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ON AXISYMMETRIC HELFRICH SURFACES

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Abstract. In this paper we study axisymmetric Helfrich surfaces. We prove the convergence of the formal power series solution of the Euler – Lagrange equation for the Helfrich functional in a neighborhood of its singular point. We also prove the following inequality

$$\lambda_v R^3 + (c^2 + 2\lambda_a)R^2 - 2cR + 1 \geqslant 0,$$

for a smooth axisymmetric Helfrich surfaces, that homeomorphic to a sphere, where c is the spontaneous curvature of the surface, λ_a and λ_v are Lagrange multipliers, R is the maximum distance between the axis of rotational symmetry and surface.

Keywords: Helfrich spheres of rotation, Delaunay surface of rotation, Willmore surface of rotation, Lobachevsky hyperbolic plane

1. Introduction

The paper [1] initiated an intensive study of extremals of the generalized Willmore functional $\int_\Sigma (2H-c)^2 dA$ in a class of closed surfaces $\Sigma \subset \mathbb{R}^3$ with a given area of Σ and fixed volume of domain V, where $\partial V = \Sigma$. Here c is a constant (spontaneous curvature), H denotes the mean curvature of Σ, dA is the induced area measure on Σ. Using the method of Lagrange multipliers this problem with the constraints is transformed into the study of *the Helfrich functional*

$$\mathcal{H}(\Sigma) = \frac{1}{2}\int_\Sigma (2H-c)^2 dA + \lambda_a \int_\Sigma dA + \lambda_v \int_V dV,$$

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where λ_a and λ_v are Lagrange multipliers [1], dV denotes the volume measure. The Euler – Lagrange equation for $\mathcal{H}$ has the following form

$$-2\triangle_\Sigma H - (2H - c)\left(2H^2 - 2K + cH\right) + 2\lambda_a H + \lambda_v = 0, \tag{1}$$

where $\triangle_\Sigma$ is the Laplace — Beltrami operator, K is the Gaussian curvature (see, for example, [2]). The solution of the Euler — Lagrange equation (1) is called a *Helfrich surface*.

Although the Helfrich surfaces are well studied numerically a list of explicit examples of the Helfrich surfaces is rather short:

1. *Biconcave discoidal surfaces* $\mathcal{B}$. This and further examples are obtained by rotating the profile curves $\gamma(y) = (x(y), y, 0)$ on the xy plane around the x-axis. The curves $\gamma(y)$ for $\mathcal{B}$ are uniquely (up to translation along the x-axis) defined by the following equation

$$\sin\psi(y) = -cy \ln\frac{y}{y_c}, \tag{2}$$

where an angle $\psi(y)$ is chosen such that a vector $(-\sin\psi(y), \cos\psi(y), 0)$ is a tangent to $\gamma(y)$, y_c is a constant (see Fig. 1 and [2]).

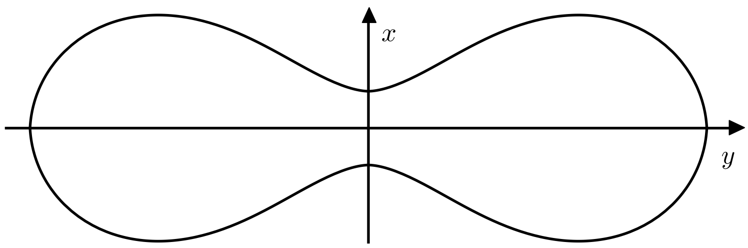

FIG. 1. Curves $\gamma(y)$ for the surface $\mathcal{B}$ at $c = -1.3$.

2. *Unduloid-like* and *nodoid-like* surfaces $\mathcal{C}$. The curves $\gamma(t)$ for surfaces $\mathcal{C}$ satisfy

$$\sin\psi(y) = ay + \frac{b}{y} + d \tag{3}$$

with some constants a, b and d. See [2] for details.

3. *Sphere, cylinder, catenoid, unduloids, nodoids* and *the Clifford torus* are the Helfrich surfaces. See [2] for details.

All these examples have the following interesting property. Considering their profile curves $\gamma(y) = (x(y), y)$ in the Lobachevsky plane with the metric

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2) \tag{4}$$

the geodesic curvature of these curves is a constant, a linear or a simple rational function. We have the following theorem.

Theorem 1. *Let the surface $\mathcal{B}$ (the surface $\mathcal{C}$) generated by rotating the curve $\gamma(y) = (\gamma(y), y)$ defined by (2) (by equation (3)). Then the geodesic curvature k of the curve $\gamma(y)$ in the Lobachevsky plane with metric (4) is equal to*

1) $\mathsf{k} = -cy$, in the case of the surface $\mathcal{B}$, where c is spontaneous curvature of $\mathcal{B}$, y is the distance from the axis of rotation.

2) $\mathsf{k} = -d - \frac{2b}{y}$, *in the case of the surface* $\mathcal{C}$, *where* b, d *are constants defined by (3),* y *is the distance from the axis of rotation.*

The idea to consider curves $\gamma(y) = (x(y), y)$ in the Lobachevsky plane in metric (4) originated from Bryant's remark in [4] on a relation of the Willmore functional $\int_\Sigma H^2 dA$ and the functional $\frac{\pi}{2}\int_\gamma \mathsf{k}^2 ds$, where k is the geodesic curvature of γ in metric (4). Later Langer and Singer proved that these functionals coincide [5].

The Euler – Lagrange equation for an axisymmetric Helfrich surface is reduced to the following equation (see [6])

$$\begin{aligned}(5)\quad &\frac{1}{y}(yf'(y) + y^2 f''(y))(1 - f^2(y)) + \frac{y}{2}(f'(y))^2 f(y)\\ &+ \frac{1}{2y}f^3(y) - \frac{1}{y}f(y) - \frac{1}{2}(c^2 + 2\lambda_a)yf(y) + cf^2(y) - \frac{\lambda_v y^2}{2} = \nu,\end{aligned}$$

where ν is some constant, $f(y)$ satisfies equation

$$\sin\psi(y) = f(y). \tag{6}$$

Equation (5) with a singular point at $y = 0$ has a formal power series solution (see [6])

$$f(y) = \sum_{k=0}^{\infty} a_{2k+1} y^{2k+1}, \tag{7}$$

where $a_1 \in \mathbb{R}$, $a_3 = \frac{1}{16}(\lambda_v + (c^2 + 2\lambda_a)a_1 - 2ca_1^2)$,

$$\begin{aligned}(8)\quad a_{2k+1} = \frac{1}{8k(k+1)}\Bigg(&(c^2 + 2\lambda_a)a_{2k-1} - 2c\tilde{\sum_{l+m=2k}} a_l a_m\\ &+ \tilde{\sum_{l+m+n=2k+1}} (l^2 - lm + m^2 - 1)a_l a_m a_n\Bigg), \quad k \in \mathbb{N}, \quad k > 1,\end{aligned}$$

and the summations $\tilde{\sum}$ are taken over all odd numbers. In the next theorem we prove the convergence of the series (7).

Theorem 2. *For any* c, λ_a, λ_v *and* a_1 *the series* (7) *converges in a neighborhood of* $y = 0$.

Suppose that Σ is a smooth axisymmetric Helfrich surface homeomorphic to a sphere. Let R be the maximum distance from the rotation axis to a point at Σ. Then we prove some restrictions on R.

Theorem 3. *The following inequality holds*

$$\lambda_v R^3 + (c^2 + 2\lambda_a)R^2 - 2cR + 1 \geqslant 0. \tag{9}$$

In the case of positive λ_v the inequality (9) means that R cannot lie between two positive roots of the left side of the equation (if they exist).

2. The proof of Theorem 1

Let $\Sigma \subset \mathbb{R}^3$ be obtained (locally) by rotating a curve $\tilde{\gamma}(y) = (x(y), y, 0)$ on the xy plane around the x-axis. We associate the curve $\gamma = (x(y), y)$ in the hyperbolic plane to the surfaces Σ. The following lemma holds.

Lemma 1. *For the curve $\gamma(y) = (x(y), y)$, $y > 0$, in the Lobachevsky plane with the metric (4) the following equation holds*

$$\mathsf{k}(y) = y\frac{d}{dy}(\sin\psi(y)) - \sin\psi(y), \tag{10}$$

where k *is the geodesic curvature* γ, ψ *is the signed angle between the* y*-axis and the vector* $\frac{d}{dy}\tilde{\gamma}(y)$ *in Euclidean space* $\mathbb{R}^3$.

Proof of Lemma 1. We derive the equation (10) from the definition of the geodesic curvature $\mathsf{k}(s) = (\nabla_{\dot\gamma}\dot\gamma, n(s))$, where s is a natural parameter on $\gamma = (x(s), y(s))$. We remark that $(\dot x, \dot y) = (y\sin\psi(s),\ y\cos\psi(s))$. The unit normal vector to γ is $n(s) = (\dot y, -\dot x)$. Using the Christoffel symbols of the hyperbolic metric (4) we have

$$\nabla_{\dot\gamma}\dot\gamma = \left(\ddot x - \frac{2}{y}\dot y\dot x, \ddot y + \frac{1}{y}(\dot x^2 - \dot y^2)\right).$$

Then the following equations hold

$$\mathsf{k}(s) = \frac{1}{y^2}(\dot y\ddot x - \ddot y\dot x) - \frac{1}{y}\dot x = \dot\psi - \sin\psi. \tag{11}$$

A straightforward computation gives

$$\dot\psi = \frac{y\frac{d}{ds}(\sin\psi(s))}{\dot y} = y(\sin\psi(y))_y.$$

Since $\dot\psi(s) = y(\sin\psi(y))_y$ from (11) it follows (10). Lemma 1 is proved.

Let us prove Theorem 1. Let the curve $\gamma_{\mathcal{B}}$ correspond to the surface $\mathcal{B}$. From (2) and Lemma 1 it follows that the geodesic curvature of $\gamma_{\mathcal{B}}$ is

$$\mathsf{k} = y\left(-cy\ln\frac{y}{y_c}\right)_y + cy\ln\frac{y}{y_c}.$$

Since $\left(cy\ln\frac{y}{y_c}\right)_y = c\ln\left(\frac{y}{y_c}\right) + c$, then we have

$$\mathsf{k} = -cy.$$

Let $\gamma_{\mathcal{C}}$ be defined by (3). This curve corresponds to the surface $\mathcal{C}$. By Lemma 1 the geodesic curvature k of $\gamma_{\mathcal{C}}$ is

$$\mathsf{k} = y\left(ay + \frac{b}{y} + d\right)_y - \left(ay + \frac{b}{y} + d\right) = -\frac{2b}{y} - d.$$

Theorem 1 is proved.

Remark 1. Let Σ obtained by rotating γ and the geodesic curvature of γ is $\mathsf{k} = by$, $b \in \mathbb{R}$. From Lemma 1 it follows, that γ defined by $\sin\psi(y) = ay + by\ln y$ with some integration constant a. By substitution this $\sin\psi(y)$ into the Euler – Lagrange equation and straightforward computations it can be shown that there are only two solutions $\sin\psi(y) = 0$ and $\sin\psi(y) = -cy\ln\frac{y}{y_c}$. See [2] for details.

Remark 2. Let us derive the geodesic curvatures k of profile curves γ of other Helfrich surfaces. A profile curves of *sphere* is a geodesic of hyperbolic plane, so $\mathsf{k} = 0$. The profile line of a *cylinder* has $\psi(s) = \pm\frac{\pi}{2}$. So, by Lemma 1 we have $\mathsf{k} = \mp 1$. For *catenoid, unduloids and nodoids* function $\sin\psi(y)$ was derived in (3). So, by Lemma 1 we obtain $\mathsf{k} = -\frac{2b}{y}$. The profile curve of *the Clifford torus* is a circle of radius r with center at a distance of $\sqrt{2}r$ from the x-axis. Hence, $\sin\psi(y) = \frac{y}{r} - \sqrt{2}$.

Therefore, by Lemma 1 geodesic curvature k of the profile curve of *the Clifford torus* is equal to $\sqrt{2}$.

3. The proof of Theorem 2

It is sufficient to prove that for some $a \in \mathbb{R}$ and for all odd k the following inequality holds $|a_k| < \frac{a^k}{6k^2}$. The proof is by induction on odd k.

Let c, λ_a, λ_v be arbitrary parameters of the Helfrich functional. We denote $\mu = c^2 + 2\lambda_a$ for brevity. Let N be the smallest integer greater than $\max\{5,\, 3\sqrt{2|\mu|}+2,\, 4\sqrt{3|c|}+1\}$. Let a_1 be an arbitrary real number. Choose $a > 1$ such that

$$|a_1| < \frac{a}{6}, \quad |a_3| < \frac{a^3}{6\cdot 3^2}, \quad \dots, \quad |a_N| < \frac{a^N}{6N^2}.$$

Assuming the following inequalities hold for odd $k > N$

$$|a_1| < \frac{a}{6}, \quad |a_3| < \frac{a^3}{6\cdot 3^2}, \quad \dots, \quad |a_{k-2}| < \frac{a^{k-2}}{6(k-2)^2}, \tag{12}$$

we will prove $|a_k| < \frac{a^k}{6k^2}$. Hence, from (8) it follows

$$|a_k| \leqslant \frac{1}{2(k^2-1)}\Bigg(|\mu|\cdot|a_{k-2}| + |2c| \sum_{l+m=k-1}^{\sim} |a_l|\cdot|a_m| \\ + \sum_{l+m+n=k}^{\sim} (l^2 - lm + m^2 - 1)\cdot|a_l|\cdot|a_m|\cdot|a_n|\Bigg).$$

Note that the terms $l^2 - lm + m^2 - 1 = lm + (l-m)^2 - 1$ in the last sum are non-negative for all natural l and m. By the induction hypothesis (12) we have

$$|a_k| < \frac{1}{2(k^2-1)}\Bigg(\frac{|\mu| a^{k-2}}{6(k-2)^2} + \frac{|2c| a^{k-1}}{36} \sum_{l+m=k-1}^{\sim} \frac{1}{l^2m^2} \\ + \frac{a^k}{6^3} \sum_{l+m+n=k}^{\sim} \frac{l^2 - lm + m^2 - 1}{l^2m^2n^2}\Bigg).$$

Let us prove the following lemma.

Lemma 2. *Let fix arbitrary odd $k > 5$ and denote*

$$I = \sum_{l+m=k-1}^{\sim} \frac{1}{l^2m^2}, \quad J = \sum_{l+m+n=k}^{\sim} \frac{l^2 - lm + m^2 - 1}{l^2m^2n^2}.$$

Then the following inequalities hold

$$I < \frac{8}{(k-1)^2}, \quad J < 32.$$

Proof of Lemma 2. A straightforward calculation yields

$$\frac{1}{l^2m^2} \leqslant \frac{2}{(l+m)^2}\left(\frac{1}{l^2} + \frac{1}{m^2}\right) \tag{13}$$

for l, $m \in \mathbb{N}$. By applying the inequality (13) to each term of I we obtain

$$I \leqslant \frac{2}{(k-1)^2} \sum_{l+m=k-1}^{\prime} \left(\frac{1}{l^2} + \frac{1}{m^2}\right).$$

The latter sum is less than $2\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{3}$. Therefore the first inequality holds

$$I<\frac{2\pi^2}{3(k-1)^2}<\frac{8}{(k-1)^2}.$$

We now estimate J. Let J_1 denote $\sum'_{l+m+n=k}\frac{l^2+m^2}{l^2m^2n^2}$. By separating J_1 apart and interchanging indices m and l we can rewrite J_1 as

$$J_1=2\sum_{\substack{l=1\\ l\notin 2\mathbb{Z}}}^{k-1}\tilde{\sum_{m+n=k-l}}\frac{1}{m^2n^2},$$

where $2\mathbb{Z}$ is the set of even numbers. By applying the first inequality of this Lemma to the each term of J_1 we have

$$J_1=2\sum_{\substack{l=1\\ l\notin 2\mathbb{Z}}}^{k-2}\tilde{\sum_{m+n=k-l}}\frac{1}{m^2n^2}<2\sum_{\substack{l=1\\ l\notin 2\mathbb{Z}}}^{k-2}\frac{8}{(k-l)^2}. \tag{14}$$

The latter sum in (14) is less than

$$2\sum_{l=1}^{k-2}\frac{8}{(k-l)^2}<2\sum_{n=1}^{\infty}\frac{8}{n^2}. \tag{15}$$

Hence from the inequality $J<J_1$, (14), (15) and $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$ it follows $J<32$.

Lemma 2 is proved.

By Lemma 2 and the previous inequality we have

$$|a_k|<\frac{1}{2(k^2-1)}\left(\frac{|\mu|}{6(k-2)^2}+\frac{16|c|}{36(k-1)^2}+\frac{32}{6^3}\right)a^k. \tag{16}$$

The choice $k>N$ guarantees

$$\frac{|\mu|}{6(k-2)^2}<\frac{2}{6^3},\quad \frac{16|c|}{36(k-1)^2}<\frac{2}{6^3}.$$

Hence from (16) it follows $|a_k|<\frac{1}{12(k^2-1)}a^k$. Consequently, $|a_k|<\frac{1}{6k^2}a^k$ for $k>5$. The induction step is completed. Thereby, the series (7) is majorized by the series $\tilde{\sum}\frac{a^ky^k}{6k^2}$ with the radius of convergence $\frac{1}{a}$.

Theorem 2 is proved.

4. The proof of Theorem 3

Let us consider the surface of rotation Σ

$$(x(t),y(t)\cos\varphi,y(t)\sin\varphi),\quad \varphi\in[0,2\pi].$$

Assume that $\gamma(t)=(x(t),y(t),0)$ is an arc length parameterized curve, i.e. $(\dot{x}(t),\dot{y}(t),0)=(-\sin\psi(t),\cos\psi(t),0)$ for some smooth function $\psi(t)$ and $(x(t),y(t))$ is a positively oriented closed curve. In this case, the induced metric of Σ takes the form

$$g_{11}=1,\quad g_{12}=g_{21}=0,\quad g_{22}=y^2(t).$$

The second fundamental form components are

$$b_{11}=\dot{\psi}(t),\quad b_{12}=b_{21}=0,\quad b_{22}=y(t)\sin\psi(t).$$

The mean and the Gaussian curvature are

$$H = \frac{1}{2}\left(\dot\psi(t) + \frac{\sin\psi(t)}{y(t)}\right), \quad K = \frac{\dot\psi(t)\sin\psi(t)}{y(t)}. \tag{17}$$

Since

$$g^{11} = 1, \quad g^{12} = g^{21} = 0, \quad g^{22} = \frac{1}{y^2(t)}, \quad \sqrt{g(t)} = \frac{1}{|y(t)|},$$

the Euler — Lagrange equation (1) takes the form

$$\begin{aligned} -\dddot\psi - \frac{2\cos\psi}{y}\ddot\psi - \frac{1}{2}\dot\psi^3 + \frac{3\sin\psi}{2y}\dot\psi^2 \\ + \frac{y(2\cos^2\psi - \sin^2\psi) + (c^2 + 2\lambda_a)y^3}{2y^3}\dot\psi - \frac{2}{y}c\,\dot\psi\sin\psi \\ - \frac{(2\cos^2\psi + \sin^2\psi)\sin\psi}{2y^3} \\ + \frac{(c^2 + 2\lambda_a)\sin\psi}{2y} + \lambda_v = 0, \quad \psi = \psi(t), \quad y = y(t). \end{aligned} \tag{18}$$

Note that in the case $\gamma(t)$ is negatively oriented, we have $H = -\frac{1}{2}\left(\dot\psi(t) + \frac{\sin\psi(t)}{y(t)}\right)$ and the Euler – Lagrange equation takes the form (18), where all terms have opposite sign except the terms $\frac{-2}{y}c\,\dot\psi\sin\psi$ and λ_v.

The equation (18) has a first integral

$$\begin{aligned} \frac{1}{y}\Big(2y^2\ddot\psi\cos\psi + y^2\dot\psi^2\sin\psi \\ + 2\dot\psi y\cos^2\psi - \lambda_v y^3 - (c^2 + 2\lambda_a)y^2\sin\psi + 2cy\sin^2\psi \\ - \sin\psi(1 + \cos^2\psi)\Big) = \nu, \quad \nu \in \mathbb{R}, \quad \psi = \psi(t), \quad y = y(t). \end{aligned} \tag{19}$$

Note that the t-derivative of (19) equals (18) times $(-2y\cos\psi)$.

Let us show that $\nu = 0$ for Σ homeomorphic to a sphere. Let t_0 denote the point of $\gamma(t) = (x(t), y(t), 0)$ at x-axis. Then $\psi(t_0)$ is equal to 0 or π. Since $\psi(t)$ and $y(t)$ are smooth functions, the limit of (19), as $t \to 0$, equals

$$2\dot\psi(t_0) - 2\lim_{t\to t_0}\left(\frac{\sin\psi(t)}{t - t_0} \times \frac{t - t_0}{y(t)}\right) = \nu. \tag{20}$$

Note that $\lim_{t\to t_0}\frac{\sin\psi(t)}{t-t_0} = (\frac{d}{dt}\sin\psi(t))|_{t=t_0} = \dot\psi(t_0)\cos\psi(t_0)$. Since $\dot y(t) = \cos\psi(t)$ by definition of $\psi(t)$, then $\lim_{t\to t_0}\frac{y(t)}{t-t_0} = \cos\psi(t_0)$. So, the left part of (20) is zero. Therefore $\nu = 0$ for any axisymmetric Helfrich sphere (see [7] for γ that y-parameterized).

Let t_1 denote one of the farthest points of $\gamma(t)$ from x-axis. The limit of (19), as $t \to t_1$, equals

$$R^2\dot\psi^2(t_1) - \left(\lambda_v R^3 + (c^2 + 2\lambda_a)R^2 - 2cR + 1\right) = 0,$$

where $\psi(t_1) = \pi/2$, $y(t_1) = R$. Therefore

$$\lambda_v R^3 + (c^2 + 2\lambda_a)R^2 - 2cR + 1 = R^2k_1^2 > 0,$$

where k_1 is the principal curvature of Σ along the meridian at $t = t_1$.

Theorem 3 is proved.

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