S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 12, стр.92–100 (2015) УДК 519.1
MSC 05C25, 05C50, 05E10

NOTE ON EXACT VALUES OF MULTIPLICITIES OF EIGENVALUES OF THE STAR GRAPH

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Abstract. The Star graph is the Cayley graph on the symmetric group Sym_n generated by the set of transpositions $\{(12), (13), \ldots, (1n)\}$. A Chapuy–Feray combinatorial approach is used to obtain multiplicities of eigenvalues. Exact values are calculated up to $n = 10$ and compared with lower bounds on multiplicities of eigenvalues for this graph.

Keywords: Cayley graphs; Star graph; graph spectrum; eigenvalues

1. Introduction

The Star graph $S_n = Cay(Sym_n, t)$, $n \geqslant 2$, is the Cayley graph on the symmetric group Sym_n of permutations $\pi = [\pi_1\pi_2 \ldots \pi_n]$, where $\pi_i = \pi(i)$, $1 \leqslant i \leqslant n$, with the generating set $t = \{t_i \in Sym_n : 2 \leqslant i \leqslant n\}$ of all transpositions t_i transposing the 1st and ith elements, $2 \leqslant i \leqslant n$, of a permutation π when multiplied on the right, i.e. $[\pi_1\pi_2 \ldots \pi_{i-1}\pi_i\pi_{i+1} \ldots \pi_n]t_i = [\pi_i\pi_2 \ldots \pi_{i-1}\pi_1\pi_{i+1} \ldots \pi_n]$. It is a connected bipartite $(n-1)$–regular graph of order $n!$ and diameter $diam(S_n) = \lfloor \frac{3(n-1)}{2} \rfloor$ [2]. Since this graph is bipartite it does not contain odd cycles but it does contain all even l–cycles where $l = 6, 8, \ldots, n!$ [6] (with the sole exception when $l = 4$). The hamiltonicity of this graph follows from results by V. Kompel'makher and V. Liskovets [7] and by P. J. Slater [12] which say that any Cayley graph on the symmetric group generated by a set of transpositions is hamiltonian.

Khomyakova, E.N., Konstantinova, E.V., Notes on exact values of multiplicities of eigenvalues of the Star graph.

The first author partially supported by RFBR Grant 15-01-05867, the second author partially supported by RFBR Grant 15-01-07906 and Grant NSh-1939.2014.1 of President of Russia for Leading Scientific Schools.

Received November, 25, 2014, published February, 20, 2015.

We consider the spectrum of the Star graph as the spectrum of its adjacency matrix. In 2009 A. Abdollahi and E. Vatandoost conjectured [1] that the spectrum of S_n is integral, moreover it contains all integers in the range from $-(n-1)$ up to $n-1$ (with the sole exception that when $n \leqslant 3$, zero is not an eigenvalue of S_n). For $n \leqslant 6$ they verified this conjecture numerically using GAP, but for $n > 6$ computer calculations are getting too hard. In 2012 R. Krakovski and B. Mohar [8] proved that the spectrum of S_n is integral. They also gave a lower bound on multiplicities of eigenvalues for this graph. At the same time, G. Chapuy and V. Feray [4] showed another approach to obtain the exact values of multiplicities of eigenvalues of S_n. Their combinatorial approach is based on the Jucys–Murphy elements and the standard Young tableaux. However, it is still complicated to find multiplicities using a computer. In this note we present exact values of multiplicities of eigenvalues of S_n for $2 \leqslant n \leqslant 10$ which were obtained manually using Chapuy–Feray approach. An oscillating behavior of multiplicities is shown with growing n.

2. Chapuy - Feray combinatorial approach

To describe a combinatorial approach for calculating multiplicities of eigenvalues of S_n, let us give basic definitions and notations on representation theory of the symmetric group [11].

A *representation of a group* G on a vector space V over a field $\mathbb{C}$ is a group homomorphism $\rho\colon G \to \mathrm{GL}(V)$, where $GL(V)$ is the general linear group on V, such that

$$\rho(g_1g_2) = \rho(g_1)\rho(g_2),$$

for all $g_1, g_2 \in G$. Given a representation ρ, any vector subspace W of V is said to be ρ-invariant if $\rho(g)(W) \subseteq W$ for all $g \in G$. A representation ρ is said to be *irreducible* if $\{0\}$ and V are the only invariant subspaces of V.

A *partition* of a natural number n is a way of writing n as a sum of natural numbers. Two sums that differ only in the order of their summands are considered as the same partition. The partition function $P(n)$ represents the number of possible partitions of a natural number n, i.e. the number of distinct ways of representing n as a sum of natural numbers.

Let $\lambda \in P(n)$ be a *partition* of n and V_λ be the *irreducible module* associated with the partition λ. It is known [11, Proposition 1.10.1] that the *regular representation* $\mathbb{C}[Sym_n]$ of the symmetric group Sym_n is decomposed into irreducible submodules as follows:

$$\mathbb{C}[Sym_n] = \oplus_{\lambda \in P(n)} dim(V_\lambda)V_\lambda. \tag{1}$$

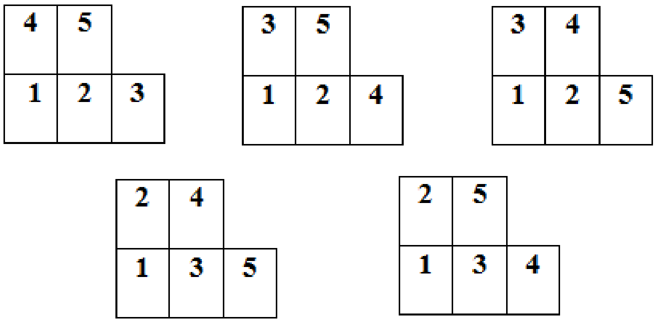
The regular representation $\mathbb{C}[Sym_n]$ is called *group algebra* of Sym_n. Group algebra of Sym_n is a vector space indexed by the elements group. Thus, if $Sym_n = \{s_1, s_2, \ldots, s_k\}, k = n!$, then

$$\mathbb{C}[Sym_n] = \{c_1s_1 + c_2s_2 + \ldots + c_ks_k : c_i \in \mathbb{C}, 1 \leqslant i \leqslant k\},$$

where $\mathbb{C}$ is the field of complex numbers.

A partition $\lambda = (t_1, t_2, \ldots, t_m), m \leqslant n$, is represented by its Young tableau. A *standard Young tableau* of shape λ is a filling of the boxes with the elements $\{1, 2, \ldots, n\}$ in such a way that elements increase along rows and columns, all elements are distinct, and each element appears exactly once.

By the representation theory, the number of different irreducible representations Sym_n over a field $\mathbb{C}$ is equal to the number of partitions of n [10]. For example, if $n = 5$ then the value of the partition function is $P(5) = 7$ and the following partitions are $(1, 1, 1, 1, 1)$, $(2, 1, 1, 1)$, $(2, 2, 1)$, $(3, 1, 1)$, $(3, 2)$, $(4, 1)$, (5). For $\lambda = (3, 2)$, the dimension of its irreducible module is $dim(V_{\lambda=(3,2)}) = 5$. Standard Young tableaux of shape $\lambda = (3, 2)$ are listed below.



Let us define values $c(i) = y - x$, where $i \in \{1, \ldots, n\}$ and y, x are the ordinate and the abscissa of the box, correspondingly. For example, the first standard Young tableau presented above has values $c(3) = 1 - 3 = -2$, $c(4) = 2 - 1 = 1$, and $c(5) = 2 - 2 = 0$.

The Jucys—Murphy elements in the group algebra $\mathbb{C}[Sym_n]$ of the symmetric group Sym_n are defined as the sum of transpositions by the following formula:

$$J_1 \equiv 0, J_2 = (1, 2), J_i = (1, i) + (2, i) + \ldots + (i - 1, i), i \in \{2, \ldots, n\}, \tag{2}$$

where a summation of transpositions is performed in terms of the summation of corresponding matrices. These elements named after A. A. Jucys [5] and G. E. Murphy [9]. For example, transpositions $(1, 2)$, $(1, 3)$, $(2, 3)$ are represented by the following matrices:

$$(1,2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (1,3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (2,3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

hence J_1 represents a zero matrix and we also have:

$$J_2 = (1,2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad J_3 = (1,3) + (2,3) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The Jucys—Murphy elements play an important role in the representation theory of the symmetric group [10]. It is known that they generate a maximal commutative subalgebra in $\mathbb{C}[Sym_n]$. Moreover, the J_n commutes with all elements of $\mathbb{C}[Sym_{n-1}]$. The action of the J_n on the irreducible modules V_λ is diagonal. The diagonalization of this algebra in each irreducible representation defines a linear Young basis, which is denoted by (v). Moreover, the spectrum of this algebra is the set of integer vectors. Thus, the bases of all irreducible representations Sym_n are labeled by standard Young tableaux.

Theorem 1. [5] *Let $\lambda \in P(n)$. Then there exists a basis (v) of the irreducible module V_λ, indexed by standard Young tableaux of shape λ, such that for all $2 \leqslant i \leqslant n$, one has:*

$$J_i v = c(i)v. \tag{3}$$

From Theorem 1 and by (1), the following result is immediately obtained.

Corollary 1. [4] *The spectrum of S_n contains only integers. The multiplicity $mul(n-k)$, where $1 \leqslant k \leqslant n-1$, of an integer $(n-k) \in \mathbb{Z}$ is given by:*

$$mul(n-k) = \sum_{\lambda \in P(n)} dim(V_\lambda) I_\lambda(n-k), \tag{4}$$

where $dim(V_\lambda)$ is the dimension of an irreducible module, $I_\lambda(n-k)$ is the number of standard Young tableaux of shape λ, satisfying $c(n) = n-k$.

R. Krakovski and B. Mohar proved the following theorem.

Theorem 2. [8] *Let $n \geqslant 2$, then for each integer $1 \leqslant k \leqslant n-1$ the values $\pm(n-k)$ are eigenvalues of S_n with multiplicity at least $\binom{n-2}{k-1}$. If $n \geqslant 4$, then 0 is an eigenvalue of S_n with multiplicity at least $\binom{n-1}{2}$.*

Using the hook–length formula presented by Theorem 3.10.2 in [11], the bound $\binom{n-2}{k-1}$ from Theorem 2 is improved as follows.

Corollary 2. [4] *Let $n \geqslant 2$, then for each integer $1 \leqslant k \leqslant n-1$ the values $\pm(n-k)$ are eigenvalues of S_n with multiplicity at least $\binom{n-2}{k-1}\binom{n-1}{k}$.*

3. Results

The exact values of multiplicities of eigenvalues of S_n for $2 \leqslant n \leqslant 10$ calculated by formula (4) are presented in Table 1. For each integer $1 \leqslant k \leqslant n-1$, the negative and positive eigenvalues $\pm(n-k)$ have the same multiplicities. Parenthetically, the lower bound multiplicities of eigenvalues calculated by formulas from Theorem 2 and Corollary 2 are also presented in the Table as (a/b), where a corresponds to Krakovski–Mohar bound, and b corresponds to Chapuy–Feray bound.

Let us note that Chapuy–Feray bound gives exact values of multiplicities for $k = 2$, since

$$mul(n-k) = \binom{n-2}{k-1}\binom{n-1}{k},$$

hence

$$mul(n-2) = (n-2)(n-1).$$

The last column of Table 1 contains summation over all multiplicities eigenvalues S_n for each n and coincides with the order of the Star graph S_n, i.e.

$$\sum_{n-k=-(n-1)}^{n-1} mul(n-k) = n!.$$

Table 1. The exact values of multiplicities of eigenvalues of S_n for $2 \leqslant n \leqslant 10$.

$n \backslash \pm(n-k)$	0	±1	±2	±3
2		1 (1/1)		
3		2 (1/2)	1 (1/1)	
4	4 (3)	3 (1/3)	6 (2/6)	1 (1/1)
5	30 (6)	4 (1/4)	28 (3/18)	12 (3/12)
6	168 (10)	30 (1/5)	120 (4/40)	105 (6/60)
7	840 (15)	468 (1/6)	495 (5/75)	830 (10/200)
8	3960 (21)	5691 (1/7)	2198 (6/126)	6321 (18/525)
9	19782 (28)	59624 (1/8)	15064 (7/196)	47544 (21/1176)
10	150640 (36)	579078 (1/9)	189936 (8/288)	358764 (28/2352)

$n \backslash \pm(n-k)$	±4	±5	±6
2			
3			
4			
5	1 (1/1)		
6	20 (4/20)	1 (1/1)	
7	276 (10/150)	30 (5/30)	1 (1/1)
8	3332 (20/700)	595 (18/315)	42 (6/42)
9	38108 (35/2450)	10024 (35/1960)	1128 (21/588)
10	427392 (56/7056)	156780 (70/8820)	25104 (56/4704)

$n \backslash \pm(n-k)$	±7	±8	±9	Σ
2				2
3				6
4				24
5				120
6				720
7				5040
8	1 (1/1)			40320
9	56 (7/56)	1 (1/1)		362880
10	1953 (28/1008)	72 (8/72)	1 (1/1)	3628800

An example of using formula (4) is given in Application.

The obtained exact values of multiplicities are also presented in Figure 1, where the abscissa and the ordinate correspond to the eigenvalues of the Star graphs and their multiplicities, correspondingly. For example, multiplicities $(1, 30, 276, 830, 495, 468, 840, 468, 495, 830, 276, 30, 1)$ of eigenvalues $(-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6)$ are given for $n = 7$. Let us note that typically the distribution of eigenvalue multiplicities for known distance-regular graphs is unimodal (see [3, Chapter 12]). However, for $2 \leqslant n \leqslant 10$ the Star graphs S_n are not distance-regular and give us another picture for a distribution of eigenvalue multiplicities which looks like an oscillating function. One can assume that this behavior of the multiplicities will be also kept for large n.

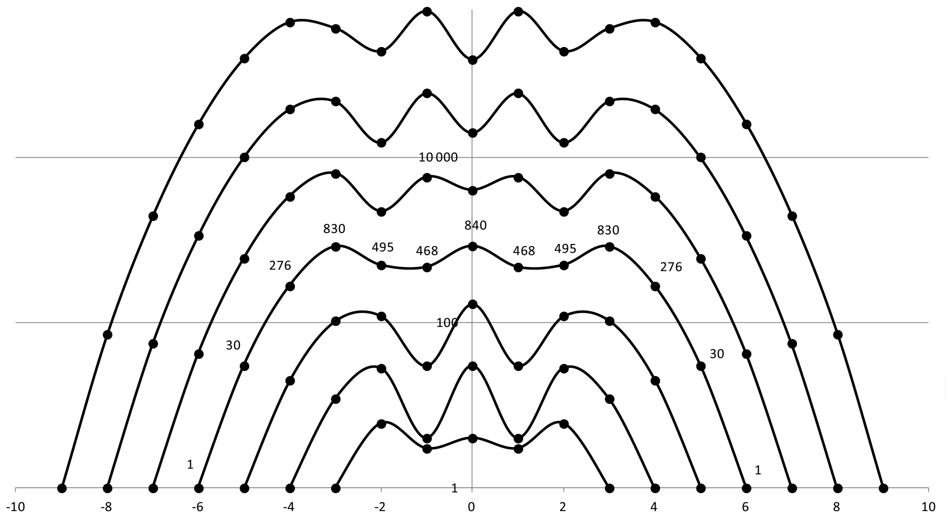


Figure 1. Graphic representations of multiplicities of eigenvalues of S_n for $4 \leqslant n \leqslant 10$.

4. Acknowledgment

The authors thank S. V. Avgustinovich for fruitful discussions on the topic and A. N. Medvedev for critical remarks and suggestions, which helped to improve the quality and clarity of the manuscript.

References

[1] A. Abdollahi, E. Vatandoost. Which Cayley graphs are integral? *The electronic journal of combinatorics*, **16** (2009) 6–7. MR2546325

[2] S. B. Akers, B. Krishnamurthy, A group–theoretic model for symmetric interconnection networks, *IEEE Trans. Comput.* **38** (4) (1989) 555–566. MR0984680

[3] A. E. Brouwer, W. H. Haemers, *Spectra of graphs*, Springer, New York, 2012.

[4] G. Chapuy, V. Feray, A note on a Cayley graph of Sym_n, *arXiv:1202.4976v2* (2012) 1–3.

[5] A. Jucys, Symmetric polynomials and the center of the symmetric group ring, *Reports Math. Phys.* **5** (1974) 107-–112. MR0419576

[6] J. S. Jwo, S. Lakshmivarahan, S. K. Dhall, Embedding of cycles and grids in star graphs, *J. Circuits, Syst., Comput.* **1** (1) (1991) 43–74.

[7] V. L. Kompel'makher, V. A. Liskovets, Successive generation of permutations by means of a transposition basis, *Kibertetika*, **3** (1975) 17–21 (in Russian). MR0498276

[8] R. Krakovski, B. Mohar, Spectrum of Cayley Graphs on the Symmetric Group generated by transposition, *Linear Algebra and its applications* **437**(2012) 1033–1039. MR2921753

[9] G. Murphy, A new construction of Young's seminormal representation of the symmetric group, *J. Algebra.* **69** (1981) 287–291. MR0617079

[10] A. Okounkov, A. Vershik, A new approach to representation theory of symmetric groups, *Selecta Math.* **2** (4) (1996) 1–15. MR1443185

[11] B. Sagan, *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions,* Springer, New York, second edition, 2001.

[12] P. J. Slater, Generating all permutations by graphical transpositions, *Ars Combinatoria* **5** (1978) 219–225. MR0504868

Application

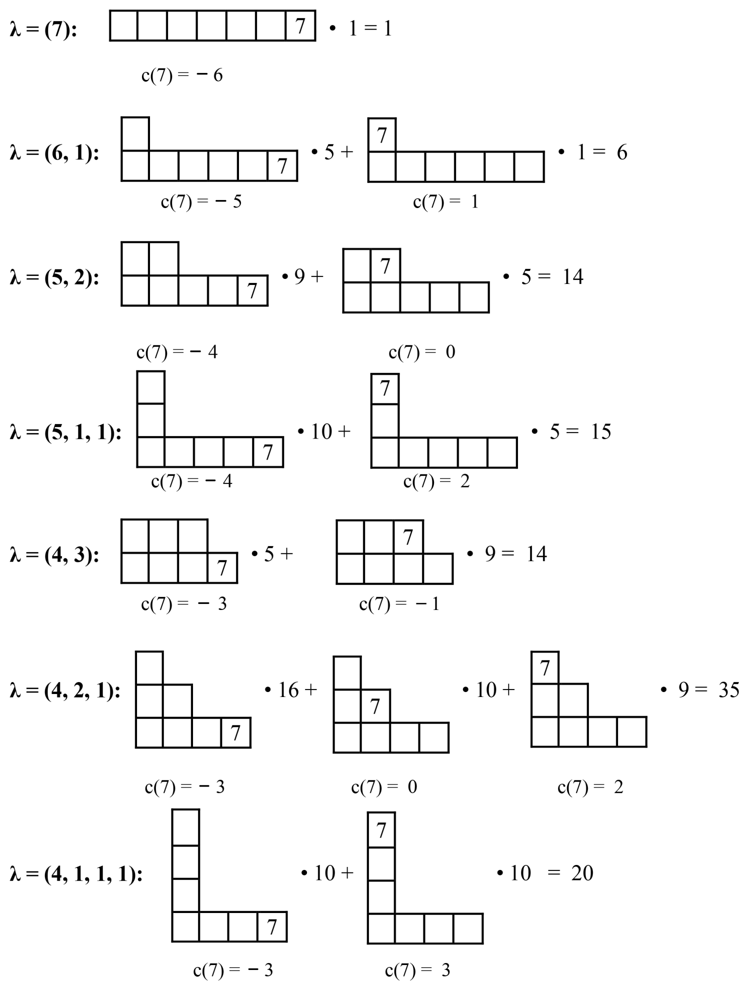
The formula (4) is used in the following way.

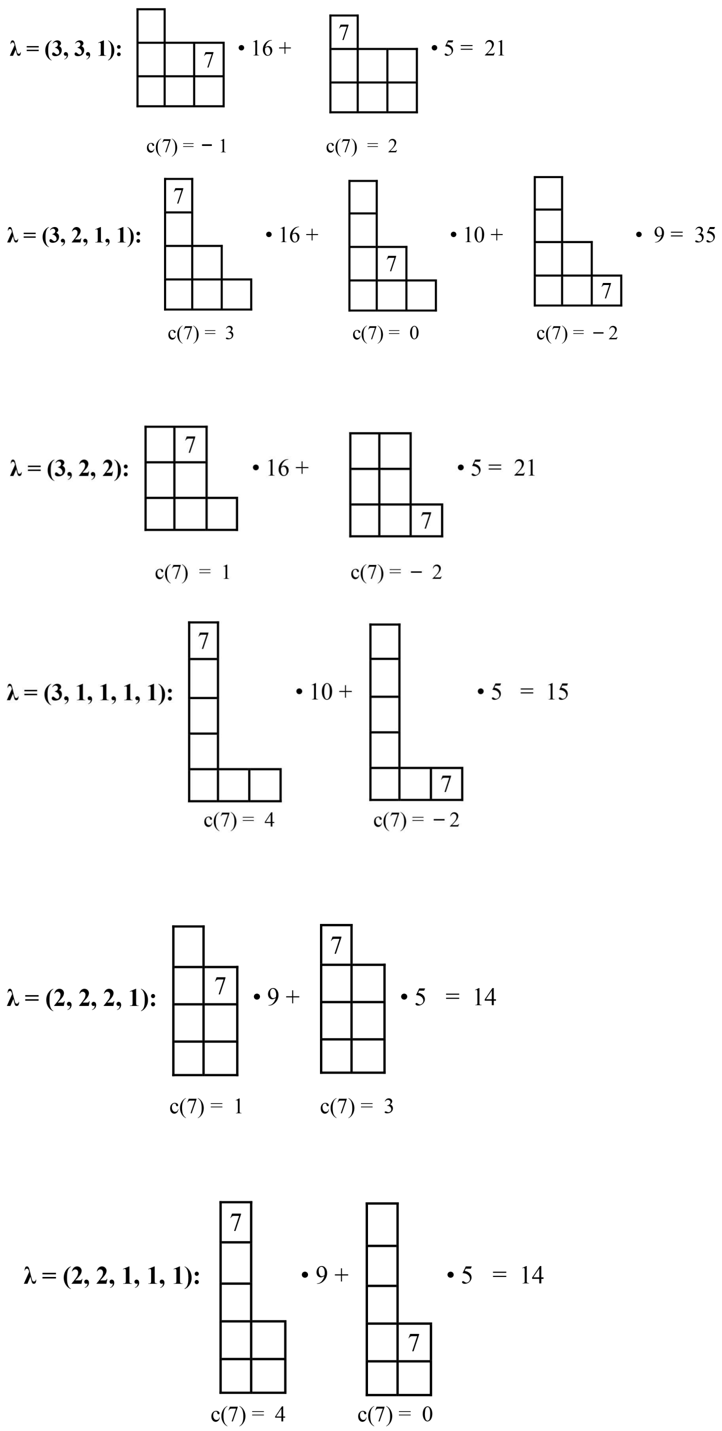
For example, if $n = 7$ then the partition function is $P(7) = 15$ with the following partitions: (7), (6, 1), (5, 2), (5, 1, 1), (4, 3), (4, 2, 1), (4, 1, 1, 1), (3, 3, 1), (3, 2, 2), (3, 2, 1, 1), (3, 1, 1, 1, 1), (2, 2, 2, 1), (2, 2, 1, 1, 1), (2, 1, 1, 1, 1, 1), (1, 1, 1, 1, 1, 1, 1).

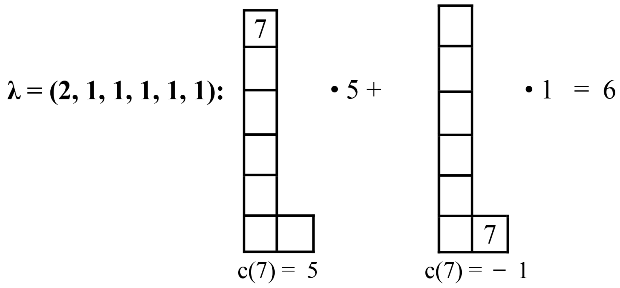
For each partition $\lambda \in P(7)$, we construct an irreducible module V_λ that is the set of standard Young tableaux of shape λ. Say, for $\lambda_1 = (7)$ the corresponding irreducible module V_{λ_1} contains one standard Young tableau, i.e. $dim(V_{\lambda_1}) = 1$, and for the partition $\lambda_2 = (4, 2, 1)$ its irreducible module V_{λ_2} contains 35 standard Young tableaux, i.e. $dim(V_{\lambda_2}) = 35$.

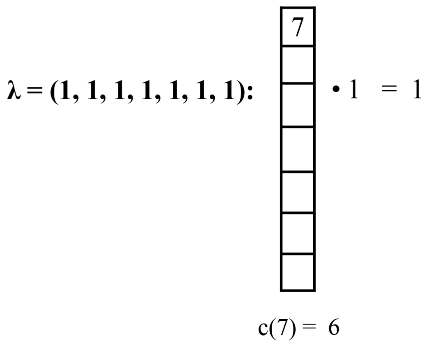
For all these partitions we obtain 232 standard Young tableaux.

The standard Young tableaux for S_7 are presented below. The values $c(7)$ are also given for each Young tableau.




λ = (3, 3, 1):
7
• 16 +
7
• 5 = 21
c(7) = − 1
c(7) = 2
λ = (3, 2, 1, 1):
7
• 16 +
7
• 10 +
7
• 9 = 35
c(7) = 3
c(7) = 0
c(7) = − 2
λ = (3, 2, 2):
7
• 16 +
7
• 5 = 21
c(7) = 1
c(7) = − 2
λ = (3, 1, 1, 1, 1):
7
• 10 +
7
• 5 = 15
c(7) = 4
c(7) = − 2
λ = (2, 2, 2, 1):
7
• 9 +
7
• 5 = 14
c(7) = 1
c(7) = 3
λ = (2, 2, 1, 1, 1):
7
• 9 +
7
• 5 = 14
c(7) = 4
c(7) = 0





Finally, by (4) we obtain:

$mul(0) = 5 \cdot 14 + 35 \cdot 10 + 35 \cdot 10 + 14 \cdot 5 = 840;$
$mul(1) = mul(-1) = 1 \cdot 6 + 16 \cdot 21 + 9 \cdot 14 = 468;$
$mul(2) = mul(-2) = 15 \cdot 5 + 35 \cdot 9 + 21 \cdot 5 = 495;$
$mul(3) = mul(-3) = 20 \cdot 10 + 35 \cdot 16 + 14 \cdot 5 = 830;$
$mul(4) = mul(-4) = 15 \cdot 10 + 14 \cdot 9 = 276;$
$mul(5) = mul(-5) = 6 \cdot 5 = 30;$
$mul(6) = mul(-6) = 1 \cdot 1 = 1.$

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