AUTOMORPHISMS OF A DISTANCE-REGULAR GRAPH WITH INTERSECTION ARRAY \(\{100, 66; 1, 1, 33, 100\}\)

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Abstract. A. A. Makhnev and D. V. Paduchikh have found intersection arrays of distance-regular graphs, in which neighborhoods of vertices are strongly-regular graphs with second eigenvalue 3. A. A. Makhnev suggested the program to research of automorphisms of these distance-regular graphs. In this paper it is obtained possible orders and subgraphs of fixed points of automorphisms of a hypothetical distance-regular graph with intersection array \(\{100, 66, 1; 1, 33, 100\}\). In particular, this graph does not vertex symmetric.

Keywords: distance-regular graph, vertex symmetric graph.

1. Introduction

We consider undirected graphs without loops and multiple edges. Given a vertex \(a\) in a graph \(\Gamma\), we denote by \(\Gamma_i(a)\) the subgraph induced by \(\Gamma\) on the set of all vertices, that are at a distance \(i\) from \(a\). The subgraph \([a] = \Gamma_1(a)\) is called the neighborhood of the vertex \(a\).

A. A. Makhnev and D. V. Paduchikh have found [1] intersection arrays of distance-regular graphs, in which neighborhoods of vertices are strongly-regular graphs with second eigenvalue 3. A. A. Makhnev suggested the program to research of automorphisms of these distance-regular graphs. In this moment only cases
Let $\Gamma$ be a strongly-regular graph with parameters $(100, 33, 8, 12)$, $g$ be an element of prime order $p$ in $\text{Aut}(\Gamma)$ and $\Delta = \text{Fix}(g)$. Then $\pi(G) \subseteq \{2, 3, 5, 11\}$ and one of the following assertions holds:

1. $\Delta$ is empty graph, $p = 2$ and $\alpha_1(g) = 20t$ or $p = 5$ and $\alpha_1(g) = 0, 50, 100$;
2. $\Delta$ is 4-clique, $p = 2$ and $\alpha_1(g) - 12$ is divided by 20, or $p = 3$ and $\alpha_1(g) - 12$ is divided by 30;
3. $\Delta$ is $\gamma$-coclique and either
   - $\gamma = 1$, $p = 3$ and $\alpha_1(g) = 3, 33, 63, 93$ or $p = 11$ and $\alpha_1(g) = 33$, or
   - $p = 3$, $\gamma = 4, 7, \ldots, 16$ and $\alpha_1(g) - 3\gamma$ is divided by 30;
4. $\Delta$ is an union of $n \geq 2$ isolated $m_i$-cliques, $p = 2$, $m_i \in \{2, 4\}$ and $|\Delta| \leq 20$;
5. $\Delta$ contains geodesic 2-path and either
   - $p = 3$, $|\Delta| = 3l + 1$, $3 \leq l \leq 8$, or
   - $p = 2$, $|\Delta| = 2e$, $3 \leq e \leq 20$ and $a^i$ is not in $\Delta$ for each vertex $a \in \Delta$.

Graph $\Gamma$ with intersection array $\{100, 66, 1; 1, 33, 100\}$ has $v = 1 + 100 + 200 + 2 = 303$ vertices and spectrum $100^1, 10^{101}, -1^{100}, -10^{101}$. By the Hoffman-Delsart’s boundary we have maximal order of clique in $\Gamma$ is not any more than $1 + 100/10 = 11$.

**Theorem 1.** Let $\Gamma$ be a distance-regular graph with intersection array $\{100, 66, 1; 1, 33, 100\}$, $G = \text{Aut}(\Gamma)$, $g$ be an element of $G$ with prime order $p$ and $\Omega = \text{Fix}(g)$ contains exactly $s$ vertices in $t$ antipodal classes. Then $\pi(G) \subseteq \{2, 3, 5, 7, 11, 29, 31, 101\}$ and one of the following assertions holds:

1. $\Omega$ is an empty graph and either $p = 101$, $\alpha_1(g) = 101$, or $p = 3$, $\alpha_1(g) = 60m + 27l + 21$;
2. $p = 31$, $\Omega$ is a distance-regular graph with intersection array $\{7, 4, 1; 1, 2, 7\}$;
3. $p = 29$, $\Omega$ is a distance-regular graph with intersection array $\{13, 8, 1; 1, 4, 13\}$;
4. $p = 11$ and $t = 2, 13, 24$;
5. $p = 7$ and either $\Omega$ is a distance-regular graph with intersection array $\{16, 10, 1; 1, 5, 16\}$, or $t = 24, 31$;
6. $p = 5$ and $t = 1, 16, 21, 26, 31$;
7. $p = 3$, $s = 3$ and $t = 2, 5, \ldots, 32$;
8. $p = 2$, $t$ is odd and either $s = 3$, $t = 1, 3, 5, \ldots, 33$, or $s = 1$ and $t = 1, 3, 5, \ldots, 11$.

**Theorem 2.** Let $\Gamma$ be a distance-regular graph with intersection array $\{100, 66, 1; 1, 33, 100\}$, in which neighbourhoods of vertices are strongly-regular graphs with parameters $(100, 33, 8, 12)$, $G = \text{Aut}(\Gamma)$, $g$ be an element of $G$ with prime order $p > 2$ and $\Omega = \text{Fix}(g)$ is not empty graph, which contains $s$ vertices in $t$ antipodal classes. Then $\pi(G) \subseteq \{2, 3, 11, 101\}$ and one of the following assertions holds:

1. $p = 11$, $s = 3$ and $t = 2$;
2. $p = 3$, $s = 3$ and either $t = 5$, $\Omega$ is an union of isolated 5-cliques, or $t = 5, 8, \ldots, 17$ and neighbourhoods of vertices in $\Omega$ are cocliques, or $t = 11, 14, \ldots, 26$ and neighbourhood of any vertex in $\Omega$ contains geodesic 2-path;
(3) $p = 2$, either $\Omega$ contained in antipodal class, or $t = 5$ and $\Omega$ is an union of isolated 5-cliques and $s = 1, 3$, or neighbourhoods of vertices in $\Omega$ are unions of isolated cliques and $s = 3, t = 3, 5$, or neighbourhood of any vertex in $\Omega$ contains geodesic 2-path and $s = 3, t = 7, 9, ..., 33$.

Corollary 1. Distance-regular graph with intersection array $\{100, 66, 1; 1, 33, 100\}$ is not vertex symmetric.

2. Automorphisms of a distance-regular graph with intersection array $\{100, 66, 1; 1, 33, 100\}$

Lemma 1. Let $\Gamma$ be a strongly-regular graph with parameters $(\nu, k, \lambda, \mu)$. Then $\Gamma$ has parameters $(16, 12, 8, 12)$, $(100, 33, 8, 12)$ or $(1000, 108, 8, 12)$.

Proof. If $\Gamma$ is a strongly-regular graph with parameters $(\nu, k, \lambda, \mu)$, then $16 + 4(k - 12) = 4\nu^2$, therefore $k = \nu^2 + 8$. Then $\Gamma$ has secondary eigenvalues $u + 2$, $-(u + 2)$ and a multiplicity of $-(u + 2)$ is equal $(u + 1)(u^2 + 8)(u^2 + u + 10)/24u$. From here $\Gamma$ has parameters $(16, 12, 8, 12)$, $(100, 33, 8, 12)$ or $(1000, 108, 8, 12)$. 

The proof of the theorem 1 uses Higmen’s method for investigation automorphisms of distance-regular graphs, represented in third chapter in Cameron’s monograph [3]. Let $\Gamma$ be a distance-regular graph of diameter $d$ with $\nu$ vertices. Then we have the symmetric association scheme $(X, R)$ with $d$ classes, where $X$ is the set of vertices of $\Gamma$ and $R_i = \{(u, v) \in X^2 \mid d(u, v) = i\}$. For vertex $u \in X$ set $k_i = |\Gamma_i(u)|$. Let $A_i$ be the adjacency matrix of the graph $\Gamma_i$. Then $A_iA_j = \sum p_{ij}^l A_l$ for some integer numbers $p_{ij}^l \geq 0$, which are called the intersection numbers. Note that $p_{ij}^l = |\Gamma_i(u) \cap \Gamma_j(w)|$ for every vertices $u, w$ with $d(u, w) = l$.

Let $P_i$ be the matrix in which in the $(j, l)$ entry there is $p_{ij}^l$. Then the eigenvalues $k = p_1(0), ..., p_1(d)$ of the matrix $P_i$ are eigenvalues of $\Gamma$ with multiplicities $m_0 = 1, ..., m_d$. Note that the matrix $P_i$ is the value of some intergral polinom of $P_1$, so the ordering of eigenvalues of the matrix $P_1$ gives the ordering of eigenvalues of $P_i$. The matrices $P$ and $Q$ with $(i, j)$ entry $p_{ij}(i)$ and $q_{ji} = m_jp_j(i)/k_i$ are called the first and the second eigenmatrix of $\Gamma$ and $PQ = QP = eI$, where $I$ is the identity matrix of order $d + 1$. Let $u_j$ and $w_j$ be the left and the right eigenvectors of matrix $P_1$ affording eigenvalue $p_1(j)$ and having the first coordinate 1. Then the multiplicity $m_j$ of the eigenvalue $p_1(j)$ is equal $v/\langle u_j, w_j \rangle$. In fact, $w_j$ are the columns of the matrix $P$ and $m_j u_j$ are the rows of the matrix $Q$.

The permutation representation of the group $G = Aut(\Gamma)$ on the vertex set of $\Gamma$ naturally gives the matrix representation $\psi$ of $G$ in $GL(\nu, \mathbb{C})$. The space $\mathbb{C}^\nu$ is the orthogonal direct sum of the eigenspaces $W_0, W_1, ..., W_d$ of the adjacent matrix $A = A_1$ of $\Gamma$. For every $g \in G$ we have $\psi(g)A = A\psi(g)$, so the subspace $W_i$ is $\psi(G)$-invariant. Let $\chi_i$ be a character of the representation $\psi_{W_i}$. Then for $g \in G$ we obtain $\chi_i(g) = v^{-1}\sum_{j=0}^{d} Q_{ij}q_{ij}(g)$, where $q_{ij}(g)$ is the numbers of vertices $x$ of $X$ such that $d(x, x^g) = j$.

Let to the end of paper $\Gamma$ be a distance-regular graph with intersection array $\{100, 66, 1; 1, 33, 100\}$, $G = Aut(\Gamma)$, $g$ be an element of prime order $p$ in $G$ and $\Omega = \text{Fix}(g)$ intersects $t$ antipodal classes along $s$ vertices. In case $p > 3$ we have $\alpha_3(g) = 0$ and each vertex from $\Gamma - \Omega$ is adjacency with $t$ vertices from $\Omega$. Let $F$ is an antipodal class, containing vertex $a \in \Omega$, $\Omega \cap F = \{a, a_2, ..., a_s\}$ and $b \in \Omega(a)$. 


Lemma 2. Let $\chi_1$ be a character of projection of the representation $\psi$ on subspace of dimension 101 (that responds to eigenvalue $\theta_1$), $\chi_2$ be a character of projection of the representation $\psi$ on subspace of dimension 100. Then $\chi_2(g) = (7\alpha_0(g) + \alpha_1(g) - 3\alpha_3(g) - 101)/20$, $\chi_2(g) = (\alpha_0(g) + \alpha_3(g))/3 - 1$, $\chi_2(g) - 101$ and $\chi_2(g) - 100$ are divided by $p$.

Proof. We have

$$Q = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 101 & 101/10 & -101/20 & -101/2 \\ 100 & -1 & -1 & 100 \\ 101 & -101/10 & 101/20 & -101/2 \end{pmatrix}.$$ 

Therefore $\chi_1(g) = (20\alpha_0(g) + 2\alpha_1(g) - \alpha_2(g) - 10\alpha_3(g))/60$. By substituting $\alpha_2(g) = 303 - \alpha_0(g) - \alpha_1(g) - \alpha_3(g)$, we obtain $\chi_1(g) = (7\alpha_0(g) + \alpha_1(g) - 3\alpha_3(g) - 101)/20$.

Similarly, $\chi_2(g) = (100\alpha_0(g) - \alpha_1(g) - \alpha_2(g) + 100\alpha_3(g))/303$. By substituting $\alpha_1(g) + \alpha_2(g) = 303 - \alpha_0(g) - \alpha_3(g)$, we obtain $\chi_2(g) = (\alpha_0(g) + \alpha_3(g))/3 - 1$.

The other conclusions are follows from the lemma 1 [4]. The lemma is proved. □

Lemma 3. The following assertions hold:

1. if $\Omega$ is an empty graph, then either $p = 101$, $\alpha_1(g) = 101$, or $p = 3$, $\alpha_1(g) = 60m + 27l + 21$.
2. if $\Omega$ is a clique, then $p = 2$ and $|\Omega| \in \{1, 3, \ldots, 11\}$.
3. if $t = 1$, then either $p = 2$, $s = 1, 3$ and $\alpha_1(g) = 40l + 10(s - 1)$, or $p = 5$, $s = 3$ and $\alpha_1(g) = 100l$.

Proof. Let $\Omega$ is an empty graph and $\alpha_i(g) = p\alpha_i$ for $i \geq 1$. As $v = 303$, then $p$ is equal 3 or 101.

Let $p = 101$, then $w_3 = 0$ and $\chi_1(g) = 101(w_1 - 1)/20$. From here $w_1 = 1$.

Let $p = 3$, then $w_1 + w_2 + w_3 = 101$, $\chi_2(g) = w_3 - 1$ and $\chi_2(g) - 100$ is divided by 3. From here $w_3 = 3l + 2$, $\chi_1(g) = 3(w_1 - 9l - 107)/20$ and $w_1 = 20m + 9l + 7$.

Let $\Omega$ is $t$-coclique, then $s = 1$, $p$ divides $3 - s$ and $t - 1 - t$. From here $p = 2$, $t$ is odd, therefore $t \in \{1, 3, \ldots, 11\}$. Let $t = 1$, then $p = 2, 5$. In case $p = 5$ we have $\chi_1(g) = \alpha_1(g)/20 - 4$ and $\alpha_1(g) = 100l$. In case $p = 2$ we have $s = 1$ and $\chi_1(g) = \alpha_1(g)/20 - 5$ and $\alpha_1(g) = 40l$ or $s = 3$ and $\chi_1(g) = \alpha_1(g)/20 - 4$ and $\alpha_1(g) = 40l + 20$. □

Lemma 4. (Theorem 5.4 from [5]). Let $\Gamma$ be a distance-regular graph with diameter 3 and with $\lambda = \mu$. Let $n^*$ is square-free part of natural number $n$. Then:

1. if $k \equiv 1 \pmod{4}$ and $r$ is even, then $p \equiv 1 \pmod{4}$ for each odd prime of number $p$ divisible by $k^*$;
2. if $k$ is even, then $(-1)^{(r-1)/2}r$ is a square by the module $p$ for each odd prime of number $p$, divisible by $k^*$.

Lemma 5. If $p > 7$, then one of the following assertions holds:

1. $p = 31$, $\Omega$ is a distance-regular graph with intersection array $\{7, 4, 1; 1, 2, 7\}$.
2. $p = 29$, $\Omega$ is a distance-regular graph with intersection array $\{13, 8, 1; 1, 4, 13\}$.
3. $p = 11$ and $t = 2, 13, 24$. 
Proof. If \( p > 31 \), then for vertices \( a, b \in \Omega \) with condition \( d(a, b) \leq 2 \), subgraph \( [a] \cap [b] \) contains in \( \Omega \). In this case \( \Omega \) is a distance-regular graph with intersection array \( \{t - 1, 66, 1; 1, 33, t - 1\} \), contradiction.

Let \( p = 31 \), then \( 101 - t \) is divided by 31 and \( t \in \{8, 39, 70\} \). As each vertex from \( \Gamma - \Omega \) is adjacency with \( t \) vertices from \( \Omega \) then \( t = 8 \). For each two vertices \( a, b \in \Omega \) such that \( d(a, b) \leq 2 \), the number \( |\Omega(a) \cap [b]| \) is equivalent with 2 by the module 31. Therefore \( \Omega(b) \) contains 3 vertices from \( a \perp \), 2 vertices from \( [a_2] \), 2 vertices from \( [a_3] \) and \( \Omega \) is a distance-regular graph with intersection array \( \{7, 4, 1; 1, 2, 7\} \).

Let \( p = 29 \), then \( 101 - t \) is divided by 29 and \( t \in \{14, 43, 72\} \). As each vertex from \( \Gamma - \Omega \) is adjacency with \( t \) vertices from \( \Omega \), then \( t = 14 \). For each two vertices \( a, b \in \Omega \) such that \( d(a, b) \leq 2 \), the number \( |\Omega(a) \cap [b]| \) is equivalent with 4 by the module 29. Therefore \( \Omega(b) \) contains 5 vertices from \( a \perp \), 4 vertices from \( [a_2] \), 4 vertices from \( [a_3] \) and \( \Omega \) is a distance-regular graph with intersection array \( \{13, 8, 1; 1, 4, 13\} \).

Contradicitions are obtained in cases \( p = 17, 19, 23 \).

Let \( p = 13 \), then \( 101 - t \) is divided by 13 and \( t = 10, 23 \). For each two vertices \( a, b \in \Omega \) such that \( d(a, b) \leq 2 \), the number \( |\Omega(a) \cap [b]| \) is equivalent with 7 by the module 13, therefore \( \Omega(b) \) contains 8 vertices from \( a \perp \), 7 vertices from \( [a_2] \), 7 vertices from \( [a_3] \) and \( \Omega \) is a distance-regular graph with intersection array \( \{22, 14, 1; 1, 7, 22\} \). By the lemma 4 this graph does not exist.

Let \( p = 11 \), then \( 101 - t \) is divided by 11 and \( t = 2, 13, 24 \).

Lemma 6. The following assertions hold:

1. if \( p = 7 \), then either \( \Omega \) is a distance-regular graph with intersection array \( \{16, 10, 1; 1, 5, 16\} \), or \( t = 24, 31 \);
2. if \( p = 5 \), then \( t = 16, 21, 26, 31 \);
3. if \( p = 3 \), then \( s = 3 \) and \( t = 2, 5, ..., 32 \);
4. if \( p = 2 \), then \( t \) is odd and either \( s = 3 \) and \( t = 3, 5, ..., 33 \), or \( s = 1 \) and \( t = 3, 5, ..., 11 \).

Proof. Let \( p = 7 \), then \( 101 - t \) is divided by 7 and \( t \in \{3, 10, ..., 31\} \). For each two vertices \( a, b \in \Omega \) such that \( d(a, b) \leq 2 \), the number \( |\Omega(a) \cap [b]| \) is equivalent with 5 by the module 7, therefore \( \Omega(b) \) contains 6 vertices from \( a \perp \), 5 vertices from \( [a_2] \), 5 vertices from \( [a_3] \), \( t \geq 17 \) and in case \( t = 17 \) graph \( \Omega \) is a distance-regular graph with intersection array \( \{16, 10, 1; 1, 5, 16\} \).

Let \( p = 5 \). Then \( 101 - t \) is divided by 5 and \( t \in \{6, 11, ..., 31\} \). For each two vertices \( a, b \in \Omega \) such that \( d(a, b) \leq 2 \), the number \( |\Omega(a) \cap [b]| \) is equivalent with 3 by the module 5, therefore \( \Omega(b) \) contains 4 vertices from \( a \perp \), 3 vertices from \( [a_2] \), 3 vertices from \( [a_3] \), \( t \geq 11 \) and in case \( t = 11 \) graph \( \Omega \) is a distance-regular graph with intersection array \( \{10, 6, 1; 1, 3, 10\} \). By the lemma 4 this graph does not exist.

Let \( p = 3 \), then \( 101 - t \) is divided by 3, \( s = 3 \) and \( t \in \{2, 5, ..., 32\} \). For each two vertices \( a, b \in \Omega \) such that \( d(a, b) \leq 2 \), the number \( |\Omega(a) \cap [b]| \) is divided by 3.

Let \( p = 2 \), then \( t \) is odd and for each vertex \( u \in \Gamma - \Omega \) the number \( |u| \cap \Omega \) is even.

If \( s = 3 \), then each vertex from \( \Gamma - \Omega \) is adjacent with \( t \) vertices from \( \Omega \), therefore \( t = 3, 5, ..., 33 \).

Let \( s = 1 \), then \( \Omega \) is clique and \( t = 3, 5, ..., 11 \). Lemma and the theorem 1 are proved.
3. Distance-regular graph with strongly regular local subgraphs

Let us prove the theorem 2. Let $\Gamma$ be a distance-regular graph with intersection array $\{100, 66, 1; 1, 33, 100\}$, in which neighbourhoods of vertices are strongly-regular with parameters $(100, 33, 8, 12)$. Then the order of clique in $\Gamma$ is not greater than 5. By the theorem 1 we have $p \in \{2, 3, 5, 11, 101\}$. Let $a \in \Omega$, $\Delta = \Omega(a)$ and $\alpha'_1(g) = |\{u \in [a] - \Delta \mid d(u, u^g) = i\}|$. By the proposition one of the following assertions holds:

1. $\Delta$ is empty graph, $p = 2$ and $\alpha'_1(g) = 20l$ or $p = 5$ and $\alpha'_1(g) = 0, 50, 100$;
2. $\Delta$ is 4-clique, $p = 2$ and $\alpha'_1(g) - 12$ is divided by 20, or $p = 3$ and $\alpha'_1(g) - 12$ is divided by 30;
3. $\Delta$ is $\gamma$-coclique and either
   - $\gamma = 1$, $p = 3$ and $\alpha'_1(g) = 3, 33, 63, 93$ or $p = 11$ and $\alpha'_1(g) = 33$, or
   - $\gamma = 4, 7, \ldots, 16$ and $\alpha'_1(g) - 3\gamma$ is divided by 30;
4. $\Delta$ is an union $\geq 2$ of isolated $m_i$-cliques, $p = 2$, $m_i \in \{2, 4\}$ and $|\Delta| \leq 20$;
5. $\Delta$ contains geodesic 2-path and either
   - $p = 3$, $|\Delta| = 3l + 1$, $3 \leq l \leq 8$, or
   - $p = 2$, $|\Delta| = 2e$, $3 \leq e \leq 20$ and $a^t$ is not contained in $\Delta$ for each vertex $a \in \Delta$.

Lemma 7. One of the following assertions holds:

1. $\Delta$ is empty graph, $p = 2$ and $\Omega$ is contained in antipodal class;
2. $\Delta$ is 4-clique, $p = 2, 3$ and $t = 5$;
3. $\Delta$ is $\gamma$-coclique and either $p = 3, 11$, $\gamma = 1$ and $t = 2$, or $p = 3$ and $t = 5, 8, \ldots, 17$;
4. $\Delta$ is an union $\geq 2$ isolated $m_i$-cliques, $p = 2$, $t = 3, 5$ and $s = 3$;
5. $\Delta$ contains geodesic 2-path and either
   - $p = 3$ and $t = 11, 14, \ldots, 26$, or
   - $p = 2$, $s = 3$, $t = 7, 9, \ldots, 33$.

Proof. Let $\Delta$ is an empty graph. Then $p = 2$ and $\Omega$ is contained in antipodal class. Let $\Delta$ is a 4-clique. Then $s = 1, t = 4$ and $p = 2, 3$.

Let $\Delta$ is a $\gamma$-coclique. Then $t = \gamma + 1$. If $\gamma = 1$, then $p = 3, 11$ and $\Omega$ contains an isolated edge. Therefore $t = 2$. If $\gamma > 1$, then $p = 3$ and $t = 5, 8, \ldots, 17$.

Let $\Delta$ is an union of $\geq 2$ isolated $m_i$-cliques, $p = 2$, $m_i \in \{2, 4\}$. Then $s = 3$ and $t = 3, 5$.

Let $\Delta$ contains geodesic 2-path and $p = 2, 3$. In case $p = 3$ we have $|\Delta| = 3l + 1$, $3 \leq l \leq 8$, therefore $t = 11, 14, \ldots, 26$. In case $p = 2$ we have $|\Delta| = 2e$, $3 \leq e \leq 20$, therefore $t = 7, 9, \ldots, 33$. Lemma and the theorem 2 are proved.

4. Distance-regular graph with intersection array $\{100, 66, 1; 1, 33, 100\}$ is not vertex symmetric

Let us prove the Corollary.

Lemma 8. One of the following assertions holds:

1. if $f$ is an element of order 101 from $G$, then $C_G(f) = \langle f \rangle$;
2. $F(G)$ is a 3-group and either $F(G) = 1$, or $\bar{G} = G/F(G)$ acts irreducibly on $F(G)$;
3. the socle $\bar{T}$ of group $\bar{G}$ is isomorphic to $L_2(101)$, $U_3(101)$ or $U_3(17)$.
Proof. By the theorem 1 subgraph \( \text{Fix}(f) \) is empty graph and \( \alpha_1(f) = 101 \). Let \( g \) be an element of prime order \( p < 101 \) from \( C_G(f) \). From action \( f \) to \( \Omega \) it follows that subgraph \( \Omega \) is empty graph and \( p = 3 \). Contradiction with action \( g \) on \( U = \{ u \mid d(u, u^f) = 1 \} \).

By the theorem of Gruenberg-Kegel [6] a soluble radical of group \( G \) is equiv to \( F(G) \). As 3 does not divide \( N_G(\langle f \rangle) \), then \( F(G) \) is a 3-group. Let \( \bar{G} = G/F(G) \). If \( F(G) \neq 1 \), then \( |F(G) : F(G)_a| = 3 \) and \( \bar{G} \) acts irreducibly on \( F(G) \).

By table 3 from [7] a group \( \bar{T} \) is isomorphic to \( L_2(101), U_3(101) \) and \( U_5(17) \). The Lemma is proved.

Let we finish the proof of the corollary. The order of a group \( U_3(101) \) is divided by \( 101^3 \). The order of a group \( U_5(17) \) is divided by 13. A group \( L_2(101) \) does not contain subgroups of index, dividing \( 101 \cdot 3 \). In all cases we have a contradiction. The Corollary is proved.

References


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