S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 12, стр. 955–959 (2015) УДК 515.16
DOI 10.17377/semi.2015.12.081 MSC 57M60, 57S17, 57S25

ABOUT THREE CONJECTURES ON FINITE GROUP ACTIONS ON 3-MANIFOLDS

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Abstract. We give a survey centering around three conjectures on finite group actions on homology 3-spheres, and also on actions on closed 3-manifolds containing a hyperelliptic rotation.

Keywords: Finite group action, 3-manifold, homology 3-sphere, hyperelliptic rotation.

1. Finite group actions on homology 3-spheres

As a general hypothesis, all 3-manifolds considered in the present paper will be closed and orientable, and all finite group actions will be smooth, orientation-preserving and faithful. By the geometrization of finite group actions on 3-manifolds due to Thurston and Perelman, every finite group acting on the 3-sphere S^3 is conjugate to a subgroup of the orthogonal group SO(4). Our first conjecture concerns finite groups acting on an integer homology 3-sphere (i.e., a closed 3-manifold with the integer homology of the 3-sphere). Recall that a finite group action is *free* if every nontrivial element has empty fixed point set. Free actions of finite groups on homology 3-spheres is a classical topic and quite well understood, the remaining problems are mainly connected to the class of Milnor-groups $Q(8a, b, c)$ (see [17]). The Milnor groups are not subgroups of the orthogonal group SO(4), so by the geometrization they do not admit actions on S^3; however, by [16], some of the Milnor groups admit a free action on a homology 3-sphere. Our first conjecture concerns *nonfree* actions on homology 3-spheres.

Zimmermann, B.P., About three conjectures on finite group actions on 3-manifolds.

Received November, 18, 2015, published December, 9, 2015.

Conjecture 1. *A finite group admitting a nonfree action on a homology 3-sphere is isomorphic to a subgroup of the orthogonal group* SO(4).

Concerning free actions, if a finite group G acts freely on a homology 3-sphere then G has periodic cohomology, of period four, and by a paper of Milnor [17] at most one involution. A list of the groups of cohomological period four with at most one involution is given in the paper of Milnor but not all of these groups admit a free action on a homology 3-sphere. Some groups of the Milnor-list have been excluded by Lee [9], and there remain then the finite subgroups of SO(4) acting freely on S^3 and some of the Milnor-groups $Q(8a, b, c)$. As noted above, the Milnor-groups do not act on S^3 but some of them admit an action on a homology 3-sphere (and some others don't; however, the situation is not completely understood here).

For arbitrary actions, a short list of the candidates of the finite *nonsolvable* groups which possibly admit a (free or nonfree) action on a homology 3-sphere is given in [11] and elaborated in [21], and this list is very close to the list of the finite, nonsolvable subgroups of SO(4) (see the discussion in [21, section 3] for some of the remaining problems). Also, by [1, Theorem 3], Conjecture 1 is true for finite groups of odd order.

We note that the analogous situation in dimension 4 is better understood: by work started in [12],[13] and completed in [2], a finite group which acts on a homology 4-sphere is isomorphic to a subgroup of SO(5).

2. Finite group actions on 3-manifolds containing a hyperelliptic rotation

By section 1 the class of groups which admit an action on an integer homology 3-sphere is quite restricted. On the other hand, it is well-known that every finite group admits free and also nonfree actions on a closed 3-manifold, and also on a rational homology sphere, but the situation becomes more difficult if one wants to say more about the types of elements with nonempty fixed point set. We say that a periodic diffeomorphism of a 3-manifold M is a *rotation* if it has nonempty, connected fixed point set (a simple closed curve) and all of its nontrivial powers have the same fixed point set, and *hyperelliptic* if the quotient space is S^3; hence M admits a hyperelliptic rotation of order n if and only if M is the n-fold cyclic branched covering of a knot in S^3. For example, if a finite group G acts on a homology 3-sphere then every nontrivial element with nonempty fixed point set is a rotation and the quotient space is again a homology 3-sphere. Taking S^3 as the universal model and supported by computational and theoretical evidence, we state the following:

Conjecture 2. *A finite group acting on a closed 3-manifold and containing a hyperelliptic rotation (a hyperelliptic rotation of order two) is isomorphic to a subgroup of the orthogonal group* SO(4).

An interesting special case of Conjecture 2 is that of a finite nonabelian simple group G acting on a closed 3-manifold M and containing a rotation or a hyperelliptic rotation of order two. If G contains a rotation of order two then, by [18, Theorem 1.1], G is isomorphic to a linear fractional group $\mathrm{PSL}_2(q)$, for an odd prime power $q \geq 5$;

we note that this result is highly nontrivial since it is based on the Gorenstein-Harada classification of the nonabelian simple groups of sectional 2-rank at most four (see [5, p.6] or [19, Theorem 6.8.12]).

If G contains a hyperelliptic rotation of order two then, by ([22, Theorem 1], G is isomorphic to the alternating or linear fractional group $\mathbb{A}_5 \cong \mathrm{PSL}_2(5)$; since this is the unique nonabelian simple subgroup of SO(4), Conjecture 2 holds for finite simple groups containing a hyperelliptic rotation of order two. We note that the reduction from $\mathrm{PSL}_2(q)$ to $\mathrm{PSL}_2(5)$ is based on purely topological arguments: the 2-fold branched covering of a knot in S^3 is a mod 2 homology 3-sphere (i.e., with coefficients in the integers mod two); one shows that in the present case M is in fact an integer homology sphere and then applies [23] to reduce from $\mathrm{PSL}_2(q)$ to $\mathrm{PSL}_2(5)$. Moreover, the result in this case is somewhat easier since the Gorenstein-Harada classification can be replaced by the much shorter Gorenstein-Walter classification of the finite simple groups with a dihedral Sylow 2-subgroup, see [15].

We note in some cases there are much shorter proofs that $\mathbb{A}_5$ is the only nonabelian simple group containing a hyperelliptic rotation of order two. Let G be a finite simple group acting on a 3-manifold M and containing a hyperelliptic rotation r of order two, in particular M is the 2-fold branched covering of a knot K in S^3. Suppose that the only finite symmetry of K is a rotation of order two (i.e., a strong inversion or a cyclic symmetry of K). Then the centralizer of the rotation r in G is an abelian group $\mathbb{Z}_2 \times \mathbb{Z}_2$ and, by [19, p.129], G is isomorphic to $\mathbb{A}_5$.

So the presence of a rotation of order two has a strong impact on the group, and even more the presence of a hyperelliptic rotation of order two. We note that the general structure of groups acting with a rotation of order two is considered in [10].

Conjecture 2 holds also for finite p-groups. If a finite p-group G acts on a 3-manifold M and contains a hyperelliptic rotation then M is the p^m-fold cyclic branched covering of a knot in S^3 and hence a mod p homology 3-sphere (i.e., with coefficients in the integers mod p, see [4, p.16]). By [3], every finite p-group which acts on a mod p homology sphere admits also an orthogonal action on a sphere of the same dimension, so in dimension three G is isomorphic to a subgroup of SO(4).

3. Actions on mod 2 homology 3-spheres and actions of $\mathrm{PSL}_2(q)$

The 2-fold branched covering of a knot in S^3 is a mod 2 homology 3-sphere, and a short list of the candidates of the finite nonsolvable groups which possibly admit an action on a mod 2 homology 3-sphere is given in [MeZ1] (using Gorenstein-Harada again). The simple groups in this list are again the linear fractional groups $\mathrm{PSL}_2(q)$, for an odd prime power $q \geq 5$ (and for the case of simple groups Gorenstein-Walter suffices, by [15]). Examples of actions of $\mathrm{PSL}_2(q)$ on mod 2 homology 3-spheres for various small values of q are constructed in [22, section 4] (and the impression is that the only obstruction here is the length of the computation).

Suppose that a group $G = \mathrm{PSL}_2(q)$ acts on a mod 2 homology 3-sphere; since G has a subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$ which cannot act freely on a mod 2 homology sphere, G contains a rotation of order two which, applying [22] as in section 2, cannot be hyperelliptic if $q > 5$. In particular, Conjecture 2 is not true just for rotations (i.e., not necessarily hyperelliptic).

Conjecture 3. *i) For an odd prime power $q \geq 5$, every linear fractional group $\mathrm{PSL}_2(q)$ admits an action on a mod 2 homology 3-sphere.*

ii) If $q > 5$, an action of $\mathrm{PSL}_2(q)$ on a closed 3-manifold does not contain a hyperelliptic rotation.

Note that part ii) is a special case of Conjecture 2.

Questions. *i) For which values of q can an action of $\mathrm{PSL}_2(q)$ on a closed 3-manifold contain a rotation of order larger than two?*

ii) What are the finite groups which admit an action with a rotation on a closed 3-manifold? Or an action with a rotation of order two?

See [10] for results on groups which contain a rotation of order two.

An interesting example of an action of a group $\mathrm{PSL}_2(q)$ containing a rotation of order greater than two is the following. There is an isometric action of $\mathrm{PSL}_2(11)$ on a closed hyperbolic 3-manifold $\mathcal{M}_0$ obtained as a regular covering of the hyperbolic tetrahedral orbifold of type $(3,5,3)$ (e.g., using the low-index subgroup procedure of GAP, the regular covering defined by a normal core, see [22]; the abelianization of the normal core, i.e., the first homology of $\mathcal{M}_0$, is isomorphic to $\mathbb{Z}^{10}$). Considering a subgroup $\mathbb{Z}_{11} \rtimes \mathbb{Z}_5$ of $\mathrm{PSL}_2(11)$ (a semidirect product with normal subgroup $\mathbb{Z}_{11}$), by [6] the subgroup $\mathbb{Z}_{11}$ acts freely on M_0 and the quotient manifold $M_0/\mathbb{Z}_{11}$ is the 5-fold cyclic branched covering of the figure-8 knot, with the induced action of $\mathbb{Z}_5$ on $M_0/\mathbb{Z}_{11}$ as the group of covering transformations, and in particular with nonempty, connected fixed point set. But then also the fixed point set of the action of $\mathbb{Z}_5$ on $\mathcal{M}_0$ is connected (since otherwise some element of $\mathbb{Z}_{11}$ would map a component of the fixed point set to another one and hence normalize $\mathbb{Z}_5$ which is not the case). Hence a generator of $\mathbb{Z}_5$ is a rotation of order five; since the first homology of $\mathcal{M}_0$ is $\mathbb{Z}^{10}$, this rotation cannot be hyperelliptic (i.e., $\mathcal{M}_0$ cannot be the 5-fold cyclic branched covering of a knot in S^3 which would be a mod 5 homology sphere).

Finally we note that explicit isometric actions of $\mathbb{A}_5 \cong \mathrm{PSL}_2(5)$ on hyperbolic homology 3-spheres which contain hyperelliptic rotations of orders two, three or five are constructed in [22, section 5]; alternatively, one may apply the G-equivariant imitation theory of Kawauchi [7] to an action of $\mathbb{A}_5$ on S^3 (as well as to an action on S^3 of any other finite subgroup of SO(4), see [14] and [8] for such applications).

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