{cap}(E; W_p^1(\mathbb{R}^n)) = 0$, where $1 < p \leq n$. Then $\mathcal{H}^s(E) = 0$ for all $s > n - p$.*

In the weighted case we consider the Muckenhoupt class A_p, $p \in (1,\infty)$, which consists of all locally integrable functions $\omega\colon \mathbb{R}^n \to [0,\infty]$ such that

$$\sup \left(\frac{1}{\mathcal{H}^n(B)} \int\limits_B \omega\,dx \right) \left(\frac{1}{\mathcal{H}^n(B)} \int\limits_B \omega^{1/(1-p)}\,dx \right)^{p-1} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

Proposition 9 ([8, Corollary 2.33])**.** *Let $\omega \in A_p$ and define $p_0 = \inf\{q \mid \omega \in A_q\}$. Then $p_0 < p$. If $\operatorname{cap}(E; W_p^1(\mathbb{R}^n,\omega)) = 0$, then $\mathcal{H}^s(E) = 0$ for all $s > n - p/p_0$.*

A more detailed discussion of classes A_p can be found in, e. g., [8], [9, Ch. V].

2.4. Path lifting. Let I be an interval in $\mathbb{R}$. This interval (possibly unbounded) may be open, closed, or half-open. *A curve* in $\mathbb{R}^n$ is a continuous mapping $\alpha\colon I \to \mathbb{R}^n$. Set $|\alpha| = \alpha(I)$.

Definition 5 ([4, Ch. II, Sec. 3])**.** Suppose $f\colon \Omega \to \mathbb{R}^n$ is a continuous, discrete, open, and sense-preserving mapping, $\beta\colon [a,b) \to \mathbb{R}^n$ is a curve and $x \in f^{-1}(\beta(a))$. A curve $\alpha\colon [a,c) \to \Omega$ is called *a maximal f-lifting of β starting at x* if (i) $\alpha(a) = x$; (ii) $f \circ \alpha = \beta|_{[a,c)}$; (iii) if $c < c' \leq b$, then there does not exist a curve $\alpha'\colon [a,c') \to \Omega$ such that $\alpha = \alpha'|_{[a,c)}$ and $f \circ \alpha' = \beta|_{[a,c')}$.

Remark 2. Analogously, we may define *a maximal f-lifting of β terminating at x* in the case $\beta\colon (b,a] \to \mathbb{R}^n$ and $x \in f^{-1}(\beta(a))$.

Proposition 10 ([4, Corollary II.3.3])**.** *Let $f\colon \Omega \to \mathbb{R}^n$ be continuous, discrete, open, and sense-preserving, let $\beta\colon [a,b) \to \mathbb{R}^n$ (respectively, $\beta\colon (b,a] \to \mathbb{R}^n$) be a curve and let $x \in f^{-1}(\beta(a))$. Then β has a maximal f-lifting starting (respectively, terminating) at x.*

Remark 3. A mapping with (θ,σ)-weighted bounded (p,q)-distortion is sense-preserving, since its Jacobian is nonnegative.

2.5. Modulus. Let Γ be a family of curves in $\mathbb{R}^n$, $n \geq 2$. A Borel function $\rho\colon \mathbb{R}^n \to [0,\infty]$ is called *admissible* for Γ if $\int_\gamma \rho\, ds \geq 1$ for every locally rectifiable curve $\gamma \in \Gamma$. The collection of all admissible functions is denoted by $\operatorname{adm}\Gamma$. Let $p \in [1,\infty)$.

Definition 6. *The ω-weighted p-modulus of the family* Γ is the number

$$\operatorname{mod}_p^\omega \Gamma = \inf_{\rho \in \operatorname{adm}\Gamma} \int\limits_{\mathbb{R}^n} \rho^p \omega\, dx.$$

For $\omega \equiv 1$, we write $\operatorname{mod}_p \Gamma$.

A more detailed information about (unweighted) moduli can be found in [4, Ch. II] or [10, Ch. 1].

It is useful to point out one simple property. We shall write $\Gamma_1 < \Gamma_2$ if each curve $\gamma \in \Gamma_2$ has a subcurve which belongs to Γ_1. If $\Gamma_1 < \Gamma_2$, then $\operatorname{mod}_p^\omega \Gamma_1 \geq \operatorname{mod}_p^\omega \Gamma_2$. Notice that $\Gamma_1 \supset \Gamma_2$ implies $\Gamma_1 < \Gamma_2$.

The following assertion is of great importance.

Proposition 11 ([4, Proposition II.10.2])**.** *Let $E = (U,K)$ be a condenser and let Γ_E be the family of all curves of the form $\gamma\colon [a,b) \to U$ with $\gamma(a) \in K$ and $|\gamma| \cap (U \setminus C) \neq \emptyset$ for every compact set $C \subset U$. Then $\operatorname{cap}_p E = \operatorname{mod}_p \Gamma_E$.*

Next we formulate the analog of Poletskiĭ's inequality for the class of mappings from Definition 1.

Proposition 12 ([11, Theorem 1])**.** *Suppose that $f\colon \Omega \to \mathbb{R}^n$ is a mapping with $(\theta,1)$-weighted bounded (p,q)-distortion, $n-1 < q \leq p < \infty$, and the weighted function $\omega(x) = \theta^{-\frac{n-1}{q-(n-1)}}(x)$ is locally integrable. If Γ is a family of curves in Ω, then*

$$(\operatorname{mod}_{p'} f(\Gamma))^{1/p'} \leq K_{p,q}^{\theta,1}(f;\Omega)^{n-1} (\operatorname{mod}_{q'}^\omega \Gamma)^{1/q'},$$

where $p' = \frac{p}{p-(n-1)}$, $q' = \frac{q}{q-(n-1)}$.

2.6. Essential singularities. An isolated boundary point $b \in \partial\Omega$ is said to be *an essential singularity* of a mapping $f\colon \Omega \to \mathbb{R}^n$ if f has no limit at b.

The following proposition is the analog of the Sokhotskiĭ—Weierstrass theorem.

Proposition 13 ([5, Corollary 6])**.** *Let $f\colon \Omega \to \mathbb{R}^n$ be a nonconstant mapping with $(\theta,1)$-weighted bounded (n,n)-distortion, and let $b \in \partial\Omega$ be an isolated boundary point which has ω-weighted n-capacity zero in $\Omega \cup \{b\}$, where $\omega(x) = \theta^{1-n}(x)$. If b is an essential singularity of f, then $\operatorname{cap}(\mathbb{R}^n \setminus f(U \setminus \{b\}); W_n^1(\mathbb{R}^n)) = 0$ for every neighborhood $U \subset \Omega \cup \{b\}$ of b.*

3. Statement and proof of main results

Unlike Section 1, in Theorem 1 a curve $\gamma\colon [0,1) \to \Omega \setminus F$ is called *asymptotic for the set* F if $\operatorname{dist}(\gamma(t_k), F) \to 0$ for some sequence $t_k \to 1-0$ as $k \to \infty$.

Theorem 1. *Suppose that* $f\colon \Omega\setminus F \to \mathbb{R}^n$ *is a mapping with* $(\theta,1)$*-weighted bounded* (p,q)*-distortion,* $F \subset \Omega$ *is closed,* $\dim F \leq n-2$, $n-1 < q < n \leq p < \frac{(n-1)^2}{n-2}$, *and the weighted function* $\omega(x) = \theta^{-\frac{n-1}{q-(n-1)}}(x)$ *is locally integrable. Let* Γ *be a family of asymptotic curves for the set* F*. If* $\operatorname{mod}^{\omega}_{q'} \Gamma = 0$ *and each point of* F *has* ω*-weighted* q'*-capacity zero in* Ω*, where* $q' = \frac{q}{q-(n-1)}$*, then the mapping* f *extends to a continuous mapping* $\tilde f\colon \Omega \to \mathbb{R}^n$ *for* $p \neq n$*. If in addition, for p=n, the inequality* $\operatorname{cap}(\mathbb{R}^n \setminus f(\Omega\setminus F); W^1_n(\mathbb{R}^n)) > 0$ *holds, then* f *extends to a continuous mapping* $\tilde f\colon \Omega \to \overline{\mathbb{R}^n}$.

Proof. Case $p \neq n$. Fix any point $x_0 \in F$ and take $R > 0$ such that $B(x_0,R)\setminus F \subset \Omega$. By virtue of Proposition 4 the set $B(x_0,R)\setminus F$ is non-empty open. Consequently, there exists a sequence $\{x_i\}_{i=1}^{\infty} \subset B(x_0,R)\setminus F$ that converges to x_0. Fix an arbitrary $i \in \mathbb{N}$. By virtue of Proposition 5 we can connect x_i and x_j with $j \geq i$ by a curve γ_{ij} lying in the domain $B(x_0,r_i)\setminus F$ for all $j \geq i$, where $r_i \in (0,R)$. Undoubtedly, we may assume that $r_i \to 0$ as $i \to \infty$.

Consider the condensers $E_{ij} = (B(x_0,R)\setminus F, \gamma_{ij})$ and $f(E_{ij}) = (f(B(x_0,R)\setminus F), f(\gamma_{ij}))$, as well as the families of curves $\Gamma_{E_{ij}}$ and $\Gamma_{f(E_{ij})}$ corresponding to these condensers in the sense of Proposition 11. Let Γ_{ij} be the family of maximal f-liftings of the curves from $\Gamma_{f(E_{ij})}$ starting at γ_{ij} (see Proposition 10). It is easy to see that $f(\Gamma_{ij}) < \Gamma_{f(E_{ij})}$. Also, $\Gamma_{ij} \subset \Gamma_{E_{ij}}$ (see the proof of [12, Lemma 1.4.1]). Therefore,

$$\operatorname{mod}_{p'}(\Gamma_{f(E_{ij})}) \leq \operatorname{mod}_{p'} f(\Gamma_{ij}) \leq \operatorname{mod}_{p'} f(\Gamma_{E_{ij}}),$$

where $p' = \frac{p}{p-(n-1)}$. Taking into account Proposition 11, we get

$$\operatorname{cap}_{p'} f(E_{ij}) \leq \operatorname{mod}_{p'} f(\Gamma_{E_{ij}}). \tag{1}$$

Clearly, the family $\Gamma_{E_{ij}}$ can be represented in the form

$$\Gamma_{E_{ij}} = \Gamma_{E_{ij},1} \cup \Gamma_{E_{ij},2}, \tag{2}$$

where $\Gamma_{E_{ij},1}$ is the family of curves $\alpha\colon [a,b) \to B(x_0,R)\setminus F$ such that $\alpha(a) \in \gamma_{ij}$ and $\operatorname{dist}(\alpha(t_k),F) \to 0$ for some sequence $t_k \to b-0$ as $k \to \infty$, and $\Gamma_{E_{ij},2}$ is the family of curves $\alpha\colon [a,b) \to B(x_0,R)\setminus F$ such that $\alpha(a) \in \gamma_{ij}$ and $\operatorname{dist}(\alpha(\tau_k), \partial B(x_0,R)) \to 0$ for some sequence $\tau_k \to b-0$ as $k \to \infty$.

We deduce from (1), (2), subadditivity of modulus (see [4, Proposition II.1.5 (1)]) and Proposition 12 that

$$\begin{aligned}&\operatorname{cap}_{p'} f(E_{ij}) \leq \\ &K^{\theta,1}_{p,q}(f;\Omega\setminus F)^{p'(n-1)}\Big((\operatorname{mod}^{\omega}_{q'}\Gamma_{E_{ij},1})^{p'/q'} + (\operatorname{mod}^{\omega}_{q'}\Gamma_{E_{ij},2})^{p'/q'}\Big).\end{aligned} \tag{3}$$

Since $\Gamma_{E_{ij},1} \subset \Gamma$, it follows that

$$\operatorname{mod}^{\omega}_{q'} \Gamma_{E_{ij},1} = 0. \tag{4}$$

Denote by $\Gamma^*_{E_{ij},2}$ the family of curves corresponding to the condenser $(B(x_0,R),\gamma_{ij})$ in the sense of Proposition 11. Obviously, $\Gamma_{E_{ij},2} \subset \Gamma^*_{E_{ij},2}$. Definition 6, Definition 3, and the inclusion $|\gamma_{ij}| \subset B(x_0, r_i)$ yield

$$\operatorname{mod}^{\omega}_{q'} \Gamma_{E_{ij},2} \leq \operatorname{mod}^{\omega}_{q'} \Gamma^{*}_{E_{ij},2} \leq \operatorname{cap}^{\omega}_{q'}(B(x_0,R),\gamma_{ij}) \leq \operatorname{cap}^{\omega}_{q'}(B(x_0,R),\overline{B(x_0,r_i)}). \tag{5}$$

Combining (4), (5), and (3), we obtain

$$\operatorname{cap}_{p'} f(E_{ij}) \leq K^{\theta,1}_{p,q}(f;\Omega\setminus F)^{p'(n-1)}\left(\operatorname{cap}^{\omega}_{q'}(B(x_0,R),\overline{B(x_0,r_i)})\right)^{p'/q'}. \tag{6}$$

Using the estimate $\operatorname{cap}_{p'}(\mathbb{R}^n, f(\gamma_{ij})) \leq \operatorname{cap}_{p'} f(E_{ij})$ and Proposition 7 (the condition $n < p < \frac{(n-1)^2}{n-2}$ ensures that $p' \in (n-1,n)$), we infer from (6) that

$$\lambda(\operatorname{diam} f(\gamma_{ij}))^{n-p'} \leq K^{\theta,1}_{p,q}(f;\Omega\setminus F)^{p'(n-1)}\left(\operatorname{cap}^{\omega}_{q'}(B(x_0,R),\overline{B(x_0,r_i)})\right)^{p'/q'}.$$

The latter relation and the fact that the point $x_0 \in F$ has ω-weighted q'-capacity zero yield $|f(x_i) - f(x_j)| \to 0$ as $i,j \to \infty$. By Cauchy's criterion, $\lim\limits_{i\to\infty} f(x_i)$ exists and is finite. Since the sequence $\{x_i\}_{i=1}^{\infty}$ was chosen arbitrarily, Heine's definition of limit of a function allows us to conclude that the limit $\lim\limits_{x\to x_0} f(x)$ exists and is finite.

Case $p = n$. Keeping in mind the previous case we shall omit some details here. Let $x_0 \in F$. Assume that f fails to have a limit in $\mathbb{R}^n \cup \{\infty\}$ at x_0. Then for some $R > 0$ we can find in $B(x_0,R) \setminus F \subset \Omega$ two sequences $\{x_i\}_{i=1}^{\infty}$ and $\{x'_i\}_{i=1}^{\infty}$ converging to x_0 such that $\operatorname{dist}(f(x_i), f(x'_i)) \geq a > 0$ for all $i \in \mathbb{N}$.

By virtue of Proposition 5 we can connect x_i and x'_i by a curve γ_i lying in $B(x_0,r_i) \setminus F$, $r_i \in (0,R)$. Again we may assume that $r_i \to 0$ as $i \to \infty$

Consider the condensers $E_i = (B(x_0,R) \setminus F, \gamma_i)$ and $f(E_i) = (f(B(x_0,R) \setminus F), f(\gamma_i))$, as well as the family of curves Γ_{E_i} and $\Gamma_{f(E_i)}$ corresponding to these condensers in the sense of Proposition 11. Arguing as in the preceding case, we arrive at the inequality of the form (6):

$$\operatorname{cap}_{n} f(E_{ij}) \leq K^{\theta,1}_{n,q}(f;\Omega\setminus F)^{n(n-1)}\left(\operatorname{cap}^{\omega}_{q'}(B(x_0,R),\overline{B(x_0,r_i)})\right)^{n/q'}.$$

Since $\operatorname{diam} f(\gamma_i) \geq a$ and $\operatorname{cap}(\mathbb{R}^n \setminus f(\Omega\setminus F); W^1_n(\mathbb{R}^n)) > 0$, in view of Proposition 6 and the inclusion $f(B(x_0,R)\setminus F) \subset f(\Omega\setminus F)$ we conclude that

$$0 < \delta \leq K^{\theta,1}_{n,q}(f;\Omega\setminus F)^{n(n-1)}\left(\operatorname{cap}^{\omega}_{q'}(B(x_0,R),\overline{B(x_0,r_i)})\right)^{n/q'}.$$

Passing to the limit as $i \to \infty$ in the latter inequality and taking into account that x_0 has ω-weighted q'-capacity zero, we get a contradiction. $\square$

Before stating the further assertions we recall that a mapping $f\colon \Omega \to \mathbb{R}^n$ has *an asymptotic value* c at a point $b \in \partial\Omega$ if $c = \lim\limits_{t\to 1-0} f(\gamma(t))$ for some curve $\gamma\colon [0,1) \to \Omega$ with $\gamma(t) \to b$ as $t \to 1-0$.

Theorem 2. *Suppose that $f\colon B(0,1) \to \mathbb{R}^n$ is a mapping with $(\theta,1)$-weighted bounded (p,q)-distortion, $n-1 < q < n \leq p < \frac{(n-1)^2}{n-2}$. Suppose also that the weighted function $\omega(x) = \theta^{-\frac{n-1}{q-(n-1)}}(x)$ is locally integrable and belongs to the Muckenhoupt class $A_{q'}$. Let $E \subset S(0,1)$ be the set of points at which f has some asymptotic value. Assume that every point of $S(0,1)$ has ω-weighted q'-capacity zero, where $q' = \frac{q}{q-(n-1)}$. If $p \neq n$, then $\operatorname{cap}(E \cap B(y,\varepsilon); W^1_{q'}(\mathbb{R}^n,\omega)) > 0$ whenever $y \in B(0,1)$*

and $\varepsilon > 0$. If $p = n$, the latter conclusion is valid under the additional condition $\operatorname{cap}(\mathbb{R}^n \setminus f(\Omega \setminus F); W_n^1(\mathbb{R}^n)) > 0$.

Proof. CASE $p \neq n$. Assume, on the contrary, that there are $y \in S(0,1)$ and $\varepsilon > 0$ such that $\operatorname{cap}(E \cap B(y,\varepsilon); W_{q'}^1(\mathbb{R}^n,\omega)) = 0$. By Proposition 9, we have that $\mathcal{H}^{n-1}(E \cap B(y,\varepsilon)) = 0$. Hence, there is a curve $\gamma\colon [0,1) \to B(0,1)$ such that $\gamma(t) \to c \in S(0,1) \cap B(y,\varepsilon/2)$ as $t \to 1-0$ and $\lim\limits_{t\to 1-0} f(\gamma(t))$ fails to exist. Consequently, we can take a sequence $t_1 < t_2 < \dots$ of positive numbers such that $t_i \to 1-0$ and $\lim\limits_{k\to\infty} f(\gamma(t_{2k})) \neq \lim\limits_{k\to\infty} f(\gamma(t_{2k+1}))$. Therefore, we may assume that there exists $a > 0$ such that $\operatorname{diam} F_k \geq a$ for all k, where $F_k = f(\gamma([t_{2k}, t_{2k+1}]))$. Denote by Γ_k the family of curves $\beta\colon [0,1) \to \mathbb{R}^n$ such that $\beta(0) \in F_k$ and there is $\lim\limits_{t\to 1-0} \beta(t) \in \mathbb{R}^n \setminus f(B(0,1))$. Consider the condenser $(\mathbb{R}^n, F_k)$, as well as the family of curves $\widetilde{\Gamma}_k$ corresponding to this condenser in the sense of Proposition 11. Evidently, $\Gamma_k < \widetilde{\Gamma}_k$, and therefore, $\operatorname{mod}_{p'} \Gamma_k \geq \operatorname{mod}_{p'} \widetilde{\Gamma}_k$, where $p' = \frac{p}{p-(n-1)}$. The condition $n < p < \frac{(n-1)^2}{n-2}$ ensures that $p' \in (n-1, n)$. The application of Proposition 7 yields

$$\operatorname{mod}_{p'} \Gamma_k \geq \operatorname{mod}_{p'} \widetilde{\Gamma}_k = \operatorname{cap}_{p'}(\mathbb{R}^n, F_k) \geq \lambda a^{n-p'} \tag{7}$$

for all k, where $\lambda > 0$ is a constant.

Since the point c has ω-weighted q'-capacity zero, we can choose $r > 0$ such that

$$\operatorname{cap}_{q'}^{\omega}(B(c,\varepsilon/2), \overline{B(c,r)}) \leq \frac{\left(\lambda a^{n-p'}\right)^{q'/p'}}{2K_{p,q}^{\theta,1}(f; B(0,1))^{q'(n-1)}}. \tag{8}$$

We claim that $B(c,\varepsilon/2) \subset B(y,\varepsilon)$. Indeed, if $z \in B(c,\varepsilon/2)$, then

$$|z-y| \leq |z-c| + |c-y| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Denote by $\widehat{\Gamma}_k$ the family of maximal f-liftings of the curves from Γ_k starting at $\gamma([t_{2k}, t_{2k+1}])$. Let l be the first k such that $\gamma([t_{2k}, t_{2k+1}]) \subset B(c,r)$ and let $\widehat{\Gamma}_l^*$ be the curves from $\widehat{\Gamma}_l$ that go outside $B(y,\varepsilon)$. If $\widehat{\Gamma}_r$ is the family of curves corresponding to the condenser $(B(c,\varepsilon/2), \overline{B(c,r)})$ in the sense of Proposition 11, then $\widehat{\Gamma}_r < \widehat{\Gamma}_l^*$, since $B(c,\varepsilon/2) \subset B(y,\varepsilon)$. From (8) and the easily verified inequality

$$\operatorname{mod}_{q'}^{\omega} \widehat{\Gamma}_r \leq \operatorname{cap}_{q'}^{\omega}(B(c,\varepsilon/2), \overline{B(c,r)})$$

we infer

$$\operatorname{mod}_{q'}^{\omega} \widehat{\Gamma}_l^* \leq \frac{\left(\lambda a^{n-p'}\right)^{q'/p'}}{2K_{p,q}^{\theta,1}(f; B(0,1))^{q'(n-1)}}. \tag{9}$$

Using (7) and Proposition 12, we obtain

$$\lambda a^{n-p'} \leq \operatorname{mod}_{p'} \Gamma_l \leq \operatorname{mod}_{p'}(f(\widehat{\Gamma}_l)) \leq K_{p,q}^{\theta,1}(f; B(0,1))^{p'(n-1)} \left(\operatorname{mod}_{q'}^{\omega} \widehat{\Gamma}_l\right)^{p'/q'},$$

and hence

$$\operatorname{mod}_{q'}^{\omega} \widehat{\Gamma}_l \geq \frac{\left(\lambda a^{n-p'}\right)^{q'/p'}}{K_{p,q}^{\theta,1}(f; B(0,1))^{q'(n-1)}}. \tag{10}$$

From (9) and (10) we get $\operatorname{mod}^{\omega}_{q'}(\widehat{\Gamma}_l \setminus \widehat{\Gamma}^*_l) \geq \mathrm{const} > 0$. Set

$$A = \left\{ x \in \mathbb{R}^n \mid \text{ there is a curve } \alpha \in \widehat{\Gamma}_l \setminus \widehat{\Gamma}^*_l \text{ such that } x = \lim_{t \to 1-0} \alpha(t) \right\}.$$

It follows from the definition of Γ_k that $A \subset B(y,\varepsilon) \cap S(0,1)$. Then we have $\operatorname{cap}(A; W^1_{q'}(\mathbb{R}^n,\omega)) \geq \mathrm{const} > 0$. To get a contradiction, it remains to note that f has a limit along every curve from $\widehat{\Gamma}_l \setminus \widehat{\Gamma}^*_l$.

Case $p = n$. We use the same arguments as in the previous case. The difference is that the condition $\operatorname{cap}(\mathbb{R}^n \setminus f(\Omega \setminus F); W^1_n(\mathbb{R}^n)) > 0$ and Proposition 6 yield $\operatorname{mod}_{p'} \Gamma_k \geq \delta$ for some $\delta > 0$. □

Theorem 3. *Suppose that $f\colon \Omega \to \mathbb{R}^n$ is a nonconstant mapping with $(\theta,1)$-weighted bounded (n,n)-distortion, and $b \in \partial\Omega$ is an isolated boundary point that has ω-weighted n-capacity zero in $\Omega \cup \{b\}$, where $\omega(x) = \theta^{1-n}(x)$. If b is an essential singularity of f, then every point of $\mathbb{R}^n \setminus f(\Omega)$ is an asymptotic value of f.*

Proof. Let $z \in \mathbb{R}^n \setminus f(\Omega)$. Without loss of generality it can be assumed that $z = 0$. Take $r > 0$ such that $\overline{B(b,r)} \subset \Omega \cup \{b\}$. Set $U = B(b,r) \setminus \{b\}$.

We claim that there exists $r' \in (0,1)$ such that

$$\overline{B(0,r')} \cap f(S(b,r)) = \emptyset. \tag{11}$$

Indeed, if this is not the case, then 0 belongs to the closure of the closed set $f(S(b,r))$, and hence, $0 \in f(S(b,r))$, which is impossible, since $0 \notin f(\Omega)$.

By Proposition 13, $\operatorname{cap}(\mathbb{R}^n \setminus f(U); W^1_n(\mathbb{R}^n)) = 0$. In view of Proposition 8 we have $\mathcal{H}^{n-1}(\mathbb{R}^n \setminus f(U)) = 0$. Consequently, $\mathcal{H}^{n-1}$-almost all points of $S(0,r')$ lie in $f(U)$. In virtue of condition (11) the set $f^{-1}(S(0,r'))$ has a connected component C such that $C \subset U$.

For $y \in S(0,1)$, we define the curve $\beta_y\colon (0,r'] \to \overline{B(0,r')}$ by the rule $\beta_y(t) = ty$. Denote by α_y the maximal f-lifting of β_y terminating at C. Arguing as in [12, p. 128–129], we can show that the curve α_y is a mapping $\alpha_y\colon (r_y, r'] \to U$ such that $\alpha_y(t) \to b$ as $t \to r_y + 0$. If we prove that $r_y = 0$ for at least one y, then this will imply that $z = 0$ is an asymptotic value of f. Below we establish the stronger fact: $r_y = 0$ for $\mathcal{H}^{n-1}$-almost all $y \in S(0,1)$.

Consider the sets $E_i = \{y \in S(0,1) \mid r_y > 1/i\}$, $i \in \mathbb{N}$. We shall prove that $\mathcal{H}^{n-1}(E_i) = 0$ for all i. To do this fix an arbitrary i and introduce the family $\Gamma_i = \{\alpha_y \mid y \in E_i\}$. As mentioned above, all curves of this family tend to b. Because of the fact that b has ω-weighted n-capacity zero, we have $\operatorname{mod}^{\omega}_n \Gamma_i = 0$. Proposition 12 yields $\operatorname{mod}_n f(\Gamma_i) = 0$.

Since $r_y > 1/i$, every curve $\gamma_y\colon [1/i, r'] \to \mathbb{R}^n$, $\gamma_y(t) = ty$, $y \in E_i$, has a subcurve that belongs to the family $f(\Gamma_i)$. Applying Hölder's inequality and taking into account the conditions $\rho \in \operatorname{adm} f(\Gamma_i)$, $y \in E_i$ (whence $|\dot\gamma_y(t)| = |y| = 1$), and $r' \in (0,1)$, we obtain

$$1 \leq \int\limits_{1/i}^{r'} \rho(ty)\,dt \leq \left(\int\limits_{1/i}^{r'} \rho^n(ty)\,dt \right)^{1/n} \left(r' - \frac{1}{i} \right)^{(n-1)/n} \leq \left(\int\limits_{1/i}^{r'} \rho^n(ty)\,dt \right)^{1/n}.$$

From this it easily follows that

$$\int_{1/i}^{r'} \rho(ty)\,dt \le \int_{1/i}^{r'} \rho^n(ty)\,dt. \tag{12}$$

After computing the integral in polar coordinates and using (12), we get

$$\begin{aligned} \int_{\mathbb{R}^n} \rho^n(x)\,dx \ge \int_{S(0,1)} \left(\int_{1/i}^{r'} t^{n-1}\rho^n(ty)\,dt \right) d\mathcal{H}^{n-1}(y) \ge \\ \frac{1}{i^{n-1}} \int_{S(0,1)} \int_{1/i}^{r'} \rho^n(ty)\,dt\,d\mathcal{H}^{n-1}(y) \ge \frac{1}{i^{n-1}} \int_{S(0,1)} \int_{1/i}^{r'} \rho(ty)\,dt\,d\mathcal{H}^{n-1}(y) \ge \\ \frac{1}{i^{n-1}} \mathcal{H}^{n-1}(E_\rho), \end{aligned} \tag{13}$$

where $E_\rho = \{y \in S(0,1) \mid \int_{1/i}^{r'} \rho(ty)\,dt \ge 1\}$. From the above arguments we deduce that $E_i \subset E_\rho$. It follows from (13) that the quantity $\mathcal{H}^{n-1}(E_\rho)$ can be made arbitrarily small, since $\mathrm{mod}_n\, f(\Gamma_i) = 0$. Thus, $\mathcal{H}^{n-1}(E_i) = 0$. $\square$

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