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AUTOMORPHISMS OF A DISTANCE-REGULAR GRAPH WITH INTERSECTION ARRAY $\{100, 66, 1; 1, 33, 100\}$

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Abstract. A. A. Makhnev and D. V. Paduchikh have found intersection arrays of distance-regular graphs, in which neighborhoods of vertices are strongly-regular graphs with second eigenvalue 3. A. A. Makhnev suggested the program to research of automorphisms of these distance-regular graphs. In this paper it is obtained possible orders and subgraphs of fixed points of automorphisms of a hypothetical distance-regular graph with intersection array $\{100, 66, 1; 1, 33, 100\}$. In particular, this graph does not vertex symmetric.

Keywords: distance-regular graph, vertex symmetric graph.

1. Introduction

We consider undirected graphs without loops and multiple edges. Given a vertex a in a graph Γ, we denote by $\Gamma_i(a)$ the subgraph induced by Γ on the set of all vertices, that are at a distance i from a. The subgraph $[a] = \Gamma_1(a)$ is called the *neighbourhood of the vertex a*.

A. A. Makhnev and D. V. Paduchikh have found [1] intersection arrays of distance-regular graphs, in which neighborhoods of vertices are strongly-regular graphs with second eigenvalue 3. A. A. Makhnev suggested the program to research of automorphisms of these distance-regular graphs. In this moment only cases

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$\{100, 66, 1; 1, 33, 100\}$, $\{176, 150, 1; 1, 25, 176\}$ and $\{256, 204, 1; 1, 51, 256\}$ are not investigated.

In this paper it is obtained possible orders and subgraphs of fixed points of automorphisms of a hypothetical distance-regular graph with intersection array $\{100, 66, 1; 1, 33, 100\}$. Possible automorphisms of a strongly-regular graph with parameters (100,33,8,12) found in [2].

Proposition 1. *Let* Γ *be a strongly-regular graph with parameters* $(100, 33, 8, 12)$*,* g *be an element of prime order* p *in* $\mathrm{Aut}(\Gamma)$ *and* $\Delta = \mathrm{Fix}(g)$*. Then* $\pi(G) \subseteq \{2, 3, 5, 11\}$ *and one of the following assertions holds:*

(1) Δ *is empty graph,* $p = 2$ *and* $\alpha_1(g) = 20t$ *or* $p = 5$ *and* $\alpha_1(g) = 0, 50, 100$*;*

(2) Δ *is* 4*-clique,* $p = 2$ *and* $\alpha_1(g) - 12$ *is divided by* 20*, or* $p = 3$ *and* $\alpha_1(g) - 12$ *is divided by* 30*;*

(3) Δ *is* γ*-coclique and either*

(*i*) $\gamma = 1$*,* $p = 3$ *and* $\alpha_1(g) = 3, 33, 63, 93$ *or* $p = 11$ *and* $\alpha_1(g) = 33$*, or*

(*ii*) $p = 3$*,* $\gamma = 4, 7, ..., 16$ *and* $\alpha_1(g) - 3\gamma$ *is divided by* 30*;*

(4) Δ *is an union of* $n \geq 2$ *isolated* m_i*-cliques,* $p = 2$*,* $m_i \in \{2, 4\}$ *and* $|\Delta| \leq 20$*;*

(5) Δ *contains geodesic* 2*-path and either*

(*i*) $p = 3$*,* $|\Delta| = 3l + 1$*,* $3 \leq l \leq 8$*, or*

(*ii*) $p = 2$*,* $|\Delta| = 2e$*,* $3 \leq e \leq 20$ *and* $a^{\perp}$ *is not in* Δ *for each vertex* $a \in \Delta$*.*

Graph Γ with intersection array $\{100, 66, 1; 1, 33, 100\}$ has $v = 1 + 100 + 200 + 2 = 303$ vertices and spectrum $100^1, 10^{101}, -1^{100}, -10^{101}$. By the Hoffman-Delsart's boundary we have maximal order of clique in Γ is not any more than $1+100/10 = 11$.

Theorem 1. *Let* Γ *be a distance-regular graph with intersection array* $\{100, 66, 1; 1, 33, 100\}$*,* $G = \mathrm{Aut}(\Gamma)$*,* g *be an element of* G *with prime order* p *and* $\Omega = \mathrm{Fix}(g)$ *contains exactly* s *vertices in* t *antipodal classes. Then* $\pi(G) \subseteq \{2, 3, 5, 7, 11, 29, 31, 101\}$ *and one of the following assertions holds:*

(1) Ω *is an empty graph and either* $p = 101$*,* $\alpha_1(g) = 101$*, or* $p = 3$*,* $\alpha_1(g) = 60m + 27l + 21$*;*

(2) $p = 31$*,* Ω *is a distance-regular graph with intersection array* $\{7, 4, 1; 1, 2, 7\}$*;*

(3) $p = 29$*,* Ω *is a distance-regular graph with intersection array* $\{13, 8, 1; 1, 4, 13\}$*;*

(4) $p = 11$ *and* $t = 2, 13, 24$*;*

(5) $p = 7$ *and either* Ω *is a distance-regular graph with intersection array* $\{16, 10, 1; 1, 5, 16\}$*, or* $t = 24, 31$*;*

(6) $p = 5$ *and* $t = 1, 16, 21, 26, 31$*;*

(7) $p = 3$*,* $s = 3$ *and* $t = 2, 5, ..., 32$*;*

(8) $p = 2$*,* t *is odd and either* $s = 3$*,* $t = 1, 3, 5, ..., 33$*, or* $s = 1$ *and* $t = 1, 3, 5, ..., 11$.

Theorem 2. *Let* Γ *be a distance-regular graph with intersection array* $\{100, 66, 1; 1, 33, 100\}$*, in which neighbourhoods of vertices are strongly-regular graphs with parameters* $(100, 33, 8, 12)$*,* $G = \mathrm{Aut}(\Gamma)$*,* g *be an element of* G *with prime order* $p > 2$ *and* $\Omega = \mathrm{Fix}(g)$ *is not empty graph, which contains* s *vertices in* t *antipodal classes. Then* $\pi(G) \subseteq \{2, 3, 11, 101\}$ *and one of the following assertions holds:*

(1) $p = 11$*,* $s = 3$ *and* $t = 2$*;*

(2) $p = 3$*,* $s = 3$ *and either* $t = 5$*,* Ω *is an union of isolated* 5*-cliques, or* $t = 5, 8, ..., 17$ *and neighbourhoods of vertices in* Ω *are cocliques, or* $t = 11, 14, ..., 26$ *and neighbourhood of any vertex in* Ω *contains geodesic* 2*-path;*

(3) $p = 2$, *either* Ω *contained in antipodal class, or* $t = 5$ *and* Ω *is is an union of isolated* 5*-cliques and* $s = 1, 3$*, or neighbourhoods of vertices in* Ω *are unions of isolated cliques and* $s = 3$*,* $t = 3, 5$*, or neighbourhood of any vertex in* Ω *contains geodesic* 2*-path and* $s = 3$*,* $t = 7, 9, ..., 33$.

Corollary 1. *Distance-regular graph with intersection array* $\{100, 66, 1; 1, 33, 100\}$ *is not vertex symmetric.*

2. Automorphisms of a distance-regular graph with intersection array $\{100, 66, 1; 1, 33, 100\}$

Lemma 1. *Let* Γ *be a strongly-regular graph with parameters* $(v, k, 8, 12)$*. Then* Γ *has parameters* $(16, 12, 8, 12)$*,* $(100, 33, 8, 12)$ *or* $(1000, 108, 8, 12)$*.*

Proof. If Γ is a strongly-regular graph with parameters $(v, k, 8, 12)$, then $16+4(k-12) = 4u^2$, therefore $k = u^2 + 8$, Γ has secondary eigenvalues $u + 2$, $-(u + 2)$ and a multiplicity of $-(u + 2)$ is equal $(u + 1)(u^2 + 8)(u^2 + u + 10)/24u$. From here Γ has parameters $(16, 12, 8, 12)$, $(100, 33, 8, 12)$ or $(1000, 108, 8, 12)$. □

The proof of the theorem 1 use Higmen's method for investigation automorphisms of distance-regular graphs, represented in third chapter in Cameron's monograph [3]. Let Γ be a distance-regular graph of diameter d with v vertices. Then we have the symmetric association scheme $(X, \mathbf{R})$ with d classes, where X is the set of vertices of Γ and $R_i = \{(u, w) \in X^2 \mid d(u, w) = i\}$. For vertex $u \in X$ set $k_i = |\Gamma_i(u)|$. Let A_i be the adjacency matrix of the graph Γ_i. Then $A_iA_j = \sum p^l_{ij} A_l$ for some integer numbers $p^l_{ij} \geq 0$, which are called the intersection numbers. Note that $p^l_{ij} = |\Gamma_i(u) \cap \Gamma_j(w)|$ for every vertices u, w with $d(u, w) = l$.

Let P_i be the matrix in which in the (j, l) entry there is p^l_{ij}. Then the eigenvalues $k = p_1(0), ..., p_1(d)$ of the matrix P_1 are eigenvalues of Γ with multiplicities $m_0 = 1, ..., m_d$. Note that the matrix P_i is the value of some integer polinom of P_1, so the ordering of eigenvalues of the matrix P_1 gives the ordering of eigenvalues of P_i. The matrices P and Q with (i, j) entry $p_j(i)$ and $Q_{ji} = m_jp_i(j)/k_i$ are called the first and the second eigenmatrix of Γ and $PQ = QP = vI$, where I is the identity matrix of order $d + 1$. Let u_j and w_j be the left and the right eigenvectors of matrix P_1 affording eigenvalue $p_1(j)$ and having the first coordinate 1. Then the multiplicity m_j of the eigenvalue $p_1(j)$ is equal $v/\langle u_j, w_j\rangle$. In fact, w_j are the columns of the matrix P and m_ju_j are the rows of the matrix Q.

The permutation representation of the group $G = \mathrm{Aut}(\Gamma)$ on the vertex set of Γ naturally gives the matrix representation ψ of G in $GL(v, \mathbb{C})$. The space $\mathbb{C}^v$ is the orthogonal direct sum of the eigenspaces $W_0, W_1, ..., W_d$ of the adjacent matrix $A = A_1$ of Γ. For every $g \in G$ we have $\psi(g)A = A\psi(g)$, so the subspace W_i is $\psi(G)$-invariant. Let χ_i be a character of the representation ψ_{W_i}. Then for $g \in G$ we obtain $\chi_i(g) = v^{-1} \sum_{j=0}^{d} Q_{ij}\alpha_j(g)$, where $\alpha_j(g)$ is the numbers of vertices x of X such that $d(x, x^g) = j$.

Let to the end of paper Γ be a distance-regular graph with intersection array $\{100, 66, 1; 1, 33, 100\}$, $G = \mathrm{Aut}(\Gamma)$, g be an element of prime order p in G and $\Omega = \mathrm{Fix}(g)$ intersects t antipodal classes along s vertices. In case $p > 3$ we have $\alpha_3(g) = 0$ and each vertex from $\Gamma - \Omega$ is adjacency with t vertices from Ω. Let F is an antipodal class, containing vertex $a \in \Omega$, $\Omega \cap F = \{a, a_2, ..., a_s\}$ and $b \in \Omega(a)$.

Lemma 2. *Let χ_1 be a character of projection of the representation ψ on subspace of dimension 101(that responds to eigenvalue θ_1), χ_2 be a character of projection of the representation ψ on subspace of dimension 100. Then $\chi_1(g) = (7\alpha_0(g)+\alpha_1(g)-3\alpha_3(g)-101)/20$, $\chi_2(g) = (\alpha_0(g)+\alpha_3(g))/3-1$, $\chi_1(g)-101$ and $\chi_2(g)-100$ are divided by p.*

Proof. We have

$$Q = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 101 & 101/10 & -101/20 & -101/2 \\ 100 & -1 & -1 & 100 \\ 101 & -101/10 & 101/20 & -101/2 \end{pmatrix}.$$

Therefore $\chi_1(g) = (20\alpha_0(g) + 2\alpha_1(g) - \alpha_2(g) - 10\alpha_3(g))/60$. By substituting $\alpha_2(g) = 303-\alpha_0(g)-\alpha_1(g)-\alpha_3(g)$, we obtain $\chi_1(g) = (7\alpha_0(g)+\alpha_1(g)-3\alpha_3(g)-101)/20$.

Similarly, $\chi_2(g) = (100\alpha_0(g) - \alpha_1(g) - \alpha_2(g) + 100\alpha_3(g))/303$. By substituting $\alpha_1(g)+\alpha_2(g) = 303-\alpha_0(g)-\alpha_3(g)$, we obtain $\chi_2(g) = (\alpha_0(g)+\alpha_3(g))/3-1$.

The other conclusions are follows from the lemma 1 [4]. The lemma is proved. □

Lemma 3. *The following assertions hold:*

(1) *if Ω is an empty graph, then either $p = 101$, $\alpha_1(g) = 101$, or $p = 3$, $\alpha_1(g) = 60m+27l+21$;*

(2) *if Ω is a clique, then $p = 2$ and $|\Omega| \in \{1,3,...,11\}$;*

(3)*if $t = 1$, then either $p = 2$, $s = 1,3$ and $\alpha_1(g) = 40l+10(s-1)$, or $p = 5$, $s = 3$ and $\alpha_1(g) = 100l$.*

Proof. Let Ω is an empty graph and $\alpha_i(g) = pw_i$ for $i \geq 1$. As $v = 303$, then p is equal 3 or 101.

Let $p = 101$, then $w_3 = 0$ and $\chi_1(g) = 101(w_1-1)/20$. From here $w_1 = 1$.

Let $p = 3$, then $w_1+w_2+w_3 = 101$, $\chi_2(g) = w_3-1$ and $\chi_2(g)-100$ is divided by 3. From here $w_3 = 3l+2$, $\chi_1(g) = 3(w_1-9l-107)/20$ and $w_1 = 20m+9l+7$.

Let Ω is t-coclique, then $s = 1$, p divides $3-s$ and $101-t$. From here $p = 2$, t is odd, therefore $t \in \{1,3,...,11\}$. Let $t = 1$, then $p = 2,5$. In case $p = 5$ we have $\chi_1(g) = \alpha_1(g)/20-4$ and $\alpha_1(g) = 100l$. In case $p = 2$ we have $s = 1$ and $\chi_1(g) = \alpha_1(g)/20-5$ and $\alpha_1(g) = 40l$ or $s = 3$ and $\chi_1(g) = \alpha_1(g)/20-4$ and $\alpha_1(g) = 40l+20$. □

Lemma 4. *(Theorem 5.4 from [5]). Let Γ be a distance-regular graph with diameter 3 and with $\lambda = \mu$. Let n^* is square-free part of natural number n. Then:*

(1) *if $k \equiv 1$ (mod 4) and r is even, then $p \equiv 1$ (mod 4) for each odd prime of number p divisible by k^*;*

(2) *if k is even, then $(-1)^{(r-1)/2}r$ is a square by the module p for each odd prime of number p, divisible by k^*.*

Lemma 5. *If $p > 7$, then one of the following assertions holds:*

(1) *$p = 31$, Ω is a distance-regular graph with intersection array $\{7,4,1;1,2,7\}$;*

(2) *$p = 29$, Ω is a distance-regular graph with intersection array $\{13,8,1;1,4,13\}$;*

(3) *$p = 11$ and $t = 2,13,24$.*

Proof. If $p > 31$, then for vertices $a, b \in \Omega$ with condition $d(a,b) \leq 2$, subgraph $[a] \cap [b]$ contains in Ω. In this case Ω is a distance-regular graph with intersection array $\{t-1, 66, 1; 1, 33, t-1\}$, contradiction.

Let $p = 31$, then $101 - t$ is divided by 31 and $t \in \{8, 39, 70\}$. As each vertex from $\Gamma - \Omega$ is adjacency with t vertices from Ωthen $t = 8$. For each two vertices $a, b \in \Omega$ such that $d(a,b) \leq 2$, the number $|\Omega(a) \cap [b]|$ is equivalent with 2 by the module 31. Therefore $\Omega(b)$ contains 3 vertices from $a^{\perp}$, 2 vertices from $[a_2]$, 2 vertices from $[a_3]$ and Ω is a distance-regular graph with intersection array $\{7, 4, 1; 1, 2, 7\}$.

Let $p = 29$, then $101-t$ is divided by 29 and $t \in \{14, 43, 72\}$. As each vertex from $\Gamma-\Omega$ is adjacency with t vertices from Ω, then $t = 14$. For each two vertices $a, b \in \Omega$ such that $d(a,b) \leq 2$, the number $|\Omega(a) \cap [b]|$ is equivalent with 4 by the module 29. Therefore $\Omega(b)$ contains 5 vertices from $a^{\perp}$, 4 vertices from $[a_2]$, 4 vertices from $[a_3]$ and Ω is a distance-regular graph with intersection array $\{13, 8, 1; 1, 4, 13\}$.

Contradictions are obtained in cases $p = 17, 19, 23$.

Let $p = 13$, then $101-t$ is divided by 13 and $t = 10, 23$. For each two vertices $a, b \in \Omega$ such that $d(a,b) \leq 2$, the number $|\Omega(a) \cap [b]|$ is equivalent with 7 by the module 13. therefore $\Omega(b)$ contains 8 vertices from $a^{\perp}$, 7 vertices from $[a_2]$, 7 vertices from $[a_3]$ and Ω is a distance-regular graph with intersection array $\{22, 14, 1; 1, 7, 22\}$. By the lemma 4 this graph does not exist.

Let $p = 11$, then $101 - t$ is divided by 11 and $t = 2, 13, 24$. □

Lemma 6. *The following assertions hold:*

(1) *if $p = 7$, then either Ω is a distance-regular graph with intersection array $\{16, 10, 1; 1, 5, 16\}$, or $t = 24, 31$;*

(2) *if $p = 5$, then $t = 16, 21, 26, 31$;*

(3) *if $p = 3$, then $s = 3$ and $t = 2, 5, ..., 32$;*

(4) *if $p = 2$, then t is odd and either $s = 3$ and $t = 3, 5, ..., 33$, or $s = 1$ and $t = 3, 5, ..., 11$.*

Proof. Let $p = 7$,then $101 - t$ is divided by 7 and $t \in \{3, 10, ..., 31\}$. For each two vertices $a, b \in \Omega$ such that $d(a,b) \leq 2$, the number $|\Omega(a) \cap [b]|$ is equivalent with 5 by the module 7. Therefore $\Omega(b)$ contains 6 vertices from $a^{\perp}$, 5 vertices from $[a_2]$, 5 vertices from $[a_3]$, $t \geq 17$ and in case $t = 17$ graph Ω is a distance-regular graph with intersection array $\{16, 10, 1; 1, 5, 16\}$.

Let $p = 5$. Then $101 - t$ is divided by 5 and $t \in \{6, 11, ..., 31\}$. For each two vertices $a, b \in \Omega$ such that $d(a,b) \leq 2$, the number $|\Omega(a) \cap [b]|$ is equivalent with 3 by the module 5. Therefore $\Omega(b)$ contains 4 vertices from $a^{\perp}$, 3 vertices from $[a_2]$, 3 vertices from $[a_3]$, $t \geq 11$ and in case $t = 11$ Ω is a distance-regular graph with intersection array $\{10, 6, 1; 1, 3, 10\}$. By the lemma 4 this graph does not exist.

Let $p = 3$, then $101 - t$ is divided by 3, $s = 3$ and $t \in \{2, 5, ..., 32\}$. For each two vertices $a, b \in \Omega$ such that $d(a,b) \leq 2$, the number $|\Omega(a) \cap [b]|$ is divided by 3.

Let $p = 2$, then t is odd and for each vertex $u \in \Gamma - \Omega$ the number $[u] \cap \Omega$ is even.

If $s = 3$, then each vertex from $\Gamma-\Omega$ is adjacent with t vertices from Ω, therefore $t = 3, 5, ..., 33$.

Let $s = 1$, then Ω is clique and $t = 3, 5, ..., 11$. Lemma and the theorem 1 are proved. □

3. Distance-regular graph with strongly regular local subgraphs

Let us prove the theorem 2. Let Γ be a distance-regular graph with intersection array $\{100, 66, 1; 1, 33, 100\}$, in which neighbourhoods of vertices are strongly-regular with parameters $(100, 33, 8, 12)$. Then the order of clique in Γ is not greater then 5. By the theorem 1 we have $p \in \{2, 3, 5, 11, 101\}$. Let $a \in \Omega$, $\Delta = \Omega(a)$ and $\alpha'_i(g) = |\{u \in [a] - \Delta \mid d(u, u^g) = i\}|$. By the proposition one of the following assertions holds:

(1) Δ is empty graph, $p = 2$ and $\alpha'_1(g) = 20t$ or $p = 5$ and $\alpha'_1(g) = 0, 50, 100$;

(2) Δ is 4-clique, $p = 2$ and $\alpha'_1(g) - 12$ is divided by 20, or $p = 3$ and $\alpha'_1(g) - 12$ is divided by 30;

(3) Δ is γ-coclique and either

(*i*) $\gamma = 1$, $p = 3$ and $\alpha'_1(g) = 3, 33, 63, 93$ or $p = 11$ and $\alpha'_1(g) = 33$, or

(*ii*) $p = 3$, $\gamma = 4, 7, ..., 16$ and $\alpha'_1(g) - 3\gamma$ is divided by 30;

(4) Δ is an union $n \geq 2$ of isolated m_i-cliques, $p = 2$, $m_i \in \{2, 4\}$ and $|\Delta| \leq 20$;

(5) Δ contains geodesic 2-path and either

(*i*) $p = 3$, $|\Delta| = 3l + 1$, $3 \leq l \leq 8$, or

(*ii*) $p = 2$, $|\Delta| = 2e$, $3 \leq e \leq 20$ and $a^\perp$ is not contained in Δ for each vertex $a \in \Delta$.

Lemma 7. *One of the following assertions holds:*

(1) Δ *is empty graph,* $p = 2$ *and* Ω *is contained in antipodal class;*

(2) Δ *is* 4*-clique,* $p = 2, 3$ *and* $t = 5$*;*

(3) Δ *is* γ*-coclique and either* $p = 3, 11$*,* $\gamma = 1$ *and* $t = 2$*, or* $p = 3$ *and* $t = 5, 8, ..., 17$*;*

(4) Δ *is an union of* $n \geq 2$ *isolated* m_i*-cliques,* $p = 2$*,* $t = 3, 5$ *and* $s = 3$*;*

(5) Δ *contains geodesic* 2*-path and either*

(*i*) $p = 3$ *and* $t = 11, 14, ..., 26$*, or*

(*ii*) $p = 2$*,* $s = 3$*,* $t = 7, 9, ..., 33$*.*

Proof. Let Δ is an empty graph. Then $p = 2$ and Ω is contained in antipodal class.

Let Δ is a 4-clique. Then $s = 1, t = 4$ and $p = 2, 3$.

Let Δ is a γ-coclique. Then $t = \gamma + 1$. If $\gamma = 1$, then $p = 3, 11$ and Ω contains an isolated edge. Therefore $t = 2$. If $\gamma > 1$, then $p = 3$ and $t = 5, 8, ..., 17$.

Let Δ is an union of $n \geq 2$ isolated m_i-cliques, $p = 2$, $m_i \in \{2, 4\}$. Then $s = 3$ and $t = 3, 5$.

Let Δ contains geodesic 2-path and $p = 2, 3$. In case $p = 3$ we have $|\Delta| = 3l + 1$, $3 \leq l \leq 8$, therefore $t = 11, 14, ..., 26$. In case $p = 2$ we have $|\Delta| = 2e$, $3 \leq e \leq 20$, therefore $t = 7, 9, ..., 33$. Lemma and the theorem 2 are proved. □

4. Distance-regular graph with intersection array $\{100, 66, 1; 1, 33, 100\}$ is not vertex symmetric

Let us prove the Corollary.

Lemma 8. *One of the following assertions holds:*

(1) *if* f *is an element of order* 101 *from* G*, then* $C_G(f) = \langle f \rangle$*;*

(2) $F(G)$ *is a* 3*-group and either* $F(G) = 1$*, or* $\bar{G} = G/F(G)$ *acts irreducibly on* $F(G)$*;*

(3) *the socle* $\bar{T}$ *of group* $\bar{G}$ *is isomorphic to* $L_2(101)$*,* $U_3(101)$ *or* $U_5(17)$*.*

Proof. By the theorem 1 subgraph Fix(f) is empty graph and $\alpha_1(f) = 101$. Let g be an element of prime order $p < 101$ from $C_G(f)$. From action f to Ω it follows that subgraph Ω is empty graph and $p = 3$. Contradiction with action g on $U = \{u \mid d(u, u^f) = 1\}$.

By the theorem of Gruenberg-Kegel [6] a soluble radical of group G is equiv to $F(G)$. As 3 does not divide $N_G(\langle f\rangle)$, then $F(G)$ is a 3-group. Let $\bar{G} = G/F(G)$. If $F(G) \neq 1$, then $|F(G) : F(G)_a| = 3$ and $\bar{G}$ acts irreducibly on $F(G)$.

By table 3 from [7] a group $\bar{T}$ is isomorphic to $L_2(101)$, $U_3(101)$ and $U_5(17)$. The Lemma is proved. □

Let we finish the proof of the corollary. The order of a group $U_3(101)$ is divided by 101^3. The order of a group $U_5(17)$ is divided by 13. A group $L_2(101)$ does not contain subgroups of index, dividing $101 \cdot 3$. In all cases we have a contradiction. The Corollary is proved.

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