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# MATHEMATICAL MODEL OF WATER-OIL DISPLACEMENT IN FRACTURED POROUS MEDIUM

A.A.KALINKIN, YU.M.LAEVSKY

Abstract. Within the mixed finite element method the numerical model for two-phase incompressible fluid filtration is designed in the terms "velocity-pressure-saturation". The main difficulty of the model is caused by fractured porous medium. Our approach allows to resolve difficulties wtith boundary condition degeneration for the saturation.

**Keywords:** Fractured porous medium, Filtration model, FEM.

## Introduction

In this paper we present one of the existing approaches for numerical modeling of filtration processes in fractured porous media of two-phase incompressible fluid, which will henceforth be interpreted as a system of "oil-water" [1]–[4]. Corresponding multidimensional mathematical model will be studied and represented as a weak mixed formulation (see.[5] and references therein) in terms of the "velocity-pressure-saturation". In this formulation, there is no problem of singularity for the boundary conditions for saturation: in mixed formulation, saturation is not required to meet any boundary conditions. At the same time there is a feature of permeability functions such that at any moment the equation for total velocity and pressure satisfies the coercivity condition on the set of solenoidal functions. Note that mixed formulation was used for finding velocity and pressure in a number of works of Yuvinga R. et al. (See, for example, [6] and references therein). Unlike [6] in this paper complete mixed formulation is studied, the main feature of which is the representation of all spatial derivatives of vector fields as

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divergences of unknown functions. Moreover, for approximations based on Raviart-Thomas elements no problems arise with the description of the saturation front due to the presence of capillary diffusion. For Laveretta-Buckley model (see., Eg, [3]) direct application of this approach would be justified only with the use of special $L_2$ projectors, that result in a direct difference schemes. Finally, to avoid overloading, here we do not address the simulation of injection and production wells. As for the modeling of fractured porous medium, here we use a popular approach with introduction of two complete sets of variables in each point - one corresponds to a porous medium, another to fractured [1], [12]. Similarly to the above-mentioned works the flow between the medias is considered to be proportional to the pressure difference. And, despite the fact that such approach is quite popular in simulation of fractured porous media, we could not find an approximation of these equations in terms of the "velocity-pressure-saturation"in the literature, which is essential for proper flow approximation.

The paper is organized as follows. The second paragraph provides basic relationships that define the filtration process of two-phase incompressible fluid, the problem of oil displacement with water is formulated as a system of equations of first order with the corresponding boundary conditions. Special attention is given to the discussion of impermeability conditions in the presence of gravity.

## 1. Equations for two-phase flow filtering

Suppose that the liquid flow occurs in domain $\Omega \subset R^d$, $d = 2, 3$ of pore space. We also assume that the flow of fluid takes place in fractured porous media and the velocity of the fluid in the fractures is much higher than in the pores, while the volumes of the spaces in the pores is significantly higher than in fractures. The standard way to describe this kind of problem — the two sets of unknowns for each point in space, the first is responsible for the movement of fluids in porous media, and the second for the movement of fluid in the fractured media. Subsequently, the variables with superscript 1 correspond porous medium, 2 — fractured.

The rate of flow of fluids between the media can be naturally defined in proportion to the pressure difference between the medias:

$$\gamma(P^2 - P^1) = q \tag{1}$$

The filtering process is determined by equation of conservation of mass written for two components of the two-phase incompressible liquid:

$$m^j \frac{\partial s_i^j}{\partial t} + \nabla \cdot \mathbf{v}_i^j = q_i^j, \qquad i = 1, 2 \tag{2}$$

and equation of conservation of momentum, according to Darcy's law:

$$\mathbf{v}_i^j = -k_i^j(s_i^j)\nabla\phi_i^j, \qquad i = 1, 2. \tag{3}$$

Hereinafter, the index $i$ is the number of phase, with $i = 1$ corresponding to the displaced phase (oil), $i = 2$ – the displacing phase (water). In (3) and (4) standard notation: $s_i$ – saturation of the phase is used that satisfies the following equation:

$$s_1^j + s_2^j \equiv 1, \tag{4}$$

$m = m(\mathbf{x})$ - porosity, $\mathbf{v}_i$ – velocity vector for each filtration phase $\phi_i$ – potential in each phase that satisfies

$$\phi_i = p_i + \rho_i g H, \qquad i = 1, 2, \tag{5}$$

here $p_i$ – phase pressure, $\rho_i$ - phase density, assumed constant due to incompressibility that was used earlier in (3), $g$ – gravitational acceleration (hereinafter, a scalar), $H$ – the distance from a point of the medium to a fixed reference surface in the direction of the gravity force. In two-dimensional case, when modelled plane contains the direction of gravity force, and in three-dimensional case we assume $H(\mathbf{x}) = x_d$, where point $\mathbf{x} \in R^d$ is $\mathbf{x} = (x_1, x_2)$ for $d = 2$ and $\mathbf{x} = (x_1, x_2, x_3)$ for $d = 3$. In planar problem, when a modelled plane is orthogonal to the direction of gravity, we assume $H = 0$. By $q_i^j$ we mean the additional weight in the $i$-th phase in the $j$-th media obtained as a consequence of flow between environments. Then, following series of equations can be written:

$$q_i^1 + q_i^2 \equiv 0, \tag{6}$$

$$q_i^1 = \begin{cases} q s_i^1 & q > 0, \\ q s_i^2 & q < 0. \end{cases} \tag{7}$$

Here the first equation is a rather obvious consequence of the law of conservation of mass, and the second is built taking into consideration the equality of concentrations in phases in the domain of higher pressure and flow between the environments. By pressure in the domain we understand the direct sum of the partial pressures:

$$P^j = s_1^j p_1^j + s_2^j p_2^j \tag{8}$$

Phase permeabilities $k_i^j(s_i^j)$ are set by equalities

$$k_i^j(s_i^j) = k_0^j \frac{f_i(s_i^j)}{\mu_i}, \tag{9}$$

here $k_0$ - absolute permeability, $\mu_i$ - coefficients of dynamic viscosity, functions $f_i(s_i)$ in accordance to [3] are determined by equalities:

$$f_i(s_i^j) = \begin{cases} 0 & 0 < s_i < \underline{s} \\ \left(\frac{s_i^j - \underline{s}}{\overline{s} - \underline{s}}\right)^3 & \underline{s} \le s_i^j \le \overline{s} \\ 1 & \overline{s} < s_i < 1, \end{cases} \quad , \quad i = 1, 2. \tag{10}$$

Here $0 < \underline{s} < 1/2$ and $\overline{s} = 1 - \underline{s}$. Finally, the difference in phase pressures caused by the presence of capillary diffusion is assumed:

$$p_1^j - p_2^j = P_k^j(s_2^j), \tag{11}$$

here $P_k(s_2^j)$ – some empirically defined piecewise smooth function that satisfies $P_k'(s_2^j) \le 0$ and $P_k'(s_2) = 0$ for $0 < s_2^j \le \underline{s}$. Negative sign of the derivative $P_k'$ corresponds to the assumption that water's wettability is higher than that of oil.

According to (4) variable $s_1^j$ can be excluded, and thereafter we denote $s^j = s_2^j$, $p^j = p_2^j$, $\phi^j = \phi_2^j = p^j + \rho_2 g H$. Thus according to (11), (5)

$$p_1^j = p^j + P_k^j(s^j), \qquad \phi_1^j = \phi^j + P_k^j(s^j) - (\rho_2 - \rho_1) g H\,.$$

Further, it is easy to derive that

$$k(s^j) \equiv k_1(1 - s^j) + k_2(s^j) \ge \frac{k_0}{(\sqrt{\mu_1} + \sqrt{\mu_2})^2} > 0, \qquad 0 < s^j < 1. \tag{12}$$

It is this inequality that ensures ellipticity of the equation for pressure or coercivity condition on the set of solenoidal functions for mixed formulation. The model of oil displacement with water we represent as a system of first order differential equations and functional dependencies of type "notation". For this purpose, we introduce function

$$\sigma(s^j) = \int\limits_{\underline{s}}^{s^j} \frac{k_1(1-\xi)}{k(\xi)} |P_k'(\xi)| d\xi, \qquad \underline{s} \le s^j \le \overline{s}. \tag{13}$$

The correctness of this formula immediately implies from (12). Similar approaches that use functions of the same type were used earlier (see para. [3]). As a new independent variable we introduce generalized potential of the displacing phase

$$\psi^j = \phi^j - \sigma(s^j). \tag{14}$$

Next, we introduce the gravity vector

$$\mathbf{G}(s^j) = \frac{k_1(1-s^j)}{k(s^j)}(\rho_2 - \rho_1) g \, \mathbf{e}_d, \tag{15}$$

here $\mathbf{e}_2 = (0,1)^T$ and $\mathbf{e}_3 = (0,0,1)^T$. It worth noting that in "oil-water"system water is a heavy phase: $\rho_2 > \rho_1$, so the gravity vector causes emersion of oil. To denote total flow velocity we use notation $\mathbf{v}^j = \mathbf{v}_1^j + \mathbf{v}_2^j$. According to (3) vector $\mathbf{v}^j$ is solenoidal. In view of equalities (13) - (15) and the fact that $\nabla H = \mathbf{e}_d$, equalities (3), (4) can be rewritten as the following system:

$$\begin{aligned}
&\frac{1}{k(s^j)}\mathbf{v}^j + \nabla\psi^j = \mathbf{G}(s^j), \\
&\nabla \cdot \mathbf{v}^j = q_1^j + q_2^j, \\
&\frac{1}{k(s^j)}\mathbf{w}^j = \nabla\sigma(s^j), \\
&\mathbf{v}_2^j - \frac{k_2(s^j)}{k(s^j)}(\mathbf{v}^j - \mathbf{w}^j) = -k_2(s^j)\mathbf{G}(s^j), \\
&m\frac{\partial s^j}{\partial t} + \nabla \cdot \mathbf{v}_2^j = q_i^j. \\
&\gamma(P^2 - P^1) = q. \\
&q_i^1 + q_2^2 \equiv 0 \\
&q_i^1 = \begin{cases} q s_i^1 & q > 0, \\ q s_i^2 & q < 0, \end{cases} \\
&P^j = s_1^j p_1^j + s_2^j p_2^j
\end{aligned} \tag{16}$$

Here, the third and fourth equalities do not represent differential equations. The third equality is simply another definition of $\mathbf{w}$. The reason for introducing this notation will become clear below while obtaining the integral form of the reduced system of equations. The fourth equation establishes a relationship between the total velocity $\mathbf{v}$ and the velocity of the displacing phase $\mathbf{v}_2$. Note that as the rest of the equations of system (16) third and fourth equalities "work" only within the domain $\Omega$, and on the border, generally speaking, can be not satisfied. In the future, these equalities will be considered in a weak form.

Let us define the boundary conditions for the system (16). Suppose $\Gamma = \partial\Omega$ – piecewise smooth boundary of the domain $\Omega$, $\Gamma^{ent}\ \Gamma^{ex}$ – disjoint parts of the border $\Gamma$:

$$\Gamma^{ent} \cup \Gamma^{ex} \subset \Gamma, \qquad \Gamma^{ent} \cap \Gamma^{ex} = \emptyset.$$

We denote $\Gamma^0 = \Gamma \setminus (\Gamma^{ent} \cup \Gamma^{ex})$ and assume $\mathrm{mes}(\Gamma^0) > 0$ and the distance between $\Gamma^{ent}$ and $\Gamma^{ex}$ is positive. Further, suppose $l^{ent} = \mathrm{mes}(\Gamma^{ent})$, $l^{ex} = \mathrm{mes}(\Gamma^{ex})$, $\mathbf{n} = \mathbf{n}(\mathbf{x})$ – unit outer-pointing (with respect to $\Omega$) normal, defined almost everywhere on $\Gamma$.

It is reasonable to assume that water enters the the oil-bearing layer as well as oil comes out in the production well exclusively through the fractured media. Also we assume that oil flows from the area of $\Omega$ through the boundary $\Gamma^{ex}$ at a rate of $q_1^{ex}(t,\mathbf{x})$, and on the remaining part of the boundary impermeability condition takes place:

$$\mathbf{v}_1^2 \cdot \mathbf{n} = q_1^{ex}(t,\mathbf{x}), \quad \mathbf{x} \in \Gamma^{ex}, \qquad \mathbf{v}_1^2 \cdot \mathbf{n} = 0, \quad \mathbf{x} \in \Gamma^0 \cup \Gamma^{ent}, \tag{17}$$

$$\mathbf{v}_1^1 \cdot \mathbf{n} = 0, \qquad \mathbf{x} \in \Gamma \tag{18}$$

At the same time the total flow rate of the oil flowing out is $Q_1^{ex}(t) = \int_{\Gamma^{ex}} q_1^{ex}(t,\mathbf{x})\,\mathrm{d}\gamma$. Further, let $q = \mathrm{const}$ be a given flow rate, entering the domain $\Omega$ through the border $\Gamma^{ent}$. Part of the water flows through the boundary $\Gamma^{ex}$ at a rate $q_2^{ex}(t,\mathbf{x})$. On the remaining part of the boundary impermeability condition is fulfilled:

$$\mathbf{v}_2^2 \cdot \mathbf{n} = -q,\ \ \mathbf{x} \in \Gamma^{ent}, \quad \mathbf{v}_2^2 \cdot \mathbf{n} = q_2^{ex}(t,\mathbf{x}),\ \ \mathbf{x} \in \Gamma^{ex}, \quad \mathbf{v}_2^2 \cdot \mathbf{n} = 0,\ \ \mathbf{x} \in \Gamma^0. \tag{19}$$

$$\mathbf{v}_2^1 \cdot \mathbf{n} = 0, \qquad \mathbf{x} \in \Gamma \tag{20}$$

Total amount of water flowing in and out is $Q = l^{ent}q$ (analogue of the well production rate) and $Q_2^{ex}(t) = \int_{\Gamma^{ex}} q_2^{ex}(t,\mathbf{x})\,\mathrm{d}\gamma$ respectively. Total flow discharges $Q$, $Q_1^{ex}(t)$and $Q_2^{ex}(t)$ have a physical dimension [m$^2$/sec] for $d = 2$ and [m$^3$/sec] for $d = 3$. The boundary conditions for the total velocity $\mathbf{v}$ are followed from (18), (20):

$$\mathbf{v}^2 \cdot \mathbf{n} = -q,\ \ \mathbf{x} \in \Gamma^{ent}, \quad \mathbf{v}^2 \cdot \mathbf{n} = q_1^{ex}(t,\mathbf{x}) + q_2^{ex}(t,\mathbf{x}),\ \ \mathbf{x} \in \Gamma^{ex}, \quad \mathbf{v}^2 \cdot \mathbf{n} = 0,\ \mathbf{x} \in \Gamma^0. \tag{21}$$

$$\mathbf{v}^1 \cdot \mathbf{n} = 0,\ \ \mathbf{x} \in \Gamma \tag{22}$$

In equalities (18)-(22) the only set value is the total flow $Q$. At the same time $q = Q/l^{ent}$. Here we introduce one of several ways to introduce flow rates $q_1^{ex}(t,\mathbf{x})$ and $q_2^{ex}(t,\mathbf{x})$. Due to the solenoidal vector field $\mathbf{v}^2$ the following equality takes place $\int_\Gamma \mathbf{v} \cdot \mathbf{n}\ d\gamma = 0$. Then

$$Q_1^{ex}(t) = \int\limits_\Gamma \mathbf{v}_1^2 \cdot \mathbf{n}\,\mathrm{d}\gamma = -\int\limits_\Gamma \mathbf{v}_2^2 \cdot \mathbf{n}\,\mathrm{d}\gamma = Q - Q_2^{ex}(t),$$

which implies that

$$Q_1^{ex}(t) + Q_2^{ex}(t) = Q, \qquad t > 0. \tag{23}$$

Suppose that for $\mathbf{x} \in \Gamma^{ex}$

$$q_1^{ex}(t, \mathbf{x}) = \frac{k_1(1-s)}{k(s)} \, \frac{l^{ent}}{l^{ex}} \, q + k_2(s)\mathbf{G}(s) \cdot \mathbf{n}, \tag{24}$$

$$q_2^{ex}(t, \mathbf{x}) = \frac{k_2(s)}{k(s)} \, \frac{l^{ent}}{l^{ex}} \, q - k_2(s)\mathbf{G}(s) \cdot \mathbf{n}. \tag{25}$$

Then

$$q_1^{ex}(t, \mathbf{x}) + q_2^{ex}(t, \mathbf{x}) \equiv \frac{1}{l^{ex}} \, Q, \tag{26}$$

and $Q_1^{ex}(t)$, $Q_2^{ex}(t)$ satisfy the equality (23). Formulas (24) and (25) are modified relations between the total flow discharges $Q_1^{ex}$ and $Q_2^{ex}$, presented in [3]. The main idea of modification is to take into account gravity on the outlet of the reservoir. In case of rectangular area when $\Gamma^{ex}$ lies entirely on the side surface, the terms corresponding to the force of gravity are absent, because in this case $\mathbf{e}_d \cdot \mathbf{n} = 0$. Until the breakthrough of water, when $s \leq \underline{s}$ at $\mathbf{x} \in \Gamma^{ex}$, the situation becomes much simpler. In this case $k_2(s)\mathbf{G}(s) \cdot \mathbf{n} = 0$, and formulas (24), (25) take form: $q_1^{ex}(t, \mathbf{x}) = l^{ent} q / l^{ex}$, $q_2^{ex}(t, \mathbf{x}) = 0$. That ensures equalities for total discharges $Q_1^{ex}(t) = Q$, $Q_2^{ex}(t) = 0$. Thus, according to (20), (22), (25) and (26) the total velocity of the two-phase fluid and the velocity of the water are defined at the boundary $\Gamma$ by the following equations

$$\mathbf{v} \cdot \mathbf{n} = -\frac{Q}{l^{ent}}, \ \mathbf{x} \in \Gamma^{ent}, \quad \mathbf{v} \cdot \mathbf{n} = \frac{Q}{l^{ex}}, \ \mathbf{x} \in \Gamma^{ex}, \quad \mathbf{v} \cdot \mathbf{n} = 0, \ \mathbf{x} \in \Gamma^0, \tag{27}$$

$$\mathbf{v}_2 \cdot \mathbf{n} = -\frac{Q}{l^{ent}}, \ \mathbf{x} \in \Gamma^{ent}, \ \mathbf{v}_2 \cdot \mathbf{n} = \frac{k_2(s^2)}{k(s^2)} \, \frac{Q}{l^{ex}} - k_2(s^2)\mathbf{G}(s^2) \cdot \mathbf{n}, \ \mathbf{x} \in \Gamma^{ex}, \quad \mathbf{v}_2 \cdot \mathbf{n} = 0, \ \mathbf{x} \in \Gamma^0. \tag{28}$$

Since $k_1(1 - s^2) = 0$ on $\Gamma^{ent}$, ie movement of oil is absent, according to formula (15) $\mathbf{G}(s^2) = \mathbf{0}$ on $\Gamma^{ent}$. Therefore, on $\Gamma^{ent} \cup \Gamma^{ex}$ an equality takes place

$$\mathbf{v}_2 \cdot \mathbf{n} = \frac{k_2(s^2)}{k(s^2)} \mathbf{v} \cdot \mathbf{n} - k_2(s^2)\mathbf{G}(s^2) \cdot \mathbf{n},$$

which coincides with the continuous extension of the fourth equality from (16) on the boundary region, in case of $\mathbf{w} \cdot \mathbf{n} = 0$ on $\Gamma^{ent} \cup \Gamma^{ex}$. Consider a part of the border $\Gamma^0$, where for each vector $\mathbf{v}$ and $\mathbf{v}_2$ impermeability conditions are set. In the case of the zero capillary pressure the fourth equation of system (16) takes the form

$$\mathbf{v}_2 = \frac{k_2(s^2)}{k(s^2)} \mathbf{v} - k_2(s^2)\mathbf{G}(s^2).$$

Continuous extension of this equation on $\Gamma^0$ can contradict the the impermeability conditions (for $s^2 > \underline{s}$ and $\mathbf{e}_d \cdot \mathbf{n}(\mathbf{x}) \neq 0$). However, this only means the impossibility of such an extension, in the sense that the latter equality takes place strictly within the domain $\Omega$, and the function $\mathbf{v}_2$ has infinitely thin boundary layer. Physically, it is fully justified: in rectangular area, when two phases of oil are moving, oil can not float above the upper limit but on arbitrarily small distance from the boundary the oil is effected by gravity vector, causing it to float. On the other hand, the source of descending by gravity water is not a border area, but the transportation of water along the border under the influence of a pressure drop. A similar situation holds for the bottom boundary if in the above reasoning water and oil changed places. The presence of capillary forces will not be reflected in the described mechanism, and therefore, it is natural to set $\mathbf{w}\cdot\mathbf{n}=0,\ \mathbf{x}\in\Gamma^0$. Then, taking into account conditions on $\Gamma^{ent}\cup\Gamma^{ex}$ assume

$$\mathbf{w}\cdot\mathbf{n}=0,\ \ \mathbf{x}\in\Gamma. \tag{29}$$

Note that (28) and (29) are not the boundary conditions for differential equations (the third and fourth equalities (16) are just reformulations) and their main purpose is to point out the set of functions, in which $L_2$ -projection will be performed in the formulation of the generalized problem.

## 2. Mixed formulation

Similar to previous paper [10] we are going to apply mixed Finite Element Method so in this paragraph weak mixed formulation is presented. Let us proceed to the construction of the projection form of the equations (16) which is the basis for the application of the finite mixed elements. The main feature of the system given below is that integral analogues of third and fourth equations of (16) are not differential equations. Corresponding integral identities define the procedure of projection in the space of vector-functions $\mathbf{L}_2(\Omega)=(L_2(\Omega))^d$ on a closed subspace

$$\mathbf{H}(\mathrm{div},\Omega)=\left\{\mathbf{u}\in\mathbf{L}_2(\Omega)\mid\nabla\cdot\mathbf{u}\in L_2(\Omega)\right\},$$

norm in which is given by

$$\|\mathbf{u}\|^2_{\mathrm{div}}=\int\limits_{\Omega}\left(\mathbf{u}\cdot\mathbf{u}+(\nabla\cdot\mathbf{u})^2\right)\mathrm{d}\mathbf{x}.$$

Further, let $\mathbf{H}_0(\mathrm{div},\Omega)$ be a closure corresponding to this norm of smooth vector functions, such that almost everywhere on $\Gamma$ the following equality takes place $\mathbf{u}\cdot\mathbf{n}=0$. In a standard way (see [5]) from problem (16), (28)-(29) we get the following weak mixed formulation for the Neumann problem: *for a given water saturation $s^0\in L_2(\Omega)$ find continuous in parameter $t>0$ functions $s(t)\in L_2(\Omega)$, $\psi(t)\in L_2(\Omega)/\mathrm{const}$, $\mathbf{v}(t)$, $\mathbf{w}(t)$, $\mathbf{v}_2(t)\in\mathbf{H}(\mathrm{div},\Omega)$ such as $s(0)=s^0$, $\partial s/\partial t(t)\in L_2(\Omega)$, boundary condition (2.21)-(2.23) holds, and for any $t>0$ following integral* *identities take place*

$$\int\limits_{\Omega} \frac{1}{k(s^j)}\,\mathbf{v^j}\cdot\mathbf{u}\,\mathrm{d}\mathbf{x} - \int\limits_{\Omega} \psi^j\,\nabla\cdot\mathbf{u}\,\mathrm{d}\mathbf{x} = \int\limits_{\Omega} \mathbf{G}(s^j)\cdot\mathbf{u}\,\mathrm{d}\mathbf{x} \quad \forall\,\mathbf{u}\in\mathbf{H}_0(\mathrm{div},\Omega),$$

$$\int\limits_{\Omega} \xi\,\nabla\cdot\mathbf{v^j}\,\mathrm{d}\mathbf{x} = \int\limits_{\Omega} \xi\,(\mathbf{q^j_1}+\mathbf{q^j_2})\,\mathrm{d}\mathbf{x} \quad \forall\,\xi\in L_2(\Omega)/\mathrm{const},$$

$$\int\limits_{\Omega} \frac{1}{k(s^j)}\,\mathbf{w^j}\cdot\mathbf{u}\,\mathrm{d}\mathbf{x} = -\int\limits_{\Omega} \sigma(s^j)\,\nabla\cdot\mathbf{u}\,\mathrm{d}\mathbf{x} \quad \forall\,\mathbf{u}\in\mathbf{H}_0(\mathrm{div},\Omega),$$

$$\int\limits_{\Omega} \mathbf{v}^j_2\cdot\mathbf{u}\,\mathrm{d}\mathbf{x} - \int\limits_{\Omega} \frac{k_2(s^j)}{k(s^j)}\,(\mathbf{v}^j-\mathbf{w}^j)\cdot\mathbf{u}\,\mathrm{d}\mathbf{x}$$

$$= \int\limits_{\Omega} k_2(s^j)\,\mathbf{G}(s^j)\cdot\mathbf{u}\,\mathrm{d}\mathbf{x} \quad \forall\,\mathbf{u}\in\mathbf{H}_0(\mathrm{div},\Omega), \tag{30}$$

$$\int\limits_{\Omega} m\frac{\partial s^j}{\partial t}\,\xi\,\mathrm{d}\mathbf{x} + \int\limits_{\Omega} \xi\,\nabla\cdot\mathbf{v}^j_2\,\mathrm{d}\mathbf{x} = \int\limits_{\Omega} \xi\,q^j_2\,\mathrm{d}\mathbf{x} \quad \forall\,\xi\in L_2(\Omega),$$

$$\int\limits_{\Omega} \gamma(P^2-P^1)\,\mathrm{d}\mathbf{x} = \int\limits_{\Omega} \mathbf{q}\,\mathrm{d}\mathbf{x},$$

$$\int\limits_{\Omega} (q^1_i+q^2_2)\,\mathrm{d}\mathbf{x} = 0,$$

$$\int\limits_{\Omega} P^j\,\mathrm{d}\mathbf{x} = \int\limits_{\Omega} (s^j_1 p^j_1 + s^j_2 p^j_2)\,\mathrm{d}\mathbf{x},$$

$$q^1_i = \begin{cases} q s^1_i, & q>0,\\ q s^2_i, & q<0. \end{cases}$$

Note that the vector function $\mathbf{w}(t)$ is actually sought as the element of the subspace $\mathbf{H}_0(\mathrm{div},\Omega)$. All spatial derivatives are included in the system (30) as the only divergence of unknown vector fields. This opens the way to the application of the mixed finite element method. It should be emphasized that an important step in the preparation of (30) having this property, was the introduction of the notation $\mathbf{w}$ (third equality in (16)). It is this step that allows us to search the saturation $s(t)$ as a function from $L_2(\Omega)$. Finally, inequality (12) provides a coercivity condition on the set of solenoidal functions: for an arbitrary vector function $\mathbf{u}\in\mathbf{H}_(\mathrm{div},\Omega)$ such as $\mathrm{div}\,\mathbf{u}=0$, the following inequality takes place

$$\int\limits_{\Omega} \frac{1}{k(s)}\,\mathbf{u}\cdot\mathbf{u}\,\mathrm{d}\mathbf{x} \geq \frac{k_0}{(\sqrt{\mu_1}+\sqrt{\mu_2})^2}\,\|\mathbf{u}\|^2_{\mathrm{div}},$$

which, in addition to *inf–sup* condition (see [5]), is the key to investigation of the unique solvability of problem(30).

## 3. Conclusion

We introduce a model that describes motion of a two-phase fluid in fractured porous media. The key difference of constructed equations from existing analogues is introduction of a two-speed medium such that in each point unknowns corresponding to both cracked and pore medium are presented. Exchange(motion) between mediums is described by analogue of Newton's law of heat transfer. All first-degree equations are written in terms of the saturation-pressure.

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Alexander Alexanderovich Kalinkin
Institute of Computational Mathematics
and Mathematical Geophysics SB RAS,
pr. Lavrenteva 6,
630090, Novosibirsk, Russia
*E-mail address*: alexander.a.kalinkin@gmail.com

Yury Mironovich Laevsky
Institute of Computational Mathematics
and Mathematical Geophysics SB RAS,
pr. Lavrenteva 6,
630090, Novosibirsk, Russia
*E-mail address*: yuri.laevsky@gmail.com